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A fully implicit finite difference scheme based on extended cubic B-splines for time fractional advection–diffusion equation

Syed Tauseef Mohyud-Din[1,2*], Tayyaba Akram[3], Muhammad Abbas[4], Ahmad Izani Ismail[3] and Norhashidah H.M. Ali[3]

*Correspondence: syedtauseefs@hotmail.com
[1]Center for Research (CFR), University of Islamabad (UoI), Islamabad, Pakistan
[2]Faculty of Engineering & Applied Sciences, Riphah International University, Islamabad, Pakistan
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate a fully implicit finite difference scheme for solving the time fractional advection–diffusion equation. The time fractional derivative is estimated using Caputo's formulation, and the spatial derivatives are discretized using extended cubic B-spline functions. The convergence and stability of the fully implicit scheme are analyzed. Numerical experiments conducted indicate that the scheme is feasible and accurate.

Keywords: Time fractional advection–diffusion equation; Extended cubic B-spline basis functions; Collocation method; Stability; Convergence

1 Introduction

Over the past few decades, several physical models have been developed in the form of fractional differential equations. Fractional differential equations have been found to be appropriate models for certain phenomena in astro-physics, fractal networks, signal processing, chaotic dynamics, turbulent flow, continuum mechanics, and wave propagation [1–7]. These models admit non-local memory effects in the mathematical formulation and thus overcome certain shortcomings in integer-based models.

An important fractional partial differential equation is the fractional advection–diffusion equation. It is important to solve this equation for a better understanding of advection and diffusion phenomena in a fractional setting, and for this purpose, numerical and approximate analytical methods are usually required. The finite element method was constructed for the space fractional advection–diffusion equation by Zheng *et al.* [8]. Wang and Wang [9] developed a fast characteristic finite difference scheme for space fractional advection–diffusion equation. For the space–time fractional advection–diffusions, explicit and implicit difference approximations were developed by Shen *et al.* [10]. Jiang *et al.* [11] presented analytical solutions for the multi-term time–space Caputo–Riesz fractional advection–diffusion equations on a finite domain with Dirichlet nonhomogeneous boundary conditions. In [11], the spectral representation of the fractional Laplacian operator was used to derive the analytical solution. A scheme based on the finite volume method for the solution of space fractional diffusion equation was investigated by Liu *et al.* [12]. A finite element multigrid method was developed for multi-term time fractional

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advection–diffusion equations by Bu *et al.* [13]. Parvizi *et al.* [14] presented a Jacobi collocation method for numerical solution of classical fractional advection–diffusion equation with a nonlinear source term. Rubab *et al.* [15] discussed analytical solutions to the time fractional advection–diffusion equation with time-dependent pulses on the boundary. In [15], the Laplace and Fourier transforms were utilized to determine the analytical solutions of fractional advection–diffusion equation with time fractional Caputo–Fabrizio derivative. Povstenko and Kyrylych [16] discussed two approaches to obtaining the space–time fractional advection–diffusion equations. In this paper, Caputo time fractional derivative and Riesz fractional Laplacian were used.

Many researchers used a spline function for solving fractional differential equations. B-spline functions can give good approximation due to their small, compact support and continuity of order 2 [17, 18]. However, there is relatively not much work on the use of B-splines for solving fractional advection–diffusion equation. B-spline collocation methods were proposed for the solutions of time fractional diffusion problems by Esen *et al.* [19, 20]. Sayevand *et al.* [21] solved anomalous time fractional diffusion problems in transport dynamic systems using a B-spline collocation scheme. In [21], the fractional derivative in Caputo sense was utilized to represent the time derivative. A cubic trigonometric B-spline collocation scheme for the time fractional diffusion problem was presented by Yaseen *et al.* [22]. In this paper, the Grunwald–Letnikov representation was used for Riemann–Liouville derivative, and the stability of the scheme (based on the finite difference method and cubic trigonometric B-spline) was discussed. Zhu and Nie [23] obtained a scheme based on exponential B-spline and wavelet operational matrix method for the time fractional convection–diffusion problem with variable coefficients. Yaseen *et al.* [24] constructed a finite difference method for solving time fractional diffusion problem via trigonometric B-spline. Zhu *et al.* [25] derived an efficient differential quadrature scheme based on modified trigonometric cubic B-spline for the solution of 1D and 2D time fractional advection–diffusion equations. Yuan and Chen [26] presented an expanded mixed finite element method for the two-sided time-dependent fractional diffusion problem with two-sided Riemann–Liouville fractional derivatives.

In this paper, a fully implicit finite difference scheme using extended cubic B-spline is formulated for the numerical solution of time fractional advection–diffusion equation. A finite difference scheme, with Caputo's formula, is applied to discretize the temporal derivative, while extended cubic B-spline is employed to discretize the spatial derivatives.

The model problem fractional advection–diffusion equation considered in this paper is given by

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + p\frac{\partial u}{\partial x} - q\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad a \le x \le b, 0 < t \le T \tag{1}$$

with initial condition

$$u(x,0) = \omega(x), \quad a \le x \le b, \tag{2}$$

and boundary conditions

$$u(a,t) = g_1(t), \qquad u(b,t) = g_2(t), \quad t \ge 0. \tag{3}$$

The advection coefficient p is a constant and the diffusivity coefficient q is a positive constant, where $g_1(t)$, $g_2(t)$, and $f(x,t)$ are continuous functions as the problem required. $\frac{\partial^\gamma}{\partial t^\gamma}$ denotes the Caputo fractional derivative of order γ for the function $u(x,t)$, described as

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{d\tau}{(t-\tau)^\gamma}.$$

The paper is organized as follows. Extended cubic B-spline basis functions are described in Sect. 2. In Sect. 3, a fully implicit finite difference scheme based on extended cubic B-spline is presented. The initial state C^0 is discussed in Sect. 4. Stability and convergence are discussed in Sect. 5 and Sect. 6, respectively. Lastly the numerical experiments and discussions are presented in Sect. 7.

