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Robust stability and stabilization of LTI fractional-order systems with poly-topic and two-norm bounded uncertainties

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Abstract

This paper investigates the problems of robust stability and stabilization of LTI fractional-order systems with poly-topic and two-norm bounded uncertainties. Firstly, some sufficient conditions of the robust asymptotical stable for such fractional-order uncertain systems are derived. Secondly, the robust stabilizing state-feedback controller is designed. All the results are obtained in terms of linear matrix inequalities (LMIs). Lastly, three numerical examples are provided to demonstrate the correctness and effectiveness of the proposed approaches.

Keywords: LTI fractional-order system; Poly-topic uncertainty; Two-norm bounded uncertainty; Robust stability; Linear matrix inequality

1 Introduction

The fractional calculus dates from the 17th century [1], and it can be defined as a classical mathematical notion and a generalization of the ordinary differentiation and integration not necessarily integer. The significance of fractional-order representation is that it is more adequate to describe real world systems than those of integer-order models [2]. Fractional-order calculus is focused on the whole time and space, but the integer-order calculus only concerned with local attribute at particular time and a certain position [3]. Due to these advantages, fractional calculus is developing fast [4, 5], and its various applications are extensively used in many fields of science and engineering: in material engineering [6], chaos systems [7–9], economic systems [10], robotics [11], and in many more [12–16].

Stability and stabilization is fundamental to all systems, certainly including fractional-order systems. Some interesting stability results have been achieved including fractional-order linear time-invariant systems, nonlinear systems, linear delayed systems, commensurate and incommensurate systems [17–22]. In [17], the key is to find a linear ordinary system that possesses the same stability property as the fractional-order system, and then the stability analysis was converted into the domain of ordinary systems which was well established and understood. In [18, 19], the method based on the Lyapunov function is used to study the stability for fractional-order system. In [20], the Linear Matrix Inequality (LMI) stability conditions are overviewed. The stability of a class of commensurate and incommensurate nonlinear fractional-order systems is studied in [21, 22].

Due to the perturbations and uncertainties in modeling, operating and manufacturing of real systems, robust stability analysis and control is much more important. As presented in [23–26], significant results have been proposed for the robust stability problem of fractional-order system with interval uncertainties. In [26], the general interval uncertainties mean that the interval uncertainties exist both in the coefficients and orders of the fractional-order system. Necessary and sufficient conditions are proposed to check the robust stability of general interval fractional-order system. The robust stability and stabilization of fractional-order system with poly-topic uncertainties are studied in [27–29]. The same problem for system with norm bounded uncertainty is considered in [30].

Reviewing the research reported above, it is obvious that the stability and stabilizing controller design for LTI fractional-order systems with poly-topic and two-norm bounded uncertainties have not been studied so far. Motivated by this consideration, the purpose of this paper is to investigate the robust stability and stabilization of dynamical systems whose characteristics are of fractional order with poly-topic and two-norm bounded uncertainties assuming that the nominal fractional-order systems are already asymptotically stable.

The rest of this paper is organized as follows. In Sect. 2, the problem formulation and some necessary preliminaries are described. In Sect. 3, the robust stability conditions of fractional-order system with poly-topic and two-norm bounded uncertainties are derived. Also, the robust stabilization via state-feedback control is proposed. And three numerical examples and their simulation results are discussed in Sect. 4 and finally conclusions are given in Sect. 5.

For convenience, the following notations are used throughout this paper: R^n and Z^+ are the set of n-dimensional Euclidean space and the set of positive integers, respectively; $R^{n\times n}$ and $C^{n\times n}$ are the set of $n \times n$ real matrices and complex matrices, separately. M^T, $\bar{M}$ and M^* are the transpose, conjugate and the transpose conjugate of M, respectively. $\mathrm{sym}(M)$ stands for the expression $M^* + M$, $\|M\|$ is the two-norm of M, I_n is the identity matrix of order n, $\bullet$ denotes the symmetric component in a matrix.

2 Model description and preliminary

In this paper the well-known Caputo definition for fractional derivatives is adopted, because the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with clear physical interpretations. The Caputo fractional derivative [2] is defined as follows:

$$\mathcal{D}^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau, \quad t > 0. \tag{1}$$

Here Γ is the Gamma function, which is defined as $\Gamma(\alpha) = (\alpha - 1)!$, and $m - 1 \leq \alpha < m$, $m \in Z^+$. From a mathematic point of view, the fractional order can be any real even complex number. In engineering applications, α is a real number related to physical parameters and lies in $(0, 2)$. Here, this paper focuses on the robust stability and stabilization problem of uncertain fractional-order system in which α is a real number in $(0, 1)$.

Consider the following uncertain LTI fractional-order system:

$$\mathcal{D}^\alpha x(t) = A(\gamma)x(t) + B(\gamma)u(t). \tag{2}$$

In this equation, $x(t) \in R^n$ is the system state vector, and $u(t) \in R^m$ is the control input vector, $A(\gamma) \in R^{n\times n}$ is the system matrix, and $B(\gamma) \in R^{n\times m}$ is the input matrix; they are both convex poly-topic sets described by

$$A(\gamma) = \sum_{i=1}^{N} \gamma_i A_i, \qquad B(\gamma) = \sum_{i=1}^{N} \gamma_i B_i, \tag{3}$$

where γ_i ($i = 1, 2, \dots, N$) are time-invariant uncertainties which satisfy $\gamma_i \geq 0$, $\sum_{i=1}^{N} \gamma_i = 1$. And two-norm uncertainties are included in A_i, B_i, they could be represented as follows:

$$A_i = \hat{A}_i + \Delta A_i, \quad \text{and} \quad \Delta A_i = D_{Ai} F_{Ai} E_{Ai}, \tag{4}$$

$$B_i = \hat{B}_i + \Delta B_i, \quad \text{and} \quad \Delta B_i = D_{Bi} F_{Bi} E_{Bi}. \tag{5}$$

