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Finite-time stability of linear non-autonomous systems with time-varying delays

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Abstract

In this paper, we investigate the problem of finite-time stability (*FTS*) of linear non-autonomous systems with time-varying delays. By constructing an appropriated function, we derive some explicit conditions in terms of matrix inequalities ensuring that the state trajectories of the system do not exceed a certain threshold over a pre-specified finite time interval. Finally, two examples are given to show the effectiveness of the main results.

Keywords: Finite-time stability; Linear non-autonomous systems; Time-varying delay; Metzler matrix

1 Introduction

During the past decades, finite time stability (*FTS*) in linear systems has received considerable attention since it was first introduced in 1960s. *FTS* is a system property which concerns the quantitative behavior of the state variables over an assigned finite-time interval. A system is *FTS* if, given a bound on the initial condition, its state trajectories do not exceed a certain threshold during a pre-specified time interval. Hence, *FTS* enables us to specify quantitative bounds on the state of a linear system and plays an important role in addressing transient performances of the systems. Therefore, in recent years, many interesting results for *FTS* have been proposed, see [1–5] for instances. It should be noticed that *FTS* and Lyapunov asymptotic stability (LAS) are completely independent concepts. Indeed, a system can be *FTS* but not LAS, and vice versa [6–8]. Asymptotic stability in dynamical systems implies convergence of the system trajectories to an equilibrium state over the infinite horizon. However, in practice, it is desirable that a dynamical system possesses *FTS*, that is, its state norm does not exceed a certain threshold in finite time. Furthermore, LAS is concerned with the qualitative behavior of a system and it does not involve quantitative information (e.g., specific estimates of trajectory bounds), whereas *FTS* involves specific quantitative information.

In the process of investigating linear systems, time delays are frequently encountered [9–12]. And in hardware implementation, time delays usually cause oscillation, instability, divergence, chaos, or other bad performances of neural networks. In recent years, various interesting results have been obtained for the *FTS* of linear autonomous systems. For linear time-invariant systems with constant delay, some finite-time stability conditions have been derived in terms of feasible linear matrix inequalities based on the Lyapunov–Krasovskii functional methods [6, 13–17]. It is worth noting that non-autonomous phenomena often occur in many realistic systems; for instance, when considering a long-term dynamical behavior of the system, the parameters of the system usually change along with time [18–22]. Moreover, stability analysis for non-autonomous systems usually requires specific and quite different tools from the autonomous ones (systems with constant coefficients). To our knowledge, there are a few results concerned with the *FTS* of non-autonomous systems with time-varying delays. In addition, it should be noted that the conditions for *FTS* of the time-varying system are usually based on the Lyapunov or Riccati matrix differential equation [7, 23, 24], which leads to indefinite matrix inequalities and lacks effective computational tools to solve them. Therefore, when dealing with the *FTS* of time-varying systems with delays, an alternative approach is clearly needed, which motivates our present investigation.

In present paper, the problems of *FTS* are investigated for linear non-autonomous systems with discrete and distributed time-varying delays. By constructing an appropriated function, some sufficient conditions are derived to guarantee the *FTS* of the addressed linear non-autonomous systems. We do not impose any restriction on the states of the system in this sense, which is better than the results in [25]. The rest of this paper is organized as follows. In Sect. 2, some notations, definitions, and a lemma are given. In Sect. 3, we present the main results. Two examples are provided in Sect. 4 to demonstrate the effectiveness of the proposed criteria. Section 5 shows the summary of this paper.

2 Preliminaries

Notations Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of positive numbers, $\mathbb{R}^n$ the n-dimensional real spaces equipped with the norm $\|x\|_\infty = \max_{i\in\underline{n}} |x_i|$ and $\mathbb{R}^{n\times m}$ the $n \times m$-dimensional real spaces. I denotes the identity matrix with appropriate dimensions and $\Lambda = \{1, 2, \dots, n\}$. For any interval $J \subseteq \mathbb{R}$, set $S \subseteq \mathbb{R}^k$ $(1 \le k \le n)$, $C(J, S) = \{\varphi : J \to S$ is continuous$\}$. $\mathscr{F} = \{\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable$\}$ and $A \vee B = \max\{A, B\}$ for constants A and B. $u = (u_i)$, $v = (v_i)$ in $\mathbb{R}^n$, $u \ge v$ iff $u_i \ge v_i$, $\forall i \in \Lambda$; $u \gg v$ iff $u_i > v_i$, $i \in \Lambda$.