2 Extended cubic B-spline functions

Assume that $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$ are the spatial knots on the interval $[a,b]$ with equal length $h = x_i - x_{i-1}$, $i = 1,\dots,N$. The extended cubic B-spline basis functions, which preserve identical properties and are twice differentiable at the knots x_i over the interval $[a,b]$, can be presented as follows [18]:

$$\phi_i(x,\lambda) = \frac{1}{24h^4} \begin{cases} 4h(1-\lambda)(x-x_{i-2})^3 + 3\lambda(x-x_{i-2})^4, & x_{i-2} \le x < x_{i-1}, \\ (4-\lambda)h^4 + 12h^3(x-x_{i-1}) + 6h^2(2+\lambda)(x-x_{i-1})^2 & \\ \quad - 12h(x-x_{i-1})^3 - 3\lambda(x-x_{i-1})^4, & x_{i-1} \le x < x_i, \\ (4-\lambda)h^4 + 12h^3(x_{i+1}-x) + 6h^2(2+\lambda)(x_{i+1}-x)^2 & \\ \quad - 12h(x_{i+1}-x)^3 - 3\lambda(x_{i+1}-x)^4, & x_i \le x < x_{i+1}, \\ 4h(1-\lambda)(x_{i+2}-x)^3 + 3\lambda(x_{i+2}-x)^4, & x_{i+1} \le x < x_{i+2}, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where x and $\lambda \in \mathbf{R}$ are a variable and a free parameter, respectively. For $-8 \le \lambda \le 1$, the extended cubic B-spline functions preserve identical properties as B-spline. When $\lambda = 0$, it should be noted that extended B-spline basis functions become a cubic B-spline basis. The splines $\phi_{-1}, \phi_0, \dots, \phi_{N+1}$ form a basis over the domain $[a,b]$.

The values of $\phi_i(x,\lambda)$ and their derivatives at different knots are as follows [18]:

$$\phi_i(x_j,\lambda) = \begin{cases} \frac{8+\lambda}{12}, & \text{if } i-j=0, \\ \frac{4-\lambda}{24}, & \text{if } i-j=\pm 1, \\ 0, & \text{else}, \end{cases} \tag{5}$$

$$\phi_i'(x_j,\lambda) = \begin{cases} 0, & \text{if } i-j=0, \\ \mp\frac{1}{2h}, & \text{if } i-j=\pm 1, \\ 0, & \text{else}, \end{cases} \tag{6}$$

$$\phi_i''(x_j,\lambda) = \begin{cases} -\frac{2+\lambda}{h^2}, & \text{if } i-j=0, \\ \frac{2+\lambda}{2h^2}, & \text{if } i-j=\pm 1, \\ 0, & \text{else}. \end{cases} \tag{7}$$

3 Description of the scheme based on extended cubic B-spline

Let $u(x,\lambda)$ be an analytical solution of the given differential equation. The approximated solution in terms of the extended cubic B-spline is defined as follows:

$$u(x_i,t_n)=\sum_{k=i-1}^{i+1} C_k^n(t)\phi_k(x,\lambda), \tag{8}$$

where $i=0,1,2,\dots,N$. The time-dependent unknowns $C_k^n(t)$'s are to be manipulated from the initial, boundary, and extended cubic B-spline collocation conditions. Each extended cubic B-spline covers four elements so that each subinterval $[x_i,x_{i+1}]$ holds only three nonzero basis functions ϕ_{i-1}, ϕ_i, ϕ_{i+1}. Thus the approximated solution and its derivatives in terms of parameters can be described as follows [17]:

$$\begin{cases} u_i^n=u(x_i,t^n)=a_1C_{i-1}^n+a_2C_i^n+a_1C_{i+1}^n,\\ (u_x)_i^n=u_x(x_i,t^n)=a_3C_{i-1}^n-a_3C_{i+1}^n,\\ (u_{xx})_i^n=u_{xx}(x_i,t^n)=a_4C_{i-1}^n+a_5C_i^n+a_4C_{i+1}^n,\end{cases} \tag{9}$$

where $a_1=\frac{4-\lambda}{24}$, $a_2=\frac{8+\lambda}{12}$, $a_3=\frac{1}{2h}$, $a_4=\frac{2+\lambda}{2h^2}$, $a_5=-\frac{2+\lambda}{h^2}$.

Caputo's formula [12] can be written as follows:

$$\frac{\partial^\gamma(x,t_n)}{\partial t^\gamma}=\frac{1}{\Gamma(2-\gamma)}\sum_{s=0}^{n-1} b_s\frac{u(x,t_{n-s})-u(x,t_{n-s-1})}{\tau^\gamma}+R_\tau^n, \tag{10}$$

where $b_s=(s+1)^{1-\gamma}-s^{1-\gamma}$. The truncation error R_τ^n is bounded, i.e.,

$$\left|R_\tau^n\right|\le I\tau^{2-\gamma}, \tag{11}$$

where I is a constant.

Lemma 3.1 *The coefficients b_s fulfill the following properties* [21]:

- $b_0=1$;
- $b_0>b_1>b_2>\cdots>b_s$, $b_s\to 0$ *as* $s\to\infty$;
- $b_s>0$ *for* $s=0,1,\dots,n$;
- $\sum_{s=0}^{n}(b_s-b_{s+1})+b_{n+1}=(1-b_1)+\sum_{s=1}^{n-1}(b_s-b_{s+1})+b_n=1$.