Here $\hat{A}_i$ and $\hat{B}_i$ are the nominal part of system matrix and input matrix, respectively; ΔA_i and ΔB_i are the additive uncertainty part of system matrix and input matrix, separately; $D_{Ai} \in R^{n\times r_1}$, $D_{Bi} \in R^{n\times r_2}$ and $E_{Ai} \in R^{q_1\times n}$, $E_{Bi} \in R^{q_2\times l}$ are known real constant matrices, $F_{Ai} \in R^{r_1\times q_1}$ and $F_{Bi} \in R^{r_2\times q_2}$ are uncertain matrices which satisfy the following:

$$\|F_{Ai}\|_2 \leq \rho_{Ai}, \qquad \|F_{Bi}\|_2 \leq \rho_{Bi}. \tag{6}$$

In order to derive the main results of the paper, the following lemmas are presented firstly.

Lemma 1 ([31]) *Let* $A \in R^{n\times n}$, $0 < \alpha < 1$, *then* $\mathcal{D}^\alpha x(t) = Ax(t)$ *is asymptotically stable if and only if* $|\arg(\operatorname{spec}(A))| > \alpha\frac{\pi}{2}$, *where* spec$(A)$ *is the spectrum* (*set of all eigenvalues*) *of* A, arg( ) *is the amplitude function*.

Lemma 2 ([32]) *Let* $A \in R^{n\times n}$, $0 < \alpha < 1$ *and* $\theta = (1-\alpha)\pi/2$. *The fractional-order system* $\mathcal{D}^\alpha x(t) = Ax(t)$ *is asymptotically stable if and only if there exists a positive definite Hermitian matrix* $X = X^*$, $X \in C^{n\times n}$, *such that*

$$\operatorname{sym}\bigl(A(rX + \bar{r}\bar{X})\bigr) < 0. \tag{7}$$

Here $r = e^{\theta i}$.

Lemma 3 ([33]) *For any matrices* X, Y *with appropriate dimension, the following inequality holds for any* $\beta > 0$:

$$X^T Y + Y^T X \leq \beta X^T X + \beta^{-1} Y^T Y. \tag{8}$$

Lemma 4 ([34]) *For a real matrix* $\Phi = \Phi^T$, *the following conditions are equivalent*:

(1) $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \bullet & \Phi_{22} \end{bmatrix} < 0,$

(2) $\Phi_{11} < 0, \qquad \Phi_{22} - \Phi_{12}^T \Phi_{11}^{-1} \Phi_{12} < 0,$

(3) $\Phi_{22} < 0, \qquad \Phi_{11} - \Phi_{12} \Phi_{22}^{-1} \Phi_{12}^T < 0.$

3 Main results

3.1 Robust stability analysis

When $u(t) = 0$, the following uncertain state space model is usually considered in the context of robustness analysis and synthesis for fractional-order systems:

$$\mathcal{D}^{\alpha}x(t) = \left(\sum_{i=1}^{N}\gamma_i(\hat{A}_i + D_{Ai}F_{Ai}E_{Ai})\right)x(t). \tag{9}$$

Theorem 3.1 *The LTI fractional-order system with poly-topic and two-norm bounded uncertainties in* (9) *is robustly asymptotically stable if there exist matrices* $X_i = X_i^T > 0$, $X_i \in C^{n\times n}$, $\varepsilon_{ij} > 0$ *for all* $i, j = 1, 2, \dots, N$ *such that*

$$\begin{bmatrix} \operatorname{sym}(\hat{A}_i(rX_j + \bar{r}\bar{X}_j)) + \varepsilon_{ij}D_{Ai}D_{Ai}^T & \bullet \\ E_{Ai}(rX_j + \bar{r}\bar{X}_j) & -\frac{\varepsilon_{ij}}{\rho_{Ai}^2}I_{q_1} \end{bmatrix} < 0. \tag{10}$$

Proof If there exists a positive definite Hermitian matrix $X_i = X_i^T > 0$, $X_i \in C^{n\times n}$, and for linear system it is obvious that $X(\gamma) = \sum_{i=1}^{N}\gamma_i X_i$, $A(\gamma) = \sum_{i=1}^{N}\gamma_i A_i$. So according to Lemma 2, the system is robustly stable if the following equation is satisfied:

$$\begin{aligned} &\operatorname{sym}\bigl(A(rX + \bar{r}\bar{X})\bigr) < 0 \\ \Leftrightarrow\quad &\operatorname{sym}\sum_{i=1}^{N}\left[\gamma_i(\hat{A}_i + D_{Ai}F_{Ai}E_{Ai})\left(r\sum_{i=1}^{N}\gamma_i X_i + \bar{r}\sum_{i=1}^{N}\gamma_i\bar{X}_i\right)\right] < 0 \\ \Leftrightarrow\quad &\sum_{i=1}^{N}\gamma_i^2\bigl[\operatorname{sym}\bigl[\hat{A}_i(rX_i + \bar{r}\bar{X}_i) + D_{Ai}F_{Ai}E_{Ai}(rX_i + \bar{r}\bar{X}_i)\bigr]\bigr] \\ &\quad + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\gamma_i\gamma_j\bigl\{\operatorname{sym}\bigl[\bigl(\hat{A}_i(rX_j + \bar{r}\bar{X}_j) + D_{Ai}F_{Ai}E_{Ai}(rX_j + \bar{r}\bar{X}_j)\bigr) \\ &\quad + \hat{A}_j(rX_i + \bar{r}\bar{X}_i) + D_{Aj}F_{Aj}E_{Aj}(rX_i + \bar{r}\bar{X}_i)\bigr]\bigr\} < 0. \end{aligned} \tag{11}$$