Consider the following linear non-autonomous system with time-varying delays:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + D(t)x(t-\tau(t)) + G(t)\int_{t-\kappa(t)}^{t} x(s)\,ds, \quad t \ge 0, \\ x(t) = \phi(t), \quad t \in [-d, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $A(t) = (a_{ij}(t)) \in \mathbb{R}^{n\times n}$, $D(t) = (d_{ij}(t)) \in \mathbb{R}^{n\times n}$, and $G(t) = (g_{ij}(t)) \in \mathbb{R}^{n\times n}$ are the system matrices; $\tau(t)$ and $\kappa(t)$ are time-varying delays satisfying $0 \le \underline{\tau} \le \tau(t) \le \bar{\tau}$, $0 \le \kappa(t) \le \bar{\kappa}$, $t \ge 0$; $\phi(t) = (\phi_i(t)) \in C([-d, 0], \mathbb{R}^n)$ is the initial condition, where $d = \bar{\tau} \vee \bar{\kappa}$. Denote $|\phi_i| = \sup_{-d\le t\le 0} |\phi_i(t)|$ and $\|\phi\|_\infty = \max_{i\in\Lambda} |\phi_i|$.

Definition 1 (Amato et al. [7]) Assume that $x(t, \phi) = x(t, 0, \phi)$ is the solution of system (1) through $(0, \phi)$. Given three positive constants r_1, r_2, T with $r_1 < r_2$, linear non-autonomous system (1) is said to be *FTS* with respect to (T, r_1, r_2) if

$$\|\phi\|_\infty \le r_1$$

implies that

$$\|x(t,\phi)\|_\infty \le r_2, \quad t \in [0,T].$$

Definition 2 (Liao et al. [26]) Let $I := [0,+\infty)$, $f(t) \in C(I,\mathbb{R})$. For any $t \in I$, the following derivative

$$D^+f(t) := \overline{\lim_{h\to 0^+}} \frac{1}{h}\big(f(t+h)-f(t)\big) = \lim_{h\to 0^+} \sup \frac{1}{h}\big(f(t+h)-f(t)\big)$$

is called right-upper derivative of $f(t)$.

Let $A(t) = (a_{ij}(t))$, $D(t) = (d_{ij}(t))$, and $G(t) = (g_{ij}(t))$ be given matrices with continuous elements. We make the following assumptions which are usually used for a time-varying system (also see [27]). For given $T > 0$, assume that:

(A_1) $a_{ii}(t) \le \bar{a}_{ii}$, $i \in \Lambda$, $|a_{ij}(t)| \le \bar{a}_{ij}$, $i \ne j$, $i,j \in \Lambda$, $t \in [0,T]$.

(A_2) $|d_{ij}(t)| \le \bar{d}_{ij}$, $|g_{ij}(t)| \le \bar{g}_{ij}$, $i,j \in \Lambda$, $t \in [0,T]$.

We denote $\mathcal{A} = (\bar{a}_{ij})$, $\mathcal{D} = (\bar{d}_{ij})$, $\mathcal{G} = (\bar{g}_{ij})$. Next, we recall here some properties of a Metzler matrix. For more details, one can refer to [28]. A matrix $A = (a_{ij})$ is called a Metzler matrix if $a_{ij} \le 0$ whenever $i \ne j$ and all principal minors of A are positive. The following lemma is used in our main results.

Lemma 1 (Hien et al. [29]) *Let $A = (a_{ij})$ be an off-diagonal non-positive matrix, $a_{ii} > 0$, $i \in \Lambda$. Then the following statements are equivalent*:

(i) *A is a nonsingular M-matrix.*

(ii) $\operatorname{Re} \lambda_k(A) > 0$ *for all eigenvalues $\lambda_k(A)$ of A.*

(iii) *There exist a matrix $B \ge 0$ and a scalar $s > \rho(B)$ such that $A = sI_n - B$, where $\rho(B) = \max\{|\lambda_k(A)|\}$ denotes the spectral radius of B.*

(iv) *There exist a vector $\xi \in \mathbb{R}^n$ and $\xi \gg 0$ such that $A\xi \gg 0$.*

(v) *There exist a vector $\eta \in \mathbb{R}^n$ and $\eta \gg 0$ such that $A^T\eta \gg 0$.*

3 Main results

We are now in a position to state our main result as follows. In this section, we shall investigate the *FTS* of the linear non-autonomous system by constructing an appropriated function and using the Metzler matrix method.