3.1 Fully implicit scheme

Let $u_i^n=u(x_i,t^n)$, $f_i^n=f(x_i,t^n)$, and $C_i^n=C_i(t^n)$ for $i=0,1,\dots,N$, $n=0,1,\dots,M$. Then, substituting (5), (6), (7) in (1), we have

$$\begin{aligned}&\frac{1}{\tau^\gamma\Gamma(2-\gamma)}\sum_{s=0}^{n} b_s\bigl[a_1\bigl(C_{i-1}^{n-s+1}-C_{i-1}^{n-s}\bigr)+a_2\bigl(C_i^{n-s+1}-C_i^{n-s}\bigr)+a_1\bigl(C_{i+1}^{n-s+1}-C_{i+1}^{n-s}\bigr)\bigr]\\ &\quad+(pa_3-qa_4)C_{i-1}^{n+1}+(-qa_5)C_i^{n+1}+(-pa_3-qa_4)C_{i+1}^{n+1}=f_i^{n+1}.\end{aligned} \tag{12}$$

After some simplification, the following recurrence relation is obtained:

$$(ra_1+pa_3-qa_4)C_{i-1}^{n+1}+(ra_2+0-qa_5)C_i^{n+1}+(ra_1-pa_3-qa_4)C_{i+1}^{n+1}$$

$$
\begin{aligned}
&= r\left(a_1 C_{i-1}^n + a_2 C_i^n + a_1 C_{i+1}^n\right) r \sum_{s=1}^{n} b_s \left[a_1\left(C_{i-1}^{n-s+1} - C_{i-1}^{n-s}\right) + a_2\left(C_i^{n-s+1} - C_i^{n-s}\right)\right. \\
&\quad \left. + a_1\left(C_{i+1}^{n-s+1} - C_{i+1}^{n-s}\right)\right] + f_i^{n+1},
\end{aligned}
$$

where $r = \frac{1}{\tau^{\gamma}\Gamma(2-\gamma)}$. The above system has $(N+1)$ linear equations and $(N+3)$ unknowns. To obtain a unique solution, two additional equations are required. These additional equations are obtained from boundary conditions. The system then becomes

$$
AC^{n+1} = B\left(b_n C^0 + \sum_{s=0}^{n-1}(b_s - b_{s+1})C^{n-s}\right) + F \tag{13}
$$

$$
A = \begin{bmatrix}
a_1 & a_2 & a_1 & \cdots & \cdots & \cdots & 0 \\
ra_1 + pa_3 - qa_4 & ra_2 - qa_5 & ra_1 - pa_3 - qa_4 & & & & \vdots \\
0 & ra_1 + pa_3 - qa_4 & ra_2 - qa_5 & ra_1 - pa_3 - qa_4 & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & ra_1 + pa_3 - qa_4 & ra_2 - qa_5 & ra_1 - pa_3 - qa_4 \\
0 & \cdots & \cdots & \cdots & a_1 & a_2 & a_1
\end{bmatrix}, \tag{14}
$$

$$
B = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
a_1 & a_2 & a_1 & 0 & \cdots & \cdots & 0 \\
0 & a_1 & a_2 & a_1 & \cdots & \cdots & 0 \\
\vdots & \cdots & \ddots & \ddots & \ddots & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & a_1 & a_2 & a_1 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0
\end{bmatrix}
$$

and $F = [g_1^{n+1}, f_0^{n+1}, \ldots, f_N^{n+1}, g_2^{n+1}]^T$. Consequently, we have $(N+3) \times (N+3)$ system of linear equations.

4 Initial state C^0

To start iteration on Eq. (12), a suitable initial vector $C^0 = [C_{-1}^0, C_0^0, \ldots, C_{N+1}^0]^T$ is constructed from the initial conditions. We utilize the initial condition together with its derivatives as follows:

- $(u_i^0)_x = \frac{d}{dx}(\omega(x_i))$, $i = 0, N$;
- $u_i^0 = u(x_i, 0) = \sum_{i=1}^{N+1} C_i^0(0)\phi(x_i)$, $i = 0, 1, \ldots, N$.

This gives a linear system of order $(N+3) \times (N+3)$. The above system can be written in the matrix form as follows:

$$DC^0 = E.$$

$$D = \begin{bmatrix} a_3 & 0 & -a_3 & 0 & \ldots & \ldots & 0 \\ a_1 & a_2 & a_1 & 0 & \ldots & \ldots & 0 \\ 0 & a_1 & a_2 & a_1 & \ldots & \ldots & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & \ldots & \vdots \\ \vdots & \ldots & \ldots & \ldots & a_1 & a_2 & a_1 \\ 0 & \ldots & \ldots & \ldots & a_3 & 0 & -a_3 \end{bmatrix}$$

and $E = [\omega_0', \omega_0, \omega_1, \ldots, \omega_N, \omega_N']^T$.

5 Stability

The concept of stability is associated with the requirement that errors which are introduced in the computational procedure die out as the procedure continues [27]. As the fractional advection–diffusion equation is linear, the stability of proposed schemes can be investigated by the Fourier method. Suppose $U(x,t)$ in the approximation of (12). We define

$$\xi_i^n = u_i^n - U_i^n, \quad i = 1, \ldots, N-1, n = 0, 1, \ldots, M, \tag{15}$$

and vector

$$\xi^n = \left[\xi_1^n, \xi_2^n, \ldots, \xi_{N-1}^n\right]^T. \tag{16}$$

Equation (15) satisfies Eq. (12), we obtain the round-off error equations as follows:

$$\begin{aligned} &(ra_1 + pa_3 - qa_4)\xi_{i-1}^{n+1} + (ra_2 - qa_5)\xi_i^{n+1} + (ra_1 - pa_3 - qa_4)\xi_{i+1}^{n+1} \\ &\quad = r\left(a_1\xi_{i-1}^n + a_2\xi_i^n + a_1\xi_{i+1}^n\right) - r\sum_{s=1}^{n} b_s\left[a_1\left(\xi_{i-1}^{n-s+1} - \xi_{i-1}^{n-s}\right) + a_2\left(\xi_i^{n-s+1} - \xi_i^{n-s}\right)\right. \\ &\qquad \left. + a_1\left(\xi_{i+1}^{n-s+1} - \xi_{i+1}^{n-s}\right)\right]. \end{aligned} \tag{17}$$

