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# Oscillation of a nonlinear impulsive differential equation system with piecewise constant argument

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**Abstract**
We deal with a nonlinear impulsive differential equation system with piecewise constant argument. We prove the existence and uniqueness of a solution. Moreover, we obtain sufficient conditions for the oscillation of the solution.

**MSC:** 34K11; 34K45

**Keywords:** Impulsive differential equation; Piecewise constant argument; Oscillation

## 1 Introduction

In this paper, we consider a nonlinear impulsive differential equations system with piecewise constant argument of the form

$$\begin{cases} x'(t) = -a(t)x(t) - x([t-1])f(y([t])) + h_1(x([t])), \\ y'(t) = -b(t)y(t) - y([t-1])g(x([t])) + h_2(y([t])), \end{cases} \quad t \neq n \in \mathbb{Z}^+, t > 0, \tag{1}$$

$$\begin{cases} \Delta x(n) = x(n^+) - x(n^-) = c_n x(n), \\ \Delta y(n) = y(n^+) - y(n^-) = d_n y(n), \end{cases} \quad n \in \mathbb{Z}^+, \tag{2}$$

with the initial conditions

$$x(-1) = x_{-1}, \qquad x(0) = x_0, \qquad y(-1) = y_{-1}, \qquad y(0) = y_0, \tag{3}$$

where $a, b : (0, \infty) \to \mathbb{R}$ are continuous functions, $f, g, h_1, h_2 \in C(\mathbb{R}, \mathbb{R})$, $c_n$ and $d_n$ are sequences of real numbers such that $c_n \neq 1$ and $d_n \neq 1$ for all $n \geq 1$, $\Delta u(n) = u(n^+) - u(n^-)$, $u(n^+) = \lim_{t\to n^+} u(t)$, $u(n^-) = \lim_{t\to n^-} u(t)$, $[\cdot]$ denotes the greatest integer function, and $x_{-1}, x_0, y_{-1}, y_0$ are given real numbers.

Differential equations with piecewise constant arguments (DEPCA) exist in a widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics, and so on. To the best of our knowledge, the first mathematical model that includes a piecewise constant argument was proposed by Busenberg and Cooke [1]. They investigated the following system describing the disease dynamics for $n = 1, 2, \ldots$:

$$\frac{dI^n}{dt}(t) = -c(t)I^n(t) + k(t)S^n(t)I^n(t),$$
$$\frac{dS^n}{dt}(t) = -c(t)S^n(t) - k(t)S^n(t)I^n(t), \quad n < t \le n+1,$$

whereas

$$I^{(1)}(1) = I_0, \qquad S^{(1)}(1) = S_0,$$

where $c$ is the death rate, and $k$ is the horizontal transmission factor. Then, oscillation and stability of DEPCA have been studied by many authors (see [2–6] and the references therein). In 1994, Dai and Singh [7] studied the oscillatory motion of spring-mass systems subject to piecewise constant forces of the form $f(x([t]))$ or $f([t])$. Later, they improved analytical and numerical methods for solving linear and nonlinear vibration problems and showed that a function $f([N(t)]/N)$ is a good approximation to the given continuous function $f(t)$ if $N$ is sufficiently large [8]. This method was also used to find numerical solutions of a nonlinear Froude pendulum and the oscillatory behavior of the pendulum [9]. On the other hand, in 1994, the case of studying discontinuous solutions of differential equations with piecewise continuous arguments has been proposed as an open problem by Wiener [10]. Due to this open problem, linear impulsive differential equations with piecewise constant arguments have been dealt with in [11–13]. Moreover, cellular neural networks with piecewise constant argument have been investigated in [14–16]. In [14], the existence and attractivity of the following cellular neural network with piecewise constant argument was studied:

$$\frac{dx_i(t)}{dt} = -a_i\big([t]\big)x_i(t) + \sum_{j=1}^{n} \big\{c_{ij}\big([t]\big)g_j\big(x_j\big([t]\big)\big)\big\} + d_i\big([t]\big),$$

where $[\cdot]$ is the greatest integer function. Recently, Chiu [17] considered the following neural network:

$$\frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{n} \left\{ b_{ij}(t)f_j\big(x_j(t)\big) + c_{ij}(t)g_j\left(x_j\left(2\left[\frac{t+1}{2}\right]\right)\right)\right\} + d_i(t),$$
$$t \ne 2k-1,$$
$$\Delta x_i|_{t=2k-1} = J_k\big(x_i\big(2k-1^-\big)\big), \quad i = 1, 2, \ldots, n, k \in \mathbb{N}.$$