Theorem 1 *Under assumptions* (A_1) *and* (A_2), *linear non-autonomous system* (1) *is* FTS *with respect to (T, r_1, r_2), if there exist three positive scalars r_1, r_2, and T with $r_1 < r_2$, a function $\mu(t) \in \mathscr{F}$, and three constants β_i, $i = 1,2,3$, satisfying*

$$\begin{gathered} \big(\mu(t) \vee 1\big) \le \frac{mr_2}{Mr_1}, \quad t \in [0,T], \\ \frac{\mu(t-\tau(t))}{\mu(t)} \le \beta_1, \qquad \frac{\int_{t-\kappa(t)}^{t} \mu(s)\,ds}{\mu(t)} \le \beta_2, \qquad \frac{\mu'(t)}{\mu(t)} \ge \beta_3. \end{gathered} \tag{2}$$

Moreover, there exists a vector $\xi \in \mathbb{R}^n$ such that

$$\mathcal{M}^0 \xi \ll 0, \tag{3}$$

where $\mathcal{M}^0 = \mathcal{A} + \beta_1\mathcal{D} + \beta_2\mathcal{G} - \beta_3 I$, $m = \min_{i\in\Lambda} \xi_i$, $M = \max_{i\in\Lambda} \xi_i$.

Proof If there exists $\xi \in \mathbb{R}^n$ satisfying (3), then we have

$$(\mathcal{A} + \beta_1 \mathcal{D} + \beta_2 \mathcal{G} - \beta_3 I)\xi \ll 0,$$

that is,

$$\sum_{j=1}^{n} (\bar{a}_{ij} + \beta_1 \bar{d}_{ij} + \beta_2 \bar{g}_{ij})\xi_j \le \beta_3 \xi_i, \quad \forall i \in \Lambda. \tag{4}$$

For convenience, let $x(t) = x(t, 0, \phi)$ be the solution of (1) through $(0, \phi)$. It follows from (1) that

$$\begin{aligned} D^+ \bigl|x_i(t)\bigr| &= \operatorname{sgn}\bigl(x_i(t)\bigr)\dot{x}_i(t) \\ &\le a_{ii}(t)\bigl|x_i(t)\bigr| + \sum_{j=1,j\ne i}^{n} \bigl|a_{ij}(t)\bigr|\bigl|x_j(t)\bigr| + \sum_{j=1}^{n} \bigl|d_{ij}(t)\bigr|\bigl|x_j\bigl(t - \tau(t)\bigr)\bigr| \\ &\quad + \sum_{j=1}^{n} \bigl|g_{ij}(t)\bigr| \int_{t-\kappa(t)}^{t} \bigl|x_j(s)\bigr|\,ds \\ &\le \bar{a}_{ii}\bigl|x_i(t)\bigr| + \sum_{j=1,j\ne i}^{n} \bar{a}_{ij}\bigl|x_j(t)\bigr| + \sum_{j=1}^{n} \bar{d}_{ij}\bigl|x_j\bigl(t - \tau(t)\bigr)\bigr| \\ &\quad + \sum_{j=1}^{n} \bar{g}_{ij} \int_{t-\kappa(t)}^{t} \bigl|x_j(s)\bigr|\,ds, \quad \forall t \ge 0, i \in \Lambda, \end{aligned} \tag{5}$$

where D^+ denotes the Dini upper-right derivative.

Denote the functions $V_i(t)$, $i \in \Lambda$, as follows:

$$V_i(t) = \begin{cases} \frac{1}{m}\|\phi\|_\infty \xi_i, & t \in [-d, 0), \\ \frac{1}{m}\|\phi\|_\infty \xi_i (\mu(t) \vee 1), & t \in [0, T], \end{cases}$$