Note that, if Eqs. (12)–(14) hold, then inequality (11) is satisfied;

$$\operatorname{sym}\bigl[\hat{A}_i(rX_i + \bar{r}\bar{X}_i) + D_{Ai}F_{Ai}E_{Ai}(rX_i + \bar{r}\bar{X}_i)\bigr] < 0 \quad (i = 1, 2, \dots, N), \tag{12}$$

$$\begin{aligned} &\operatorname{sym}\bigl[\hat{A}_i(rX_j + \bar{r}\bar{X}_j) + D_{Ai}F_{Ai}E_{Ai}(rX_j + \bar{r}\bar{X}_j)\bigr] < 0 \\ &\quad (i = 1, 2, \dots, N-1; j = i+1, \dots, N), \end{aligned} \tag{13}$$

$$\begin{aligned} &\operatorname{sym}\bigl[\hat{A}_j(rX_i + \bar{r}\bar{X}_i) + D_{Aj}F_{Aj}E_{Aj}(rX_i + \bar{r}\bar{X}_i)\bigr] < 0 \\ &\quad (i = 1, 2, \dots, N-1; j = i+1, \dots, N). \end{aligned} \tag{14}$$

And based on Lemma 3, there exist $\varepsilon_{ii} > 0$ $(i = 1, 2, \dots, N)$; $\varepsilon_{ij} > 0$, $\varepsilon_{ji} > 0$ $(i = 1, 2, \dots, N-1;$ $j = i+1, \dots, N)$ such that

$$\operatorname{sym}\bigl(D_{Ai}F_{Ai}E_{Ai}(rX_i + \bar{r}\bar{X}_i)\bigr) \le \varepsilon_{ii}D_{Ai}D_{Ai}^T + \varepsilon_{ii}^{-1}\rho_{Ai}^2(rX_i + \bar{r}\bar{X}_i)^T E_{Ai}^T E_{Ai}(rX_i + \bar{r}\bar{X}_i), \tag{15}$$

$$\operatorname{sym}\bigl(D_{Ai}F_{Ai}E_{Ai}(rX_j + \bar{r}\bar{X}_j)\bigr) \le \varepsilon_{ij}D_{Ai}D_{Ai}^T + \varepsilon_{ij}^{-1}\rho_{Ai}^2(rX_j + \bar{r}\bar{X}_j)^T E_{Ai}^T E_{Ai}(rX_j + \bar{r}\bar{X}_j), \tag{16}$$

$$\operatorname{sym}\bigl(D_{Aj}F_{Aj}E_{Aj}(rX_i + \bar{r}\bar{X}_i)\bigr) \le \varepsilon_{ji}D_{Aj}D_{Aj}^T + \varepsilon_{ji}^{-1}\rho_{Aj}^2(rX_i + \bar{r}\bar{X}_i)^T E_{Aj}^T E_{Aj}(rX_i + \bar{r}\bar{X}_i). \tag{17}$$

Substitute (15) into (12), (16) into (13), and (17) into (14), respectively. Then based on Lemma 4, inequalities (12)–(14) can be written as

$$\begin{bmatrix} \operatorname{sym}(\hat{A}_i(rX_i + \bar{r}\bar{X}_i)) + \varepsilon_{ii} D_{Ai} D_{Ai}^T & \bullet \\ E_{Ai}(rX_i + \bar{r}\bar{X}_i) & -\frac{\varepsilon_{ii}}{\rho_{Ai}^2} I_{q_1} \end{bmatrix} < 0, \quad \varepsilon_{ii} > 0, i = 1, 2, \ldots, N, \tag{18}$$

$$\begin{bmatrix} \operatorname{sym}(\hat{A}_i(rX_j + \bar{r}\bar{X}_j)) + \varepsilon_{ij} D_{Ai} D_{Ai}^T & \bullet \\ E_{Ai}(rX_j + \bar{r}\bar{X}_j) & -\frac{\varepsilon_{ij}}{\rho_{Ai}^2} I_{q_1} \end{bmatrix} < 0,$$

$$\varepsilon_{ij} > 0, i = 1, 2, \ldots, N-1; j = i+1, \ldots, N, \tag{19}$$

$$\begin{bmatrix} \operatorname{sym}(\hat{A}_j(rX_i + \bar{r}\bar{X}_i)) + \varepsilon_{ji} D_{Aj} D_{Aj}^T & \bullet \\ E_{Aj}(rX_i + \bar{r}\bar{X}_i) & -\frac{\varepsilon_{ji}}{\rho_{Aj}^2} I_{q_1} \end{bmatrix} < 0,$$

$$\varepsilon_{ji} > 0, i = 1, 2, \ldots, N-1; j = i+1, \ldots, N. \tag{20}$$

Note that the above condition (18)–(20) can be further rewritten as

$$\begin{bmatrix} \operatorname{sym}(\hat{A}_i(rX_j + \bar{r}\bar{X}_j)) + \varepsilon_{ij} D_{Ai} D_{Ai}^T & \bullet \\ E_{Ai}(rX_j + \bar{r}\bar{X}_j) & -\frac{\varepsilon_{ij}}{\rho_{Ai}^2} I_{q_1} \end{bmatrix} < 0 \quad \text{for all } i, j = 1, 2, \ldots, N. \tag{21}$$