Although nonlinear differential equations with piecewise constant arguments have many applications in real-world problems, there are only a few papers on the oscillation of nonlinear differential equations with piecewise constant arguments [18–20]. So, we study oscillation of system (1)–(2).

In Sect. 2, we prove the existence and uniqueness of the solutions, Sect. 3 consists of our main results. Moreover, we give some examples to illustrate our results.

## 2 Existence of solutions

In this section, we obtain the solution of (1)–(3) in terms of the corresponding difference equations system.

**Definition 1** A pair of functions $(x(t), y(t))$ is said to be a *solution* of (1)–(2) if it satisfies the following conditions:

(i) $x : \mathbb{R}^+ \cup \{-1\} \to \mathbb{R}$ and $y : \mathbb{R}^+ \cup \{-1\} \to \mathbb{R}$ are continuous with a possible exception at the points $[t] \in [0, \infty)$,
(ii) $x(t)$ and $y(t)$ are right continuous and have left-hand limits at the points $[t] \in [0, \infty)$,
(iii) $x(t)$ and $y(t)$ are differentiable and satisfy (1) for any $t \in \mathbb{R}^+$ with a possible exception at the points $[t] \in [0, \infty)$ where one-sided derivatives exist,
(iv) $(x(n), y(n))$ satisfies (2) for $n \in \mathbb{Z}^+$.

**Theorem 1** *If $c_n \neq 1$ and $d_n \neq 1$ for all $n \geq 1$ then the initial value problem* (1)–(3) *has a unique solution $(x(t), y(t))$ on $[0, \infty) \cup \{-1\}$, which can be formulated on the interval $n \leq t < n+1, n \in \mathbb{N} = \{0, 1, 2, \ldots, \}$, as*

$$\begin{cases} x(t) = \exp(-\int_n^t a(u)\,du)\{x(n) \\ \qquad + [-x(n-1)f(y(n)) + h_1(x(n))] \int_n^t \exp(\int_n^s a(u)\,du)\,ds\}, \\ y(t) = \exp(-\int_n^t b(u)\,du)\{y(n) + [-y(n-1)g(x(n)) \\ \qquad + h_2(y(n))] \int_n^t \exp(\int_n^s b(u)\,du)\,ds\}, \end{cases} \tag{4}$$

*where $(x(n), y(n))$ is the unique solution of the difference equations system*

$$\begin{cases} x(n+1) = \frac{1}{1-c_{n+1}} \exp(-\int_n^{n+1} a(u)\,du)\{x(n) + [-x(n-1)f(y(n)) \\ \qquad + h_1(x(n))] \int_n^{n+1} \exp(\int_n^s a(u)\,du)\,ds\}, \\ y(n+1) = \frac{1}{1-d_{n+1}} \exp(-\int_n^{n+1} b(u)\,du)\{y(n) + [-y(n-1)g(x(n)) \\ \qquad + h_2(y(n))] \int_n^{n+1} \exp(\int_n^s b(u)\,du)\,ds\} \end{cases} \tag{5}$$

*for $n \geq 0$ with initial conditions* (3).