we have

$$\begin{aligned} &\bar{a}_{ii} V_i(t) + \sum_{j=1,j\ne i}^{n} \bar{a}_{ij} V_j(t) + \sum_{j=1}^{n} \bar{d}_{ij} V_j\bigl(t - \tau(t)\bigr) + \sum_{j=1}^{n} \bar{g}_{ij} \int_{t-\kappa(t)}^{t} V_j(s)\,ds \\ &\quad = \frac{1}{m}\|\phi\|_\infty \left(\bar{a}_{ii}\mu(t)\xi_i + \sum_{j=1,j\ne i}^{n} \bar{a}_{ij}\mu(t)\xi_j + \sum_{j=1}^{n} \bar{d}_{ij}\mu\bigl(t - \tau(t)\bigr)\xi_j + \sum_{j=1}^{n} \bar{g}_{ij}\xi_j \int_{t-\kappa(t)}^{t} \mu(s)\,ds \right) \\ &\quad = \frac{1}{m}\|\phi\|_\infty \mu(t) \left(\bar{a}_{ii}\xi_i + \sum_{j=1,j\ne i}^{n} \bar{a}_{ij}\xi_j + \sum_{j=1}^{n} \bar{d}_{ij} \frac{\mu(t - \tau(t))}{\mu(t)}\xi_j + \sum_{j=1}^{n} \bar{g}_{ij}\xi_j \frac{\int_{t-\kappa(t)}^{t} \mu(s)\,ds}{\mu(t)} \right) \\ &\quad \le \frac{1}{m}\|\phi\|_\infty \mu(t) \left(\bar{a}_{ii}\xi_i + \sum_{j=1,j\ne i}^{n} \bar{a}_{ij}\xi_j + \sum_{j=1}^{n} \bar{d}_{ij}\beta_1\xi_j + \sum_{j=1}^{n} \bar{g}_{ij}\beta_2\xi_j \right) \\ &\quad \le \frac{1}{m}\|\phi\|_\infty \mu(t) \sum_{j=1}^{n} (\bar{a}_{ij} + \beta_1 \bar{d}_{ij} + \beta_2 \bar{g}_{ij})\xi_j \end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{m}\|\phi\|_\infty \mu(t)\beta_3\xi_i \\
&\le \frac{1}{m}\|\phi\|_\infty \mu(t)\frac{\mu'(t)}{\mu(t)}\xi_i \\
&= \frac{1}{m}\|\phi\|_\infty \mu'(t)\xi_i, \quad \forall t\in[0,T], i\in\Lambda.
\end{aligned} \tag{6}$$

Thus, it follows from (6) that

$$\dot{V}_i(t) \ge \bar{a}_{ii}V_i(t) + \sum_{j=1,j\ne i}^{n} \bar{a}_{ij}V_j(t) + \sum_{j=1}^{n}\bar{d}_{ij}V_j\bigl(t-\tau(t)\bigr) + \sum_{j=1}^{n}\bar{g}_{ij}\int_{t-\kappa(t)}^{t} V_j(s)\,ds, \quad t\ge 0.$$

We claim that

$$\bigl|x_i(t)\bigr| \le V_i(t), \quad \forall t\in[0,T], i\in\Lambda.$$

Let

$$\rho_i(t) = \bigl|x_i(t)\bigr| - V_i(t), \quad t\ge -d.$$

Then we have

$$\bigl|x_i(t)\bigr| \le |\phi_i| \le \|\phi\|_\infty \le \frac{1}{m}\xi_i\|\phi\|_\infty = V_i(t), \quad t\in[-d,0],$$

and hence

$$\rho_i(t)\le 0, \quad t\in[-d,0], i\in\Lambda.$$

Next, we claim

$$D^+\rho_i(t)\le 0, \quad t\in[0,T].$$

If not, assume that there exist an index $i\in\Lambda$ and $t_1\in(0,T]$ such that

$$\rho_i(t_1)=0, \qquad \rho_i(t)>0, \quad t\in(t_1,t_1+\delta), \delta>0$$

and

$$\rho_j(t)\le 0, \quad \forall t\in[-d,t_1], j\in\Lambda.$$

Then

$$D^+\rho_i(t_1)>0.$$

However, it follows from (5) and (6) that for $t\in[0,t_1]$,

$$\begin{aligned}
D^+\rho_i(t) &\le \bar{a}_{ii}\rho_i(t) + \sum_{j=1,j\ne i}^{n}\bar{a}_{ij}\rho_j(t) + \sum_{j=1}^{n}\bar{d}_{ij}\rho_j\bigl(t-\tau(t)\bigr) + \sum_{j=1}^{n}\bar{g}_{ij}\int_{t-k(t)}^{t}\rho_j(s)\,ds \\
&\le \bar{a}_{ii}\rho_i(t),
\end{aligned}$$

therefore,

$$D^+\rho_i(t_1) \leq 0,$$

which yields a contradiction. This shows that

$$\rho_i(t) \leq 0, \quad t \in [0,T], i \in \Lambda,$$

thus, we obtain

$$\left|x_i(t)\right| \leq \frac{1}{m}\|\phi\|_\infty \xi_i\big(\mu(t) \vee 1\big), \quad t \in [0,T], i \in \Lambda.$$