Then initial and boundary conditions become

$$\xi_i^0 = \omega(x_i), \quad i = 1, 2, \ldots, N, \tag{18}$$

and

$$\xi_0^n = g_1(t_n), \qquad \xi_N^n = g_2(t_n), \quad n = 0, 1, \ldots, M. \tag{19}$$

Define grid functions based on the Fourier method as follows:

$$\xi^n = \begin{cases} \xi_i^n, & x_i - \frac{h}{2} < x \le x_i + \frac{h}{2}, i = 1, 2, \ldots, N-1, \\ 0, & a \le x \le a + \frac{h}{2} \text{ or } b - \frac{h}{2} \le x \le b. \end{cases} \tag{20}$$

Then $\xi^n(x)$ can be expressed in the form of Fourier series

$$\xi^n(x) = \sum_{-\infty}^{\infty} \eta_n(m) \exp\bigl(i2\pi mx/(b-a)\bigr), \quad n = 1, 2, \ldots, M, \tag{21}$$

where

$$\eta_n(m) = \frac{1}{b-a} \int_a^b \xi^n(x) \exp\bigl(-i2\pi mx/(b-a)\bigr)\, dx. \tag{22}$$

Note the natural definition of norm:

$$\begin{aligned} \|\xi^n\|_2 &= \left(\sum_{i=1}^{N-1} h |\xi_i^n|^2 \right)^{1/2} \\ &= \left[\int_a^{a+h/2} |\xi^n|^2\, dx + \sum_{i=1}^{M-1} \int_{x_i-h/2}^{x_i+h/2} |\xi^n|^2\, dx + \int_{b-h/2}^{b} |\xi^n|^2\, dx \right]^{1/2} \\ &= \left[\int_a^b |\xi^n|^2\, dx \right]^{1/2}. \end{aligned}$$

Using the Parseval equality [28], we have

$$\int_a^b |\xi^n|^2\, dx = \sum_{-\infty}^{\infty} |\eta_n(m)|^2\, dx,$$

we get

$$\|\xi^n\|_2^2 = \sum_{-\infty}^{\infty} |\eta_n(m)|^2\, dx. \tag{23}$$

5.1 Stability for a fully implicit scheme

Let the solution in the form of Fourier series analysis be described as follows:

$$\xi_j^n = \eta_n e^{i\sigma jh}, \tag{24}$$

where $i = \sqrt{-1}$ and $\sigma = 2\pi m/(b-a)$. Using expression (24) in (17), we obtain

$$\begin{aligned} &(ra_1 + pa_3 - qa_4)\eta_{n+1} e^{i\sigma(j-1)h} + (ra_2 - qa_5)\eta_{n+1} e^{i\sigma jh} + (ra_1 - pa_3 - qa_4)\eta_{n+1} e^{i\sigma(j+1)h} \\ &\quad = r\bigl(a_1 \eta_n e^{i\sigma(j-1)h} + a_2 \eta_{n+1} e^{i\sigma jh} + a_1 \eta_{n+1} e^{i\sigma(j+1)h}\bigr) - r \sum_{s=1}^{n} b_s \bigl[a_1\bigl(\eta_{n-s+1} - \eta^{n-s}\bigr) e^{i\sigma(j-1)h} \\ &\qquad + a_2(\eta_{n-s+1} - \eta_{n-s}) e^{i\sigma jh} + a_1(\eta_{n-s+1} - \eta_{n-s}) e^{i\sigma(j+1)h}\bigr]. \end{aligned}$$

After some calculation and collection of likewise terms, we obtain

$$\eta_{n+1} = \frac{1}{w_1}\eta_n - \frac{1}{w_1}\sum_{s=1}^{n} b_s(\eta_{n-s+1} - \eta_{n-s}), \tag{25}$$

where $w_1 = 1 + \frac{24q(2+\lambda)\sin^2(\sigma h/2) - 12phi\sin(\sigma h)}{rh^2[12+(\lambda-4)2\sin^2(\sigma h/2)]}$, clearly $w_1 \geq 1$ for $\lambda > -2$.

Proposition 5.1 *Suppose that η_n, $n = 1, 2, \ldots, T \times M$, is the solution of* (25), *we have*

$$|\eta_n| \leq |\eta_0|, \quad n = 1, 2, \ldots, T \times M. \tag{26}$$

Proof Apply the mathematical induction to verify inequality (26). Put $n = 0$ in (25) which now takes the form

$$|\eta_1| = \frac{1}{w_1}|\eta_0| \leq |\eta_0|, \qquad \frac{1}{w_1} \geq 1.$$

Suppose that $|\eta_n| \leq |\eta_0|$ is true for $n = 1, 2, \ldots, T \times M - 1$. From Eq. (25) we have

$$\begin{aligned} |\eta_{n+1}| &\leq \frac{1}{w_1}|\eta_n| - \frac{1}{w_1}\sum_{s=1}^{n}\bigl(|\eta_{n+1-s}| - |\eta_{n-s}|\bigr) \\ &\leq \frac{1}{w_1}|\eta_0| - \frac{1}{w_1}\sum_{s=1}^{n}\bigl(|\eta_0| - |\eta_0|\bigr) \\ &\leq |\eta_0|. \end{aligned}$$

Hence (26) is true. □

Theorem 1 *The implicit scheme* (12) *is unconditionally stable.*

Proof Utilizing the above proposition and noticing (23), we obtain

$$\|\xi^n\|_2 \leq \|\xi^0\|_2, \quad n = 0, 1, \ldots, M,$$

which shows that implicit scheme (12) with initial and boundary conditions is unconditionally stable. □

6 Convergence

In this section, we follow Kadalbajoo and Arora's [29] technique to examine the convergence of the proposed method.