*Proof* Let $(x_n(t), y_n(t)) \equiv (x(t), y(t))$ be a solution of (1)–(2) on $n \leq t < n+1$. So, system (1) can be rewritten in the form

$$\begin{cases} x'(t) + a(t)x(t) = -x(n-1)f(y(n)) + h_1(x(n)), \\ y'(t) + b(t)y(t) = -y(n-1)g(x(n)) + h_2(y(n)). \end{cases} \tag{6}$$

From (6), for $n \leq t < n+1$, we get

$$\begin{cases} x_n(t) = \exp(-\int_n^t a(u)\,du)\{x(n) + [-x(n-1)f(y(n)) \\ \qquad + h_1(x(n))] \int_n^t \exp(\int_n^s a(u)\,du)\,ds\}, \\ y_n(t) = \exp(-\int_n^t b(u)\,du)\{y(n) + [-y(n-1)g(x(n)) \\ \qquad + h_2(y(n))] \int_n^t \exp(\int_n^s b(u)\,du)\,ds\}. \end{cases} \tag{7}$$

On the other hand, for $n-1 \le t < n$, we have

$$\begin{cases} x_{n-1}(t) = \exp(-\int_{n-1}^{t} a(u)\,du)\{x(n-1) + [-x(n-2)f(y(n-1)) \\ \qquad + h_1(x(n-1))] \int_{n-1}^{t} \exp(\int_{n-1}^{s} a(u)\,du)\,ds\}, \\ y_{n-1}(t) = \exp(-\int_{n-1}^{t} b(u)\,du)\{y(n-1) + [-y(n-2)g(x(n-1)) \\ \qquad + h_2(y(n-1))] \int_{n-1}^{t} \exp(\int_{n-1}^{s} b(u)\,du)\,ds\}. \end{cases} \tag{8}$$

Using impulse conditions (2), from (7) and (8) we obtain difference equations system (5).

Considering initial conditions (3), the solution of system (5) is obtained uniquely. Thus, the solution of (1)–(3) is obtained as (4). □

## 3 Oscillatory solutions

**Definition 2** A function $x(t)$ defined on $[0,\infty)$ is called *oscillatory* if there exist two real-valued sequences $\{t_n\}_{n\ge 0}$, $\{t'_n\}_{n\ge 0} \subset [0,\infty)$ such that $t_n \to +\infty$, $t'_n \to +\infty$ as $n \to +\infty$ and $x(t_n) \le 0 \le x(t'_n)$ for $n \ge N$, where $N$ is sufficiently large. Otherwise, $x(t)$ is called *nonoscillatory*.

*Remark* 1 According to Definition 2, a piecewise continuous function $x : [0,\infty) \to \mathbb{R}$ can be oscillatory even if $x(t) \ne 0$ for all $t \in [0,\infty)$.

**Definition 3** A sequence $\{y_n\}_{n\ge -1}$ is said to be *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is called *nonoscillatory*.

**Definition 4** The solution of problem (1)–(3) is called *oscillatory* if each components is oscillatory.

The following result is clear.

**Corollary 1** *If the solution* $(x(n), y(n)), n \ge -1$, *of the difference equation system* (5) *with initial conditions* (3) *is oscillatory, then the solution* $(x(t), y(t))$ *of* (1)–(3) *is also oscillatory.*

*Remark* 2 If $c_n > 1$ and $d_n > 1$ for all $n \in \mathbb{Z}^+$, then from the impulse conditions (2) it is clear that the solution $(x(t), y(t))$ of problem (1)–(3) is already oscillatory.

**Theorem 2** *Assume that there exist* $M_1 > 0$ *and* $M_2 > 0$ *such that* $f(u) \ge M_1$ *and* $g(u) \ge M_2$ *for all* $u \in \mathbb{R}, uh_1(u) < 0$ *and* $uh_2(u) < 0$ *for* $u \ne 0$, *and* $c_n < 1$ *and* $d_n < 1$ *for* $n \in \mathbb{Z}^+$. *If the following conditions are satisfied, then all solutions of system* (5) *are oscillatory*:

$$\limsup_{n\to\infty} (1-c_n) \int_n^{n+1} \exp\left(\int_{n-1}^{s} a(u)\,du\right) ds > \frac{1}{M_1}, \tag{9}$$

$$\limsup_{n\to\infty} (1-d_n) \int_n^{n+1} \exp\left(\int_{n-1}^{s} b(u)\,du\right) ds > \frac{1}{M_2}. \tag{10}$$