By Lemma 2, the system is robustly asymptotically stable. This ends the proof. □

3.2 Robust stabilization of fractional-order uncertain linear systems

In this section, we consider the linear state-feedback control for fractional-order systems (2), that is,

$$u(t) = Kx(t). \tag{22}$$

Theorem 3.2 *The LTI fractional-order system with poly-topic and two-norm bounded uncertainties in* (2) *is robustly asymptotically stable if there exist matrices* $X_i = X_i^T > 0$, $X_i \in C^{n\times n}$, $R \in R^{n\times m}$, $\varepsilon_{1ij} > 0$, $\varepsilon_{2ij} > 0$, *for all* $i, j = 1, 2, \ldots, N$, *such that*

$$\begin{bmatrix} \operatorname{sym}(\hat{A}_i(rX_j + \bar{r}\bar{X}_j) + \hat{B}_i R_j) + \varepsilon_{1ij} D_{Ai} D_{Ai}^T + \varepsilon_{2ij} D_{Bi} D_{Bi}^T & \bullet & \bullet \\ E_{Ai}(rX_j + \bar{r}\bar{X}_j) & -\frac{\varepsilon_{1ij}}{\rho_{Ai}^2} I_{q_1} & \bullet \\ E_{Bi} R_j & 0 & -\frac{\varepsilon_{2ij}}{\rho_{Bi}^2} I_{q_2} \end{bmatrix} < 0. \tag{23}$$

The state-feedback controller gain matrix is determined as

$$K = \sum_{i=1}^{N} \gamma_i R_i \sum_{i=1}^{N} (rX_i + \bar{r}\bar{X}_i)^{-1}. \tag{24}$$

Proof Since it is obvious that $X(\gamma) = \sum_{i=1}^{N} \gamma_i X_i$, $A(\gamma) = \sum_{i=1}^{N} \gamma_i A_i$, according to Lemma 2, the system is robustly stable if the following equation is satisfied:

$$\begin{aligned}
&\operatorname{sym}\bigl((A+BK)(rX+\bar{r}\bar{X})\bigr) < 0 \\
&\quad\Leftrightarrow\quad \sum_{i=1}^{N} \gamma_i^2 \bigl\{\operatorname{sym}\bigl[\hat{A}_i(rX_i+\bar{r}\bar{X}_i) + \hat{B}_i R_i + D_{Bi}F_{Bi}E_{Bi}R_i + D_{Ai}F_{Ai}E_{Ai}(rX_i+\bar{r}\bar{X}_i)\bigr]\bigr\} \\
&\qquad + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \gamma_i\gamma_j \bigl\{\operatorname{sym}\bigl[\bigl(\hat{A}_i(rX_j+\bar{r}\bar{X}_j) + D_{Ai}F_{Ai}E_{Ai}(rX_j+\bar{r}\bar{X}_j) \\
&\qquad + \hat{B}_i R_j + D_{Bi}F_{Bi}E_{Bi}R_j\bigr) \\
&\qquad + \hat{A}_j(rX_i+\bar{r}\bar{X}_i) + D_{Aj}F_{Aj}E_{Aj}(rX_i+\bar{r}\bar{X}_i) + \hat{B}_j R_i + D_{Bj}F_{Bj}E_{Bj}R_i\bigr]\bigr\} < 0.
\end{aligned} \tag{25}$$

If Eqs. (26)–(28) hold, then inequality (25) is satisfied;

$$\begin{aligned}
&\operatorname{sym}\bigl[\hat{A}_i(rX_i+\bar{r}\bar{X}_i) + \hat{B}_i R_i + D_{Bi}F_{Bi}E_{Bi}R_i + D_{Ai}F_{Ai}E_{Ai}(rX_i+\bar{r}\bar{X}_i)\bigr] < 0 \\
&\quad (i = 1,2,\ldots,N),
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\operatorname{sym}\bigl[\hat{A}_i(rX_j+\bar{r}\bar{X}_j) + D_{Ai}F_{Ai}E_{Ai}(rX_j+\bar{r}\bar{X}_j) + \hat{B}_i R_j + D_{Bi}F_{Bi}E_{Bi}R_j\bigr] < 0 \\
&\quad (i = 1,2,\ldots,N-1; j = i+1,\ldots,N),
\end{aligned} \tag{27}$$

$$\begin{aligned}
&\operatorname{sym}\bigl[\hat{A}_j(rX_i+\bar{r}\bar{X}_i) + D_{Aj}F_{Aj}E_{Aj}(rX_i+\bar{r}\bar{X}_i) + \hat{B}_j R_i + D_{Bj}F_{Bj}E_{Bj}R_i\bigr] < 0 \\
&\quad (i = 1,2,\ldots,N-1; j = i+1,\ldots,N).
\end{aligned} \tag{28}$$

According to Lemma 3, there exist $\varepsilon_{1ii}, \varepsilon_{2ii} > 0$ $(i = 1,2,\ldots,N)$; $\varepsilon_{1ij}, \varepsilon_{2ij} > 0$, $\varepsilon_{1ji}, \varepsilon_{2ji} > 0$ $(i = 1,2,\ldots,N-1; j = i+1,\ldots,N)$, such that

$$\operatorname{sym}\bigl(D_{Ai}F_{Ai}E_{Ai}(rX_i+\bar{r}\bar{X}_i)\bigr) \le \varepsilon_{1ii}D_{Ai}D_{Ai}^T + \varepsilon_{1ii}^{-1}\rho_{Ai}^2(rX_i+\bar{r}\bar{X}_i)^T E_{Ai}^T E_{Ai}(rX_i+\bar{r}\bar{X}_i), \tag{29}$$

$$\operatorname{sym}(D_{Bi}F_{Bi}E_{Bi}R_i) \le \varepsilon_{2ii}D_{\mathrm{B}i}D_{\mathrm{B}i}^T + \varepsilon_{2ii}^{-1}\rho_{Bi}^2 R_i^T E_{Bi}^T E_{Bi}R_i, \tag{30}$$

$$\operatorname{sym}\bigl(D_{Ai}F_{Ai}E_{Ai}(rX_j+\bar{r}\bar{X}_j)\bigr) \le \varepsilon_{1ij}D_{Ai}D_{Ai}^T + \varepsilon_{1ij}^{-1}\rho_{Ai}^2(rX_i+\bar{r}\bar{X}_i)^T E_{Ai}^T E_{Ai}(rX_i+\bar{r}\bar{X}_i), \tag{31}$$