Consequently,

$$\left\|x(t)\right\|_\infty \leq \frac{1}{m}\|\phi\|_\infty\|\xi\|_\infty\big(\mu(t) \vee 1\big) \leq \frac{M}{m}\|\phi\|_\infty\big(\mu(t) \vee 1\big), \quad t \in [0,T]. \tag{7}$$

If $\|\phi\|_\infty \leq r_1$, then it follows from (2) and (7) that

$$\left\|x(t)\right\|_\infty \leq r_2, \quad \forall t \in [0,T].$$

This shows that system (1) is *FTS* with respect to (T,r_1,r_2). The proof is complete. □

Corollary 1 *Under assumptions* (A_1) *and* (A_2), *linear non-autonomous system* (1) *is* FTS *with respect to* (T,r_1,r_2) *if there exist three positive scalars* r_1, r_2, *and* T *with* $r_1 < r_2$, *a function* $\mu(t)$ *that is monotonous increasing and* $\mu(t) \geq 1$, $t \in [0,T]$, *and three constants* β_i, $i = 1,2,3$, *satisfying*

$$\mu(T) \leq \frac{mr_2}{Mr_1}, \quad t \in [0,T],$$

$$\frac{\mu(t-\tau(t))}{\mu(t)} \leq \beta_1, \qquad \frac{\int_{t-\kappa(t)}^t \mu(s)\,ds}{\mu(t)} \leq \beta_2 = \bar{\kappa}, \qquad \frac{\mu'(t)}{\mu(t)} \geq \beta_3.$$

Moreover, there exists a vector $\xi \in \mathbb{R}^n$ *such that*

$$\mathcal{M}^0\xi \ll 0,$$

where $\mathcal{M}^0 = \mathcal{A} + \beta_1\mathcal{D} + \bar{\kappa}\mathcal{G} - \beta_3 I$, $m = \min_{i\in\Lambda}\xi_i$, $M = \max_{i\in\Lambda}\xi_i$.

Corollary 2 *Under assumptions* (A_1) *and* (A_2), *linear non-autonomous system* (1) *is* FTS *with respect to* (T,r_1,r_2), *if there exist three positive scalars* r_1, r_2, *and* T *with* $r_1 < r_2$, *a function* $\mu(t) \equiv \mu > 0$, *and three scalars* β_i, $i = 1,2,3$, *satisfying*

$$\mu \leq \frac{mr_2}{Mr_1}, \quad t \in [0,T],$$

$$1 \leq \beta_1, \qquad \kappa(t) \leq \beta_2, \qquad \beta_3 \leq 0.$$

Moreover, there exists a vector $\xi \in \mathbb{R}^n$ *such that*

$$\mathcal{M}^0\xi \ll 0,$$

where $\mathcal{M}^0 = \mathcal{A} + \beta_1\mathcal{D} + \beta_2\mathcal{G} - \beta_3 I$, $m = \min_{i\in\Lambda}\xi_i$, $M = \max_{i\in\Lambda}\xi_i$.

Remark 1 In [25], Hien considered the *FTS* of system (1) and derived some conditions for exponential estimation. In this paper, we study the *FTS* of system (1) via the auxiliary function μ, and some new sufficient conditions for *FTS*, which are different from the results in [25], are derived. In other words, our development result is more general than the result in [25].

4 Example

In this section, we present two numerical examples to illustrate the effectiveness of the proposed results.

Example 1 Consider the following system:

$$\dot{x}(t) = A(t)x(t) + D(t)x\bigl(t - \tau(t)\bigr) + G(t)\int_{t-\kappa(t)}^{t} x(s)\,ds, \quad t \geq 0, \tag{8}$$

where

$$A(t) = \begin{pmatrix} -4 & |\cos t| \\ \sin^2 2t & -4 \end{pmatrix}, \qquad D(t) = \begin{pmatrix} \sin^2 t & |\cos 2\sqrt{t}| \\ 0 & \cos^2 t \end{pmatrix},$$
$$G(t) = \begin{pmatrix} |\sin\sqrt{t}| & 0 \\ \sin^2 3t & |\cos 2t| \end{pmatrix},$$

and $\tau(t) = |\sin 4t|$, $\kappa(t) = |\cos t|$.