*Proof* Let $(x(n), y(n))$ be a solution of (5). Suppose that $x(n) > 0, x(n-1) > 0$, and $x(n-2) > 0$ for $n > N$, where $N$ is sufficiently large. From the first equation of (5) we have

$$\begin{aligned}(1-c_n)\exp\left(\int_{n-1}^{n} a(u)\,du\right)x(n) &= x(n-1) + \bigl[-x(n-2)f\bigl(y(n-1)\bigr) \\ &\quad + h_1\bigl(x(n-1)\bigr)\bigr]\int_{n-1}^{n}\exp\left(\int_{n-1}^{s} a(u)\,du\right)ds \\ &< x(n-1).\end{aligned}$$

Multiplying both sides of this inequality by $-f(y(n))\int_n^{n+1}\exp(\int_n^s a(u)\,du)\,ds < 0$ and adding $x(n) + h_1(x(n))\int_n^{n+1}\exp(\int_n^s a(u)\,du)\,ds$, we obtain from (5) that

$$\begin{aligned}&-(1-c_n)\exp\left(\int_{n-1}^{n} a(u)\,du\right)x(n)f\bigl(y(n)\bigr)\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds + x(n) \\ &\quad + h_1\bigl(x(n)\bigr)\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds \\ &\quad > (1-c_{n+1})x(n+1)\exp\left(\int_n^{n+1} a(u)\,du\right) > 0.\end{aligned}\tag{11}$$

Since $x(n) > 0, n > N$, and $h_1(x(n)) < 0$, from (11) we get

$$1 > (1-c_n)f\bigl(y(n)\bigr)\int_n^{n+1}\exp\left(\int_{n-1}^{s} a(u)\,du\right)ds.$$

So, we have

$$\frac{1}{M_1} \geq \limsup_{n\to\infty}(1-c_n)\int_n^{n+1}\exp\left(\int_{n-1}^{s} a(u)\,du\right)ds,\tag{12}$$

which contradicts (9).

If $x(n) < 0, x(n-1) < 0$, and $x(n-2) < 0$ for $n > N$, then we obtain the same contradiction. So the component $x(n)$ of the solution $(x(n), y(n))$ is oscillatory. Similarly, we can show that the component $y(n)$ is oscillatory under condition (10). Hence, the proof is complete. □

**Corollary 2** *Under the hypotheses of Theorem* 2, *all solutions of system* (1)–(2) *are oscillatory.*

**Theorem 3** *Assume that there exist constants* $K_1, K_2, M_1, M_2 > 0$ *such that* $f(u) \geq M_1$, $g(u) \geq M_2$ *for all* $u \in \mathbb{R}$, $c_n \leq 1 - K_1, d_n \leq 1 - K_2$ *for* $n \in \mathbb{N}$, *and* $uh_1(u) < 0, uh_2(u) < 0$ *for* $u \neq 0$. *Suppose that the following conditions are satisfied*:

$$\frac{1}{4K_1M_1} < \liminf_{n\to\infty}\exp\left(\int_n^{n+1} a(u)\,du\right)\liminf_{n\to\infty}\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds < \infty,\tag{13}$$

$$\frac{1}{4K_2M_2} < \liminf_{n\to\infty}\exp\left(\int_n^{n+1} b(u)\,du\right)\liminf_{n\to\infty}\int_n^{n+1}\exp\left(\int_n^s b(u)\,du\right)ds < \infty.\tag{14}$$

*Then all solutions of* (5) *are oscillatory.*

*Proof* Let $(x(n), y(n))$ be a solution of (5). We need to show that under condition (13), $x(n)$ is oscillatory. Assume that $x(n) > 0, x(n-1) > 0$ for $n > N$, where $N$ is sufficiently large. From the first equation of (5) we obtain that

$$\begin{aligned}1 &= (1-c_{n+1})\frac{x(n+1)}{x(n)}\exp\left(\int_n^{n+1} a(u)\,du\right)\\&\quad+\frac{x(n-1)}{x(n)}f\big(y(n)\big)\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds\\&\quad-\frac{h_1(x(n))}{x(n)}\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds.\end{aligned}$$

Let $v_n = \frac{x(n)}{x(n-1)}$. Since $v(n) > 0, \liminf_{n\to\infty} v_n \geq 0$ and

$$1 \geq (1-c_{n+1})v_{n+1}\exp\left(\int_n^{n+1} a(u)\,du\right)+\frac{1}{v_n}M_1\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds. \tag{15}$$

So, we need to consider two cases.