$$\operatorname{sym}(D_{Bi}F_{Bi}E_{Bi}R_j) \le \varepsilon_{2ij}D_{\mathrm{B}i}D_{\mathrm{B}i}^T + \varepsilon_{2ij}^{-1}\rho_{Bi}^2 R_j^T E_{Bi}^T E_{Bi}R_j, \tag{32}$$

$$\operatorname{sym}\bigl(D_{Aj}F_{Aj}E_{Aj}(rX_i+\bar{r}\bar{X}_i)\bigr) \le \varepsilon_{1ji}D_{Aj}D_{Aj}^T + \varepsilon_{1ji}^{-1}\rho_{Aj}^2(rX_i+\bar{r}\bar{X}_i)^T E_{Aj}^T E_{Aj}(rX_i+\bar{r}\bar{X}_i), \tag{33}$$

$$\operatorname{sym}(D_{Bj}F_{Bj}E_{Bj}R_i) \le \varepsilon_{2ji}D_{Bj}D_{Bj}^T + \varepsilon_{2ji}^{-1}\rho_{Bj}^2 R_i^T E_{Bj}^T E_{Bj}R_i. \tag{34}$$

Substitute (29)–(30) into (26), (31)–(32) into (27), and (33)–(34) into (28), respectively. Then based on Lemma 4, for $\varepsilon_{1ii}, \varepsilon_{2ii} > 0$ $(i = 1,2,\ldots,N)$; $\varepsilon_{1ij}, \varepsilon_{2ij} > 0$, $\varepsilon_{1ji}, \varepsilon_{2ji} > 0$ $(i = 1,2,\ldots,N-1; j = i+1,\ldots,N)$ inequalities (26)–(28) can be written as

$$\begin{bmatrix}
\operatorname{sym}(\hat{A}_i(rX_i+\bar{r}\bar{X}_i) + \hat{B}_i R_i) + \varepsilon_{1ii}D_{Ai}D_{Ai}^T + \varepsilon_{2ii}D_{Bi}D_{Bi}^T & \bullet & \bullet \\
E_{Ai}(rX_i+\bar{r}\bar{X}_i) & -\frac{\varepsilon_{1ii}}{\rho_{Ai}^2}I_{q_1} & \bullet \\
E_{Bi}R_i & 0 & -\frac{\varepsilon_{2ii}}{\rho_{Bi}^2}I_{q_2}
\end{bmatrix} < 0, \tag{35}$$

$$\begin{bmatrix} \mathrm{sym}(\hat{A}_i(rX_j + \bar{r}\bar{X}_j) + \hat{B}_i R_j) + \varepsilon_{1ij} D_{Ai} D_{Ai}^T + \varepsilon_{2ij} D_{Bi} D_{Bi}^T & \bullet & \bullet \\ E_{Ai}(rX_j + \bar{r}\bar{X}_j) & -\frac{\varepsilon_{1ij}}{\rho_{Ai}^2} I_{q1} & \bullet \\ E_{Bi} R_j & 0 & -\frac{\varepsilon_{2ij}}{\rho_{Bi}^2} I_{q2} \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} \mathrm{sym}(\hat{A}_j(rX_i + \bar{r}\bar{X}_i) + \hat{B}_j R_i) + \varepsilon_{1ji} D_{Aj} D_{Aj}^T + \varepsilon_{2ji} D_{Bj} D_{Bj}^T & \bullet & \bullet \\ E_{Aj}(rX_i + \bar{r}\bar{X}_i) & -\frac{\varepsilon_{1ji}}{\rho_{Aj}^2} I_{q1} & \bullet \\ E_{Bj} R_i & 0 & -\frac{\varepsilon_{2ji}}{\rho_{Bj}^2} I_{q2} \end{bmatrix} < 0. \tag{37}$$

Note that the above condition (35)–(37) is equivalent to inequality (23). By Lemma 2, the system is robustly asymptotically stable. This completes the proof. □

4 Numerical examples

In this section, three numerical examples are given to illustrate the use of the proposed theoretical results. Some comparisons with recent relevant publications are shown in the first example, then the second example deals with the stability analysis of fractional-order linear systems with poly-topic and two-norm bounded uncertainties. And a feedback controller is designed in the third example. The objective is to show the application of the obtained results.

4.1 Example 1

In this example, the method is applied to an uncertain fractional-order linear system with the parameters described in the following [30]. This system has two-norm bounded uncertainty only, it is a special case of Theorem 3.1 in this paper. We have

$$N = 1, \qquad \alpha = 0.8,$$

$$\hat{A}_1 = \begin{bmatrix} -3 & 2 & -1 \\ -3.0 & 1.5 & -1 \\ 2.0 & 2.0 & -1 \end{bmatrix}, \qquad D_{A1} = \begin{bmatrix} 0.5 & 1 & 0 \\ -0.4 & 0.2 & 0 \\ 0.1 & -0.1 & -0.6 \end{bmatrix},$$

$$E_{A1} = \begin{bmatrix} 0.1 & 0.3 & 0.4 \\ 0 & 0.1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix}, \qquad F_{A1} = \begin{bmatrix} 0.0083 & 0.0166 & 0.0249 \\ 0.0332 & 0.0415 & 0.0499 \\ 0.0582 & 0.0665 & 0.0748 \end{bmatrix}.$$

Using Theorem 3.1, for $i = j = 1$, the time response of selected system within the uncertainty is shown in Fig. 1.

This is the same result as in Ref. [30]; it indicates the correctness of Theorem 3.1. But in [30], the norm bounded uncertainty is only considered, and the poly-topic uncertainty is not taken into consideration. This paper is just to deal with the two-norm bounded and poly-topic uncertainties simultaneously. But in [30], the robust stability bounds on the uncertainties are also derived. That is also our next work to do in the future.