It is easy to see that (A_1) and (A_2) hold, and then we have

$$\mathcal{A} = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, \qquad \mathcal{D} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let

$$\mu(t) = 1 + 0.01t, \quad t \geq 0.$$

Thus, we have

$$\frac{\mu(t-\tau(t))}{\mu(t)} = \frac{1 + 0.01(t-\tau(t))}{1 + 0.01t} \leq 1,$$
$$\frac{\int_{t-\kappa(t)}^{t}\mu(s)\,ds}{\mu(t)} = \frac{\int_{t-\kappa(t)}^{t}(1 + 0.01s)\,ds}{1 + 0.01t} \leq \frac{(1 + 0.01t)\kappa(t)}{1 + 0.01t} = \kappa(t) \leq \bar{\kappa},$$
$$\frac{\mu'(t)}{\mu(t)} = \frac{0.01}{1 + 0.01t} \geq \frac{0.01}{1 + 0.01T}.$$

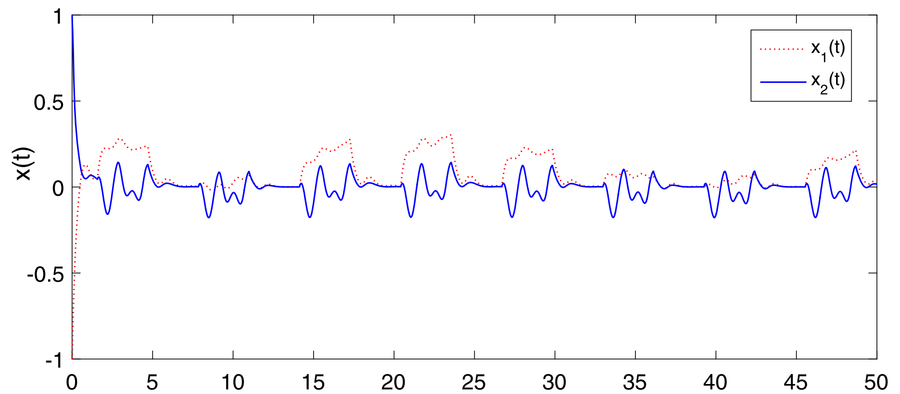

Figure 1 Simulation for Example 1 when $(T, r_1, r_2) = (21.3144, 1, 1.213144)$

Let

$$\beta_1 = 1, \qquad \beta_2 = \bar{\kappa}, \qquad \beta_3 = \frac{0.01}{1 + 0.01T}.$$

It should be noted that system (8) does not satisfy the Lyapunov stability conditions proposed in [27]. More precisely, in this case the matrix $\mathcal{M} = \mathcal{A} + \mathcal{D} + \bar{\kappa}\mathcal{G}$ is not invertible, and hence it does not satisfy conditions of Theorem 2.5 in [27]. However, $\mathcal{M}^0 = \mathcal{M} - \beta_3 I$ satisfies (3) and the domain of the solution $\xi \in R^2$ of (3) is defined by $\frac{2}{2+\beta_3}\xi_1 < \xi_2 < \frac{2+\beta_3}{2}\xi_1$.

Case I. Let us take $r_1 = 1$, $r_2 = 1.25$, and then system (8) is *FTS* with respect to (T, r_1, r_2) for any finite time $0 < T \leq T_{\max} = 25$, and in this case $\beta_3 = 0.08$. Note that in [25], the maximum value of T is $T_{\max} = 21.3144$. Hence, our result is more general than [25].

Case II. Let us take $r_1 = 1$, when $T = 21.3144$, we obtain $\beta_3 = 0.8243$ and $r_2 = 1.213144$.

It should be mentioned that the simulation in Fig. 1 of Example 1 is *FTS* with respect to (T, r_1, r_2), but not LAS.