*Case* 1. Let $\liminf_{n\to\infty} v_n = v = +\infty$. Then, from (15) we get

$$\begin{aligned}1 &\geq \liminf_{n\to\infty}(1-c_{n+1})\liminf_{n\to\infty} v_{n+1}\liminf_{n\to\infty}\exp\left(\int_n^{n+1} a(u)\,du\right)\\&\quad+M_1\liminf_{n\to\infty}\frac{1}{v_n}\liminf_{n\to\infty}\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds,\end{aligned}$$

which is a contradiction. So, we consider the second case.

*Case* 2. Let $\liminf_{n\to\infty} v_n < \infty$. If the first equation of (5) is divided by $x(n-1)$, then we have

$$\begin{aligned}\frac{x(n)}{x(n-1)} &= (1-c_{n+1})\frac{x(n+1)}{x(n-1)}\exp\left(\int_n^{n+1} a(u)\,du\right)\\&\quad+f\big(y(n)\big)\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds\\&\quad-\frac{h_1(x(n))}{x(n-1)}\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds,\end{aligned}$$

and then we obtain that

$$v_n \geq (1-c_{n+1})v_nv_{n+1}\exp\left(\int_n^{n+1} a(u)\,du\right)+M_1\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds. \tag{16}$$

Taking the inferior limit on both sides of inequality (16), we get

$$\begin{aligned}v &\geq v^2\liminf_{n\to\infty}(1-c_{n+1})\liminf_{n\to\infty}\exp\left(\int_n^{n+1} a(u)\,du\right)\\&\quad+M_1\liminf_{n\to\infty}\int_n^{n+1}\exp\left(\int_n^s a(u)\,du\right)ds.\end{aligned}$$

Let $\liminf_{n\to\infty} \exp(\int_n^{n+1} a(u)\,du) = A$ and $\liminf_{n\to\infty} \int_n^{n+1} \exp(\int_n^s a(u)\,du)\,ds = B$. Then the last inequality can be rewritten as

$$v \geq \liminf_{n\to\infty}(1 - c_{n+1})v^2 A + M_1 B. \tag{17}$$

Now we consider two subcases:

(i) If $\liminf_{n\to\infty}(1 - c_{n+1}) = \infty$, then we have a contradiction from (17).

(ii) Assume that $0 < K_1 \leq \liminf_{n\to\infty}(1 - c_{n+1}) < \infty$. Then from (17) we have

$$AK_1 v^2 - v + M_1 B \leq 0$$

or

$$AK_1\left[\left(v - \frac{1}{2K_1 A}\right)^2 + \frac{4M_1 B K_1 A - 1}{4K_1^2 A^2}\right] \leq 0.$$

Since $A > 0$ and $K_1 > 0$, we have

$$\frac{4M_1 B K_1 A - 1}{4K_1^2 A^2} \leq 0,$$

which contradicts condition (13).

In the case of $x(n) < 0, x(n-1) < 0$ for sufficiently large $n > N$, the proof is similar, and we obtain the same contradiction.

On the other hand, if we assume rgar $y(n)$ is a nonoscillatory sequence, then we have a contradiction to condition (14). Hence, $(x(n), y(n))$ is an oscillatory solution of system (5). □

**Corollary 3** *Under the hypothesis of Theorem* 3, *all solutions of system* (1)–(2) *are oscillatory.*

*Remark* 3 In the case of $a(t) \equiv a$ and $b(t) \equiv b$, conditions (9) and (10) are reduced to the following forms, respectively:

$$\limsup_{n\to\infty}(1 - c_n) > \frac{ae^{-a}}{M_1(e^a - 1)},$$
$$\limsup_{n\to\infty}(1 - d_n) > \frac{be^{-b}}{M_2(e^b - 1)}.$$