Theorem 2 ([30, 31]) *Assume that $u(x,t) \in C^4[a,b]$, $f \in C^2[a,b]$, and $\Omega = [a = x_0, x_1, \ldots, x_N = b]$ is the equidistant partition of $[a,b]$ with step size h. If $\hat{U}(x,t)$ is the unique spline interpolating the solution of the proposed problem at knots $x_0, \ldots, x_N \in \Omega$, then there is a constant m_i independent of h, so that for every $t \geq 0$, we have*

$$\bigl\|D^i\bigl(u(x,t) - U(x,t)\bigr)\bigr\|_\infty \leq m_i h^{4-i}, \quad i = 0, 1, 2. \tag{27}$$

Lemma 6.1 *The extended B-spline set* $\{\phi_{-1}, \phi_0, \dots, \phi_{N+1}\}$ *described in definition* (4) *fulfills the inequality*

$$\sum_{i=-1}^{N+1} \bigl|\phi_i(x,\lambda)\bigr| \le \frac{7}{4}, \quad 0 \le x \le 1. \tag{28}$$

Proof By the triangular inequality, we obtain

$$\left|\sum_{i=-1}^{N+1} \phi_i(x,\lambda)\right| \le \sum_{i=-1}^{N+1} \bigl|\phi_i(x,\lambda)\bigr|.$$

For any knot x_i, we have

$$\sum_{i=-1}^{N+1} \bigl|\phi_i(x_i,\lambda)\bigr| = \bigl|\phi_{i-1}(x_i,\lambda)\bigr| + \bigl|\phi_i(x_i,\lambda)\bigr| + \bigl|\phi_{i+1}(x_i,\lambda)\bigr| = \frac{4-\lambda}{24} + \frac{8+\lambda}{12} + \frac{4-\lambda}{24} = 1 < \frac{7}{4}.$$

Also, for $x \in [x_i, x_{i+1}]$, we have

$$\begin{aligned} \bigl|\phi_i(x,\lambda)\bigr| &\le \frac{8+\lambda}{12}, \qquad \bigl|\phi_{i+1}(x,\lambda)\bigr| \le \frac{8+\lambda}{12}, \\ \bigl|\phi_{i-1}(x,\lambda)\bigr| &\le \frac{4-\lambda}{24}, \qquad \bigl|\phi_{i+2}(x,\lambda)\bigr| \le \frac{4-\lambda}{24}. \end{aligned}$$

Then, for any point $x \in [x_i, x_{i+1}]$, we have

$$\sum_{i=-1}^{N+1} \bigl|\phi_i(x,\lambda)\bigr| = \bigl|\phi_{i-1}(x,\lambda)\bigr| + \bigl|\phi_i(x,\lambda)\bigr| + \bigl|\phi_{i+1}(x,\lambda)\bigr| + \bigl|\phi_{i+2}(x,\lambda)\bigr| = \frac{20+\lambda}{12}.$$

Since $-8 \le \lambda \le 1$, thus we have $1 \le \frac{20+\lambda}{12} \le \frac{7}{4}$. □

Theorem 3 *The approximate solution* $U(x,t)$ *to the exact solution* $u(x,t)$ *of the time-dependent fractional partial differential problem* (1)–(3) *exists. Moreover, if* $f \in C^2[0,1]$, *we have*

$$\bigl\|u(x,t) - U(x,t)\bigr\|_\infty \le Mh^2 \tag{29}$$

for every $t \ge 0$ *and sufficiently small* h, *where* M *is a positive constant independent of* h.

Proof Let $\hat{U}(x,t)$ be the computed spline approximation to the approximated solution $U(x,t)$, where $\hat{U}(x,t) = \sum_{i=-1}^{N+1} d_i(t)\phi_i(x)$. By the triangular inequality, we can write it as follows:

$$\bigl\|u(x,t) - U(x,t)\bigr\|_\infty \le \bigl\|u(x,t) - \hat{U}(x,t)\bigr\|_\infty + \bigl\|\hat{U}(x,t) - U(x,t)\bigr\|_\infty. \tag{30}$$

Using Theorem 2 error approximation, we get

$$\bigl\|D^i\bigl(u(x,t) - \hat{U}(x,t)\bigr)\bigr\|_\infty \le m_i h^{4-i}, \quad i = 0,1,2. \tag{31}$$

Using the above estimate inequality (31), we obtain

$$\left\| u(x,t) - U(x,t) \right\|_\infty \le m_0 h^4 + \left\| \hat{U}(x,t) - U(x,t) \right\|_\infty. \tag{32}$$

The collocation conditions are

$$Lu(x_i,t) = LU(x_i,t) = f(x_i,t), \quad i = 0,1,\dots,N.$$

Let

$$L\hat{U}(x,t) = \hat{f}(x_i,t), \quad i = 0,1,\dots,N.$$

Thus the given problem in the form of difference equation $L(\hat{U}(x_i,t) - U(x_i,t))$ at any time level n can be written as follows:

$$\begin{aligned}
&(ra_1 + pa_3 - qa_4)\delta_{i-1}^{n+1} + (ra_2 - qa_5)\delta_i^{n+1} + (ra_1 - pa_3 - qa_4)\delta_{i+1}^{n+1} \\
&\quad = r\big(a_1\delta_{i-1}^n + a_2\delta_i^n + a_1\delta_{i+1}^n\big) - r\sum_{s=1}^{n} b_s\big[a_1\big(\delta_{i-1}^{n-s+1} - \delta_{i-1}^{n-s}\big) + a_2\big(\delta_i^{n-s+1} - \delta_i^{n-s}\big) \\
&\qquad + a_1\big(\delta_{i+1}^{n-s+1} - \delta_{i+1}^{n-s}\big)\big] + f^{n+1},
\end{aligned} \tag{33}$$

and the boundary conditions are as follows:

$$a_1\delta_{i-1}^{n+1} + a_2\delta_i^{n+1} + a_1\delta_{i+1}^{n+1} = 0, \quad i = 0,N,$$

where

$$\delta_i^n = C_i^n - d_i^n, \quad i = -1,0,1,\dots,N+1.$$

From inequality (31), we obtain

$$\beta_i^2 = h^2\big[f_i^n - \hat{f}_i^n\big] \le mh^4, \quad i = 0,1,\dots,N.$$