Example 2 Consider system (8) with parameters as follows:

$$A(t) = \begin{pmatrix} -4 & |\cos t| & |\sin 2t| \\ \sin^2 3t & -3 & -2 \\ |\sin 4t| & -2 & -4 \end{pmatrix}, \qquad D(t) = \begin{pmatrix} \cos^2 t & 0 & |\cos 4t| \\ 0 & \cos^2 3t & 0 \\ \sin^2 4t & 0 & |\sin 5t| \end{pmatrix},$$

$$G(t) = \begin{pmatrix} |\sin t| & \sin^2 t & 0 \\ \sin^2 2t & 0 & 0 \\ 0 & 0 & |\sin 5t| \end{pmatrix},$$

and $\tau(t) = |\cos 2t|$, $\kappa(t) = |\cos 3t|$.

It is easy to see that (A_1) and (A_2) hold, and we have

$$\mathcal{A} = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -3 & -2 \\ 1 & -2 & -4 \end{pmatrix}, \qquad \mathcal{D} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

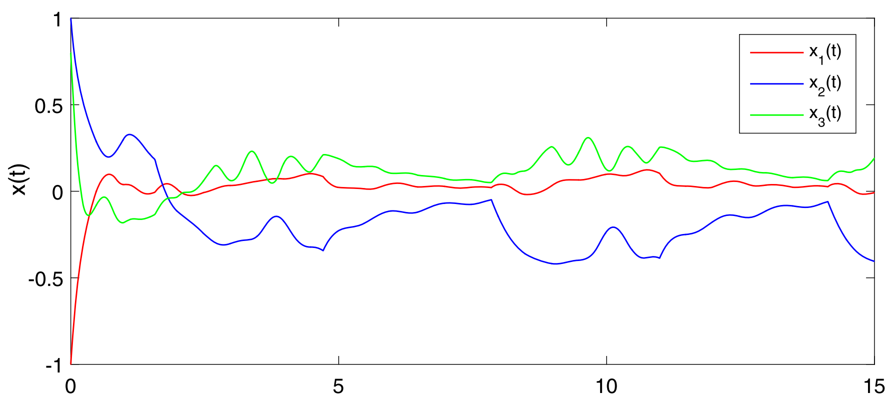

Figure 2 Simulation for Example 2 when $(T, r_1, r_2) = (15, 1, 2.5)$

Let

$$\mu(t) = 0.5 + 0.05t, \quad t \geq 0.$$

Thus, we have

$$\frac{\mu(t-\tau(t))}{\mu(t)} = \frac{0.5 + 0.05(t-\tau(t))}{0.5 + 0.05t} \leq 1,$$
$$\frac{\int_{t-\kappa(t)}^{t} \mu(s)\,ds}{\mu(t)} = \frac{\int_{t-\kappa(t)}^{t} (0.5 + 0.05s)\,ds}{0.5 + 0.05t} \leq \frac{(0.5 + 0.05t)\kappa(t)}{0.5 + 0.05t} = \kappa(t) \leq \bar{\kappa},$$
$$\frac{\mu'(t)}{\mu(t)} = \frac{0.05}{0.5 + 0.05t} \geq \frac{0.05}{0.5 + 0.05T}.$$

Let

$$\beta_1 = 1, \qquad \beta_2 = \bar{\kappa}, \qquad \beta_3 = \frac{0.05}{0.5 + 0.05T}.$$

In this case the matrix $\mathcal{M} = \mathcal{A} + \mathcal{D} + \bar{\kappa}\mathcal{G}$ is not invertible, and hence it does not satisfy conditions of Theorem 2.5 in [27]. However, $\mathcal{M}^0 = \mathcal{M} - \beta_3 I$ satisfies (3) and the domain of the solution $\xi \in R^3$ of (3) is defined by

$$2\xi_2 + 2\xi_3 < (2 + \beta_3)\xi_1, \qquad 2\xi_1 - 2\xi_3 < (2 + \beta_3)\xi_2, \qquad 2\xi_1 - 2\xi_2 < (2 + \beta_3)\xi_3.$$

Let us take $r_1 = 1$, $r_1 = 2.5$, and then system (8) is *FTS* with respect to (T, r_1, r_2) for any finite time $0 < T \leq T_{\max} = 15$, and in this case $\beta_3 = 0.04$, see Fig. 2.

5 Conclusion

In the present paper, we have investigated the *FTS* of a class of non-autonomous systems with time-varying delays. Some new sufficient conditions for *FTS* have been derived in terms of inequalities for a type of Metzler matrixes. Finally, two examples were provided to show the effectiveness of the proposed method.

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Competing interests
The authors declare that none of the authors have any competing interests in the manuscript.

Authors' contributions
The main idea of this paper was proposed by YX and LX. YX prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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