4.2 Example 2

Consider a fractional-order linear system (9) with the following parameters:

$$N = 2, \qquad \alpha = 0.9, \qquad \gamma_1 = 0.1, \qquad \gamma_2 = 0.9,$$

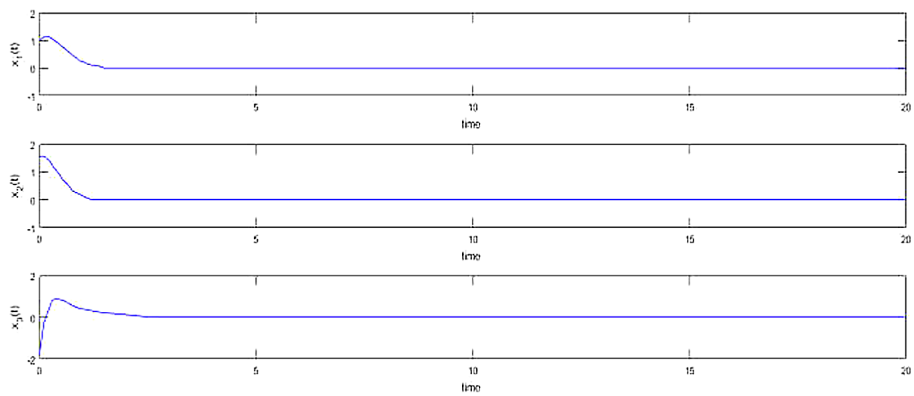

Figure 1 The time response of the system in paper [30]

$$\hat{A}_1 = \begin{bmatrix} -1 & 0.2 \\ 0.3 & -1.2 \end{bmatrix}, \qquad D_{A1} = \begin{bmatrix} 0.11 & 0.12 \\ 0.21 & 0.11 \end{bmatrix},$$
$$E_{A1} = \begin{bmatrix} 0.12 & 0.21 \\ 0.65 & 0.51 \end{bmatrix}, \qquad F_{A1} = \begin{bmatrix} 0.12 & 0.21 \\ 0.65 & 0.51 \end{bmatrix},$$
$$\hat{A}_2 = \begin{bmatrix} -0.9 & 0.7 \\ 0.5 & -1.5 \end{bmatrix}, \qquad D_{A2} = \begin{bmatrix} 0.05 & 0.16 \\ 0.18 & 0.76 \end{bmatrix},$$
$$E_{A2} = \begin{bmatrix} 0.17 & 0.31 \\ 0.68 & 0.76 \end{bmatrix}, \qquad F_{A2} = \begin{bmatrix} 0.19 & 0.25 \\ 0.58 & 0.49 \end{bmatrix}.$$

According to Theorem 3.1, it follows that the above fractional-order system is robustly asymptotically stable. By using the LMI toolbox provided by MATLAB, a feasible solution of an LMI condition is obtained as follows:

$$\varepsilon_{11} = 60.7672, \qquad \varepsilon_{12} = 0.9883, \qquad \varepsilon_{21} = 87.6725, \qquad \varepsilon_{22} = 0.8390,$$
$$X_1 = \begin{bmatrix} 33.2398 & 1.8577 \\ 1.8577 & 32.1176 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} 0.4608 & 0.0287 \\ 0.0287 & 0.3303 \end{bmatrix}.$$

The time response of the fractional-order system with initial state $x_{1\text{initial}} = x_{2\text{initial}} = 0.1$ is described in Fig. 2, which shows that all the states are convergent.

From Theorem 3.1, it is obvious that the robust stability condition has no relationship with γ_i. Three randomly selected cases for comparison are:

(1) Case 1: $\gamma_1 = 0.1$, $\gamma_2 = 0.9$,
(2) Case 2: $\gamma_1 = 0.9$, $\gamma_2 = 0.1$,
(3) Case 3: $\gamma_1 = 0.4$, $\gamma_2 = 0.6$.

The time response of the system in these cases are simulated in Fig. 3.

From Fig. 3 it can be seen that all $x_1(t)$, $x_2(t)$ are stable regardless of the value of γ_i, which is consistent with Theorem 3.1. The validity of the theorem is illustrated from another point of view.