Define $\beta^n = \max\{|\beta_i^n|; 0 \le i \le N\}$, $e_i^n = |\delta_i^n|$ and $e^n = \max\{|e_i^n|; 0 \le i \le N\}$. Now Eq. (33) becomes

$$\begin{aligned}
&(ra_1 + pa_3 - qa_4)\delta_{i-1}^1 + (ra_2 - qa_5)\delta_i^1 + (ra_1 - pa_3 - qa_4)\delta_{i+1}^1 \\
&\quad = r\big(a_1\delta_{i-1}^0 + a_2\delta_i^0 + a_1\delta_{i+1}^0\big) + \frac{1}{h^2}\beta_i^1, \quad i = 0,1,\dots,N.
\end{aligned}$$

From the initial condition, $e^0 = 0$.

$$(ra_2 - qa_5)\delta_i^1 = -(ra_1 - qa_4)\big(\delta_{i-1}^1 + \delta_{i+1}^1\big) - pa_3\big(\delta_{i-1}^1 - \delta_{i+1}^1\big) + \frac{1}{h^2}\beta_i^1.$$

Taking absolute values of β_i^n and δ_i^n with sufficiently small h gives

$$e_i^1 \le \frac{6mh^4}{2rh^2(2+\lambda) + 12(2+\lambda)q + 12ph}, \quad i = 0,1,\dots,N.$$

From the boundary conditions, we get

$$e_{-1}^{1} \leq \left(\frac{20+\lambda}{(4-\lambda)((rh^2+6q)(2+\lambda)+12ph)}\right)3mh^4,$$
$$e_{N+1}^{1} \leq \left(\frac{20+\lambda}{(4-\lambda)((rh^2+6q)(2+\lambda)+12ph)}\right)3mh^4.$$

This implies

$$e^1 \leq m_1 h^2, \tag{34}$$

where m_1 is independent of h. Here mathematical induction on n is utilized. Suppose that $e_i^l \leq m_l h^2$ for $l = 1, 2, \dots, n$ is true and $m = \max\{m_l : 0 \leq l \leq n\}$, then from Eq. (33), we have

$$\begin{aligned}
&(ra_1 + pa_3 - qa_4)\delta_{i-1}^{n+1} + (ra_2 - qa_5)\delta_i^{n+1} + (ra_1 + pa_3 - qa_4)\delta_{i+1}^{n+1} \\
&\quad = r\big[(b_0 - b_1)\big(a_1\delta_{i-1}^{n} + a_2\delta_i^{n} + a_1\delta_{i+1}^{n}\big) + (b_1 - b_2)\big(a_1\delta_{i-1}^{n-1} + a_2\delta_i^{n-1} + a_1\delta_{i+1}^{n-1}\big) + \cdots \\
&\qquad + (b_{n-1} - b_n)\big(a_1\delta_{i-1}^{1} + a_2\delta_i^{1} + a_1\delta_{i+1}^{1}\big) + b_n\big(a_1\delta_{i-1}^{0} + a_2\delta_i^{0} + a_1\delta_{i+1}^{0}\big)\big] + \frac{1}{h^2}\beta^2.
\end{aligned}$$

Taking absolute values of δ_i^n and β_i^n, we have

$$e_i^{n+1} \leq \frac{6mh^2}{2rh^2(2+\lambda) + 12(2+\lambda)q + 12ph}\left(r\sum_{s=0}^{n-1}(b_s - b_{s+1})mh^2 + mh^2\right)$$

from the boundary conditions

$$e_i^{n+1} \leq mh^2, \quad i = -1, N+1.$$

Then, for every n, we have

$$e_i^{n+1} \leq mh^2. \tag{35}$$

Now we can write, from inequality (35) and Lemma 6.1,

$$\hat{U}(x,t) - U(x,t) = \sum_{i=-1}^{N+1}\big(d_i(t) - C_i(t)\big)\phi_i(x,\lambda).$$

Taking the norm, we obtain

$$\big\|\hat{U}(x,t) - U(x,t)\big\|_\infty \leq 1.75mh^2.$$

From Eq. (35) and the above inequality, we obtain

$$\big\|u(x,t) - \hat{U}(x,t)\big\|_\infty + \big\|\hat{U}(x,t) - U(x,t)\big\|_\infty \leq m_0h^4 + 1.75mh^2 = Mh^2,$$

where $M = m_0h^2 + 1.75m$. □

From relation (11) and the above theorem, it is deduced that the present scheme is convergent, i.e.,

$$\left\| u(x,t) - U(x,t) \right\|_\infty \leq Mh^2 + I\tau^{2-\gamma},$$

where M and I are constants.

7 Illustrative examples and discussions

Some numerical experiments are described in this section to illustrate the performance of the present scheme. The calculated absolute errors are found by absolute $\|e\|_\infty$ and Euclidean $\|e\|_2$ norms, i.e.,

$$\|e\|_\infty = \left\| U(x_i,t) - u(x_i,t) \right\|_\infty = \max_{0\leq i\leq N} \left| u(x_i,t) - u(x_i,t) \right|,$$

$$\|e\|_2 = \left\| U(x_i,t) - u(x_i,t) \right\|_2 = \sqrt{h\sum_{i=0}^{N} \left| u(x_i,t) - u(x_i,t) \right|^2}.$$

The numerical order of convergence is calculated by the following formula [32]:

$$Order = \frac{\log(\|e\|_\infty(N_i)) - \log(\|e\|_\infty(N_{i+1}))}{\log(N_{i+1}) - \log(N_i)},$$

where $\|e\|_\infty(N_i)$ and $\|e\|_\infty(N_{i+1})$ are the absolute error at the number of partitioning N_i and N_{i+1}, respectively.