*Remark* 4 In the case of $a(t) \equiv a$ and $b(t) \equiv b$, conditions (13) and (14) are reduced to the following conditions, respectively:

$$\frac{1}{4K_1 M_1} < \frac{e^a(e^a - 1)}{a},$$
$$\frac{1}{4K_2 M_2} < \frac{e^b(e^b - 1)}{b}.$$

If $c_n \equiv d_n \equiv 0$, $n \in \mathbb{Z}^+$, then we have the nonimpulsive equation system

$$\begin{cases} x'(t) = -a(t)x(t) - x([t-1])f(y([t])) + h_1(x([t])), \\ y'(t) = -b(t)y(t) - y([t-1])g(x([t])) + h_2(y([t])), \end{cases} \quad t > 0. \tag{18}$$

In this case, the following results are clear.

**Corollary 4** *Assume that there exist* $M_1 > 0$ *and* $M_2 > 0$ *such that* $f(u) \geq M_1$ *and* $g(u) \geq M_2$ *for all* $u \in \mathbb{R}$, *and* $uh_1(u) < 0$ *and* $uh_2(u) < 0$ *for* $u \neq 0$. *Suppose that the following conditions are satisfied*:

$$\limsup_{n\to\infty} \int_n^{n+1} \exp\left(\int_{n-1}^s a(u)\,du\right) ds > \frac{1}{M_1},$$
$$\limsup_{n\to\infty} \int_n^{n+1} \exp\left(\int_{n-1}^s b(u)\,du\right) ds > \frac{1}{M_2}.$$

*Then all solutions of system* (18) *are oscillatory.*

**Corollary 5** *Assume that there exist the constants* $M_1, M_2 > 0$ *such that* $f(u) \geq M_1, g(u) \geq M_2$ *for all* $u \in \mathbb{R}$, $0 < K_1 \leq 1$, $0 < K_2 \leq 1$, *and* $uh_1(u) < 0$ *and* $uh_2(u) < 0$ *for* $u \neq 0$. *Suppose that the following conditions are satisfied*:

$$\begin{aligned} \frac{1}{4K_1M_1} &< \liminf_{n\to\infty} \exp\left(\int_n^{n+1} a(u)\,du\right) \liminf_{n\to\infty} \int_n^{n+1} \exp\left(\int_n^s a(u)\,du\right) ds \\ &< \infty, \\ \frac{1}{4K_2M_2} &< \liminf_{n\to\infty} \exp\left(\int_n^{n+1} b(u)\,du\right) \liminf_{n\to\infty} \int_n^{n+1} \exp\left(\int_n^s b(u)\,du\right) ds \\ &< \infty. \end{aligned}$$

*Then all solutions of* (18) *are oscillatory.*

Now, let us consider following nonlinear differential equation:

$$\begin{cases} x'(t) = -a(t)x(t) - x([t-1])f(x([t])) + h(x([t])), \\ \Delta x(n) = c_n x(n), \quad n = 1, 2, \ldots, \end{cases} \quad t \neq n, t > 0, \tag{19}$$

which is investigated in [18] with $h(u) \equiv 0$.

**Corollary 6** *Assume that there exists* $M > 0$ *such that* $f(u) \geq M$ *for* $u \in \mathbb{R}$, $uh(u) < 0$ *for* $u \neq 0$, *and* $c_n < 1$ *for* $n \in \mathbb{Z}^+$. *If*

$$\limsup_{n\to\infty} (1 - c_n) \int_n^{n+1} \exp\left(\int_{n-1}^s a(u)\,du\right) ds > \frac{1}{M},$$

*then all solutions of equation* (19) *are oscillatory.*

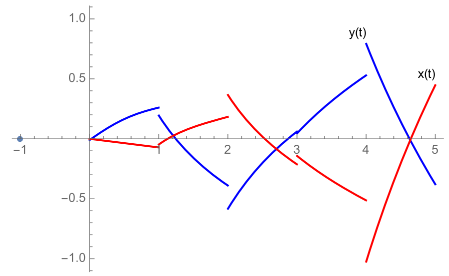


**Figure 1** Oscillatory solutions of the system (20)–(21) with the initial conditions $x(-1) = 0$, $x(0) = 0.01, y(-1) = 0, y(0) = -0.01$