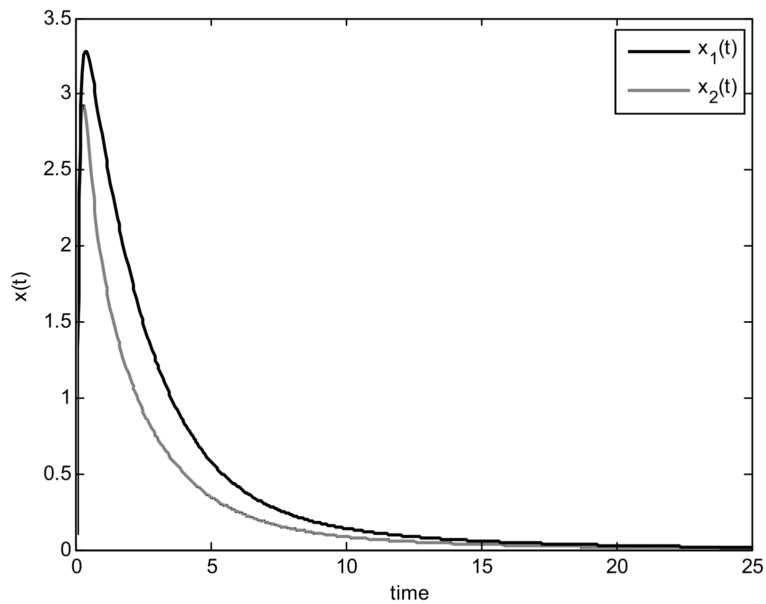

Figure 2 The time response of the system with $\gamma_1 = 0.1$, $\gamma_2 = 0.9$

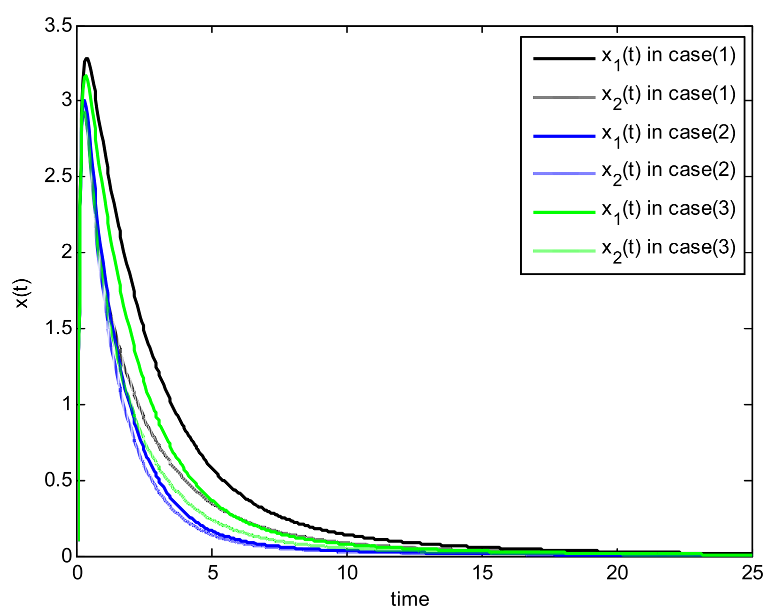

Figure 3 The time response of the system in 3 cases

4.3 Example 3

Consider the following fractional-order linear system:

$$\hat{A}_1 = \begin{bmatrix} -2.5 & 5.2 & -1.5 \\ -3.3 & 4.5 & -0.8 \\ 3.0 & 4.0 & -7.0 \end{bmatrix}, \qquad D_{A1} = \begin{bmatrix} 0.1 & 0.12 & 0.2 \\ 0.21 & 0.11 & 0.3 \\ 0.9 & 0.7 & 0.5 \end{bmatrix},$$

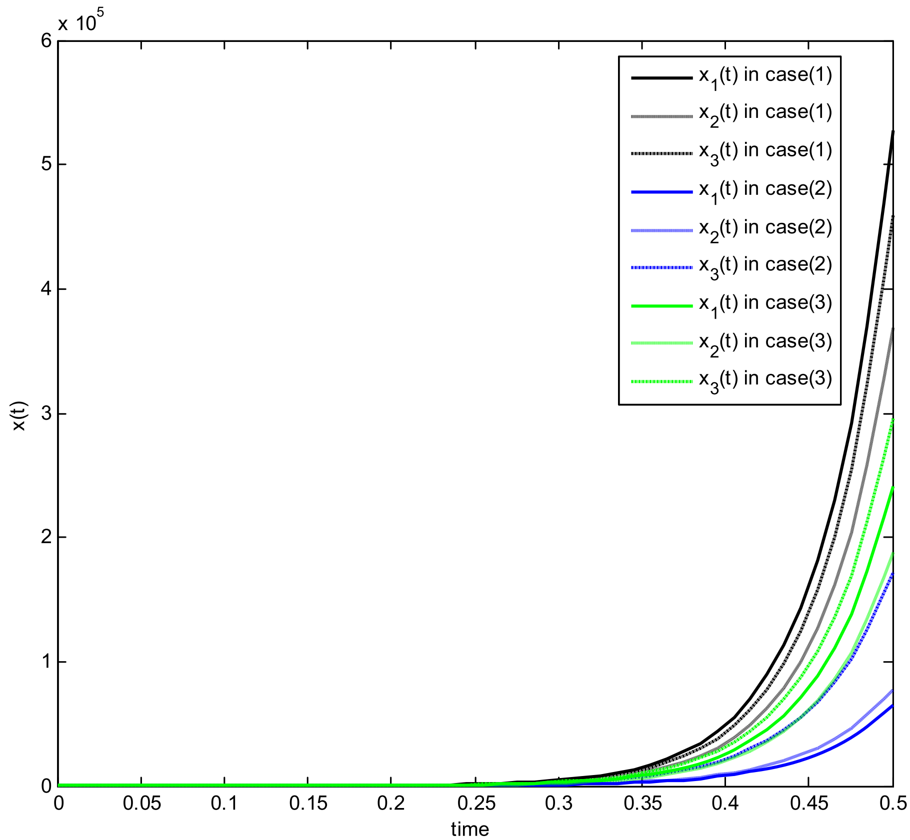

Figure 4 The time response of the system in three cases

$$E_{A1}=\begin{bmatrix}0.12 & 0.65 & 0.76\\ 0.68 & 0.76 & 0.87\\ 0.45 & 0.21 & 0.39\end{bmatrix},\qquad F_{A1}=\begin{bmatrix}2 & 1 & 1.5\\ 5 & 1 & 6\\ 3 & 4 & 1\end{bmatrix},$$

$$B_1=\begin{bmatrix}0.8\\ 2\\ 1.2\end{bmatrix},\qquad D_{B1}=\begin{bmatrix}0.98 & 0.35 & 0.23\\ 0.15 & 0.24 & 0.39\\ 0.65 & 0.28 & 0.43\end{bmatrix},$$

$$E_{B1}=\begin{bmatrix}0.12\\ 0.65\\ 0.28\end{bmatrix},\qquad F_{B1}=\begin{bmatrix}0.76 & 0.92 & 0.21\\ 0.73 & 0.45 & 0.65\\ 0.76 & 0.27 & 0.19\end{bmatrix},$$

$$\hat{A}_2=\begin{bmatrix}-3.5 & 5 & -1\\ -2 & 5 & -0.8\\ 4.0 & 2.6 & -7\end{bmatrix},\qquad D_{A2}=\begin{bmatrix}0.1 & 0.65 & 0.83\\ 0.52 & 0.54 & 0.2\\ 0.13 & 0.43 & 0.65\end{bmatrix},$$