7.1 Problem 1

Consider $p = 1$, $q = 2$, solve (1)–(3) with initial and boundary conditions $\omega(x) = e^x$, $g_1(t) = E(t^\gamma)$, $g_2(t) = eE(t^\gamma)$, respectively, and the homogeneous source term is considered on $[0,1]$. The exact analytical solution [25] is

$$u(x,t) = e^x E\left(t^\gamma\right),$$

where E_γ is the Mittag–Leffler function

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k+1)}, \quad 0 < \gamma < 1.$$

Tables 1–3 show the comparison of $\|\cdot\|_\infty$ and $\|\cdot\|_2$ between MCTB-DQM [25] and the proposed method based on extended cubic B-spline for different values of λ. Our technique yields better accuracy compared to MCTB-DQM method with $O(\tau^3 + h^2)$ [25]. By choosing $N = 100$, $\gamma = 0.5$ at time $T = 1$, Table 4 shows the comparison at different values of x. Figure 1 depicts the comparison between approximated and exact values for a fully implicit scheme. Table 5 reflects the comparison of max error ($\|\cdot\|_\infty$) and Euclidean norm ($\|\cdot\|_2$) at $T = 1$ for problem-2.

Table 1 A comparison of maximum error ($\| \cdot \|_\infty$) and Euclidean norm ($\| \cdot \|_2$) at $T = 1$ for problem-1

N	$\tau = 1.0 \times 10^{-2}, \gamma = 0.2$					
	MCTB-DQM [25]		Proposed method			
	$\| \cdot \|_\infty$	$\| \cdot \|_2$	$\| \cdot \|_\infty$	$\| \cdot \|_2$	Order	CPU time
08	1.4902e–02	1.0412e–02	7.0982e–04	5.2421e–04	…	0.09360
16	3.8827e–03	2.6898e–03	6.9478e–05	5.0417e–05	3.35283	0.14040
32	1.0156e–03	6.6522e–04	3.4560e–05	2.5203e–05	1.00747	0.26520
64	2.5720e–04	1.4842e–04	1.7410e–06	1.2739e–06	4.31108	0.73321
128	6.3504e–05	2.2129e–05	3.8083e–07	1.9860e–07	2.19272	2.07481

Table 2 A comparison of maximum error ($\| \cdot \|_\infty$) and Euclidean norm ($\| \cdot \|_2$) at $T = 1$ for problem-1

N	$\tau = 1.0 \times 10^{-2}, \gamma = 0.5$					
	MCTB-DQM [25]		Proposed method			
	$\| \cdot \|_\infty$	$\| \cdot \|_2$	$\| \cdot \|_\infty$	$\| \cdot \|_2$	Order	CPU time
08	6.3092e–03	4.4047e–03	1.9311e–04	1.4246e–04	…	0.062400
16	1.6452e–03	1.1394e–03	7.0386e–05	5.1038e–05	1.45609	0.10920
32	4.3121e–04	2.8317e–04	2.6417e–05	1.9079e–05	1.41382	0.23400
64	1.0956e–04	6.4521e–05	5.4923e–06	3.8494e–06	2.26599	0.63960
128	2.7227e–05	1.0443e–05	5.7211e–07	3.0277e–07	3.26304	2.01241

Table 3 A comparison of maximum error ($\| \cdot \|_\infty$) and Euclidean norm ($\| \cdot \|_2$) at $T = 1$ for problem-1

N	$\tau = 1.0 \times 10^{-2}, \gamma = 0.8$					
	MCTB-DQM [25]		Proposed method			
	$\| \cdot \|_\infty$	$\| \cdot \|_2$	$\| \cdot \|_\infty$	$\| \cdot \|_2$	Order	CPU time
08	4.1559e–03	2.9052e–03	2.6864e–05	1.4246e–04	…	0.07800
16	1.0852e–03	7.5335e–04	1.0356e–05	7.4698e–06	1.37526	0.14040
32	2.8491e–04	1.8911e–04	1.1811e–06	7.8660e–07	3.13225	0.24960
64	7.2683e–05	4.4967e–05	5.3813e–07	3.2574e–07	1.13407	0.65520
128	1.8220e–05	8.7572e–06	2.4165e–07	1.5278e–07	1.15502	2.01241

Table 4 A comparison of exact solution and approximated solution at $T = 1$ for problem-1

	$\tau = 1.0 \times 10^{-3}, \gamma = 0.5, \lambda = 0.00001, N = 100$		
x	Exact solution	Approximated solution	Error
0.1	5.53577911	5.53577900	1.1363e–07
0.2	6.11798209	6.11798186	2.2948e–07
0.3	6.76141588	6.76141555	3.3011e–07
0.4	7.47252019	7.47251979	4.0100e–07
0.5	8.25841200	8.25841157	4.3141e–07
0.6	9.12695678	9.12695636	4.1533e–07
0.7	10.0868472	10.0868468	3.5263e–07
0.8	11.1476902	11.1476900	2.5034e–07
0.9	12.3201030	12.3201029	1.2413e–07

Table 5 A comparison of maximum error ($\| \cdot \|_\infty$) and Euclidean norm ($\| \cdot \|_2$) at $T = 1$ for problem-2

N	$\tau = 1.25 \times 10^{-3}, \gamma = 0.3$							
	CBSCM [21]		MCTB-DQM [25]		Proposed method			
	$\| \cdot \|_\infty$	$\| \cdot \|_2$	$\| \cdot \|_\infty$	$\| \cdot \|_2$	$\| \cdot \|_\infty$	$\| \cdot \|_2$	Order	CPU time
08	4.8273e–02	3.4134e–02	1.5762e–02	9.4300e–03	2.2761e–05	5.6903e–06	…	6.08404
16	1.2351e–02	8.7334e–03	2.1670e–03	1.1924e–03	7.4956e–06	1.3251e–06	1.60246	8.01845
32	3.1048e–03	2.1955e–03	2.8541e–04	1.5040e–04	1.7463e–06	2.1829e–07	2.1017	15.3193
64	7.7721e–04	5.4957e–04	3.6701e–05	1.8925e–05	1.3761e–07	1.2163e–08	3.6656	34.5386
128	1.9430e–04	1.3739e–04	4.6559e–06	2.3752e–06	2.2313e–08	1.3945e–09	2.62468	93.4914