**Corollary 7** *Assume that there exist $M, K > 0$ such that $f(u) \geq M$ for $u \in \mathbb{R}$, $c_n \leq 1 - K$ for $n \in \mathbb{Z}^+$, and $uh(u) < 0$ for $u \neq 0$. If*

$$\frac{1}{4KM} < \liminf_{n\to\infty} \exp\left(\int_n^{n+1} a(u)\,du\right) \liminf_{n\to\infty} \int_n^{n+1} \exp\left(\int_n^s a(u)\,du\right) ds < \infty,$$

*then all solutions of equation* (19) *are oscillatory.*

*Remark* 5 It is clear that the problem considered in this paper is more general than the problem investigated in [18]. When $h(t) \equiv 0$, Corollaries 6 and 7 coincide with Corollaries 1 and 3 in [18], respectively.

Here we give two numerical examples to illustrate our results. Mathematica® software is used to get the figures.

*Example* 1 Let us consider the following nonlinear impulsive differential equations system with piecewise constant argument and variable coefficient:

$$\begin{cases} x'(t) = -\frac{2}{t}x(t) - x([t-1])(e^{-y([t])} + 2) - \sqrt[3]{x([t])}, \\ y'(t) = -\frac{2t}{t^2+1}y(t) - y([t-1])(x^2([t]) + 2) - \sqrt[5]{y([t])}, \end{cases} \quad t \neq n, t > 0, \tag{20}$$

$$\begin{cases} \Delta x(n) = \frac{(-1)^n}{2}x(n), \\ \Delta y(n) = \frac{(-1)^n}{3}y(n), \end{cases} \quad n \in \mathbb{Z}^+. \tag{21}$$

It is clear that $a(t) = \frac{2}{t}$, $f(u) = \exp(-u) + 2$, $h_1(u) = -u^{1/3}$, and $b(t) = \frac{2t}{t^2+1}$, $g(u) = u^2 + 2$, $h_2(u) = -u^{1/5}$ satisfy all hypotheses of Theorems 2 and 3. Then all solutions of system (20)–(21) are oscillatory. The solution $(x_n(t), y_n(t))$ of system (20)–(21) with initial conditions $x(-1) = 0, x(0) = 0.01, y(-1) = 0, y(0) = -0.01$ is shown in Fig. 1.

*Example* 2 Consider the following system:

$$\begin{cases} x'(t) = -(\ln 2)x(t) - x([t-1])(y^2([t]) + 1) - x^3([t]), \\ y'(t) = -(\ln 3)y(t) - y([t-1])(x^4([t]) + 1) - y([t]), \end{cases} \quad t \neq n, t > 0, \tag{22}$$

$$\begin{cases} \Delta x(n) = \frac{1}{2^n}x(n), \\ \Delta y(n) = \frac{1}{3^n}y(n), \end{cases} \quad n = \mathbb{Z}^+. \tag{23}$$

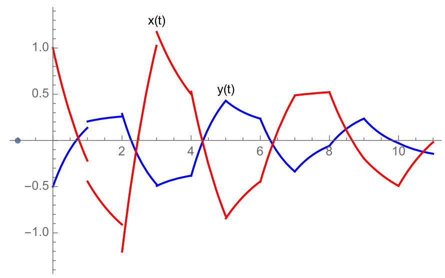


**Figure 2** Oscillatory solutions of system (22)–(23) with the initial conditions $x(-1) = 0, x(0) = 1, y(-1) = 0$, $y(0) = -0.5$

Since all hypotheses of Theorem 2 are satisfied for $a(t) = \ln 2, f(u) = u^2 + 1, h_1(u) = -u^3$ and $b(t) = \ln 3, g(u) = u^4 + 1, h_2(u) = -u$, all solutions of system (22)–(23) are oscillatory. The solution $(x_n(t), y_n(t))$ of system (22)–(23) with the initial conditions $x(-1) = 0, x(0) = 1, y(-1) = 0, y(0) = -0.5$ is shown in Fig. 2.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The main idea of this paper was proposed by FK. The manuscript prepared initially and all steps of the proof are performed by FK, AU, and HB. All authors have read and approved the final manuscript.

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