$$E_{A2}=\begin{bmatrix}0.12 & 0.65 & 0.76\\ 0.68 & 0.76 & 0.87\\ 0.45 & 0.21 & 0.39\end{bmatrix},\qquad F_{A2}=\begin{bmatrix}1 & 0.8 & 1.2\\ 1.1 & 3.1 & 2.5\\ 2.2 & 3.5 & 0.8\end{bmatrix},$$

$$B_2=\begin{bmatrix}1.2\\ 2.4\\ 1.0\end{bmatrix},\qquad D_{B2}=\begin{bmatrix}0.89 & 0.53 & 0.32\\ 054 & 0.41 & 0.87\\ 0.66 & 0.54 & 0.68\end{bmatrix},$$

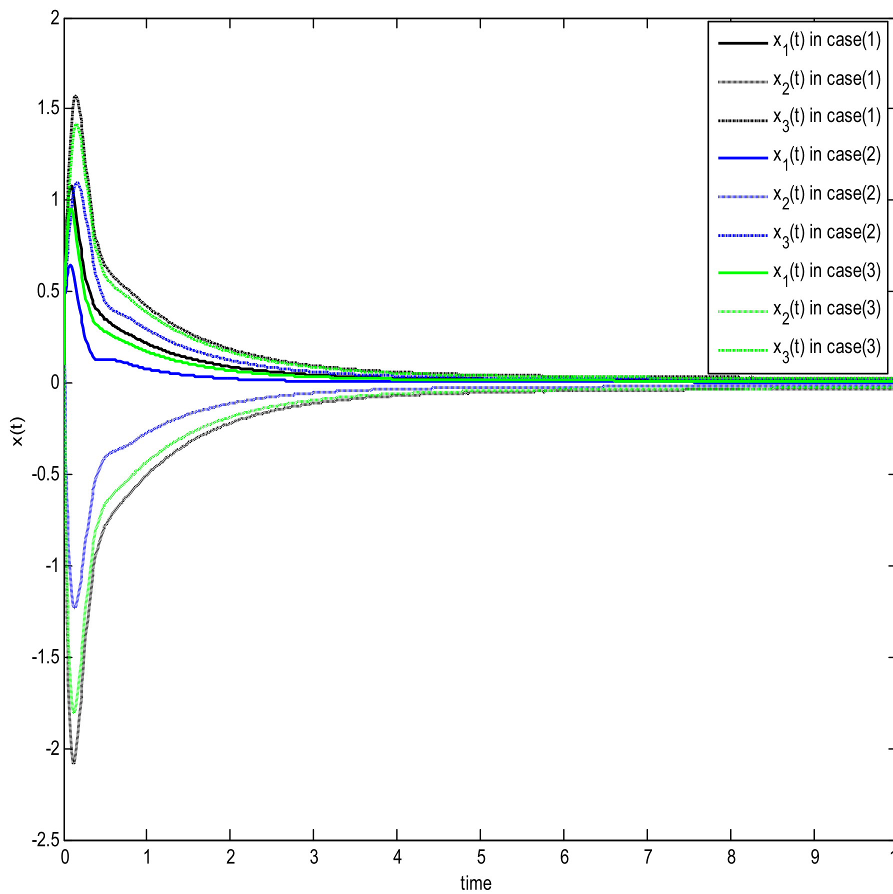

Figure 5 The time response of the system in three cases

$$E_{B2} = \begin{bmatrix} 0.12 \\ 0.65 \\ 0.28 \end{bmatrix}, \qquad F_{B2} = \begin{bmatrix} 0.67 & 0.29 & 0.12 \\ 0.37 & 054 & 0.56 \\ 0.67 & 0.72 & 0.91 \end{bmatrix},$$

$$N = 2, \qquad \alpha = 0.45.$$

Also we consider the cases for comparison:

(1) Case 1: $\gamma_1 = 0.1$, $\gamma_2 = 0.9$,

(2) Case 2: $\gamma_1 = 0.9$, $\gamma_2 = 0.1$,

(3) Case 3: $\gamma_1 = 0.4$, $\gamma_2 = 0.6$.

According to Theorem 3.1, the system is not robustly asymptotically stable. The time response of the fractional-order system is described in Fig. 4, which shows that all its states are not convergent when the control input $u(t) = 0$.

By using Theorem 3.2, the fractional-order system with $u(t) = Kx(t)$ is determined to be asymptotically stabilized, which is shown in Fig. 5.

Different γ_i are corresponding to different feedback gain matrices, and the stabilizing state-feedback gain matrices are obtained:

$$\text{(1) Case 1,} \quad K = \begin{bmatrix} -40.7700 & -40.3679 & -27.1365 \end{bmatrix};$$

$$(2)\ \text{Case 2,} \quad K = \begin{bmatrix} -28.2832 & -35.7695 & -27.0709 \end{bmatrix};$$

$$(3)\ \text{Case 3,} \quad K = \begin{bmatrix} -36.0874 & -38.6435 & -27.1119 \end{bmatrix}.$$

5 Conclusion

In this paper, some new results have been proposed for the robust stability and stabilization of the fractional-order system with poly-topic and two-norm bounded uncertainties when $0 < \alpha < 1$. Firstly, a sufficient condition for robust stabilization has been derived; secondly, the corresponding linear state-feedback stabilizing controller has been designed if the robust stabilization condition is not satisfied. Both of them are presented using the linear matrix inequality approaches, with the help of the LMI toolbox in MATLAB, and three numerical examples have been given to demonstrate the effectiveness of the proposed method.

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Competing interests
The author declares to have no competing interest.

Authors' contributions
The author wrote, read and approved the final manuscript.

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