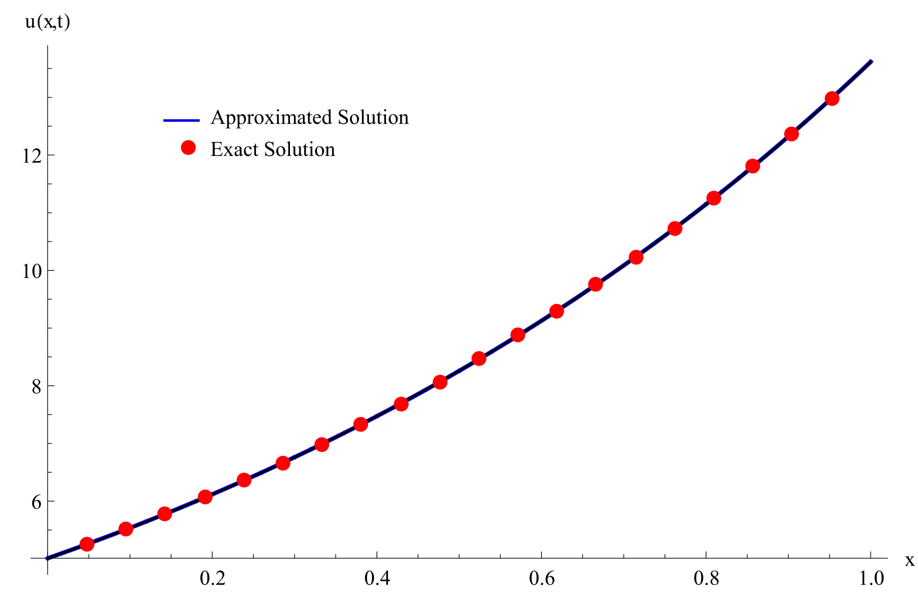

Figure 1 Comparison graph of approximated values and exact values with $\gamma = 0.5$, $N = 50$, $\tau = 0.01$

Table 6 A comparison of exact values and approximated values at different knots

x	Exact solution	Approximated solution	Absolute error
0.1	0.58778525	0.58778972	4.4636e–06
0.2	0.95105652	0.95106374	7.2223e–06
0.3	0.95105652	0.95106374	7.2223e–06
0.4	0.58778525	0.58778972	4.4636e–06
0.5	0.00000000	2.96897923	2.9690e–14
0.6	–0.5877853	–0.5877897	4.4636e–06
0.7	–0.9510565	–0.9510637	7.2223e–06
0.8	–0.9510565	–0.9510637	7.2223e–06
0.9	–0.5877853	–0.5877897	4.4636e–06

7.2 Problem 2

Consider $p = 0$, $q = 1$, solve (1)–(3) with initial and boundary conditions $\omega(x) = 0$, $g_1(t) = 0$, $g_2(t) = 0$, respectively, and the homogeneous source term is

$$f(x,t) = \frac{2t^{2-\gamma}\sin(2\pi x)}{\Gamma(3-\gamma)} + 4\pi^2 t^2 \sin(2\pi x)$$

on $[0,1]$. The exact analytical solution [25] takes the form

$$u(x,t) = t^2 \sin(2\pi x).$$

Table 6 displays the errors between exact analytical solutions and approximated solutions at different knots corresponding to $N = 100$, $\gamma = 0.5$, $\lambda = -0.00065$, $\tau = 1.0 \times 10^{-2}$, and $T = 1$. Table 7 shows the absolute error for problem 2 corresponding to $N = 50$, $\gamma = 0.3$, $\lambda = -0.0026305$, $\tau = 0.1$, and $T = 10$. All the graphical results can also be seen in Figs. 2 and 3.

Table 7 A comparison of exact values and approximated values at different knots at time $T = 10$

x	Exact solution	Approximated solution	Absolute error
0.1	58.7785252	58.7785251	1.2446e–07
0.2	95.1056516	95.1056514	2.0138e–07
0.3	95.1056516	95.1056514	2.0138e–07
0.4	58.7785252	58.7785251	1.2446e–07
0.5	0.00000000	–5.8308913	5.8309e–13
0.6	–58.778525	–58.778525	1.2446e–07
0.7	–95.105652	–95.105651	2.0138e–07
0.8	–95.105652	–95.105651	2.0138e–07
0.9	–58.778525	–58.778525	1.2446e–07

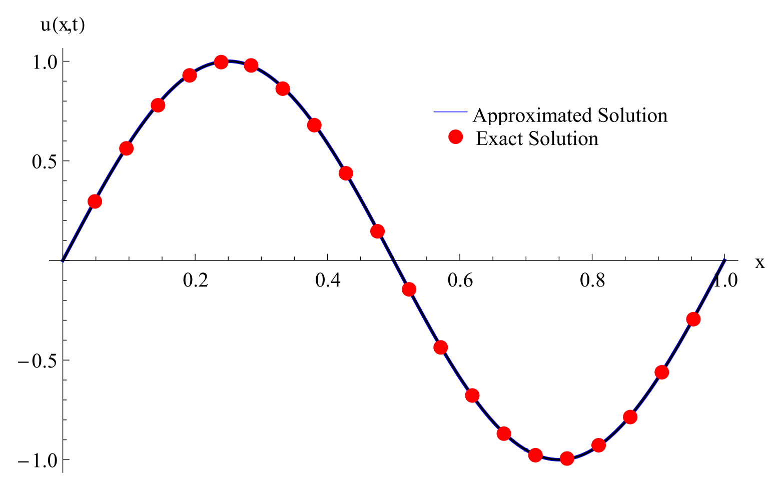

Figure 2 Comparison of approximated and exact solution for $N = 16$, $\gamma = 0.3$, $\tau = 0.01$ of problem-2

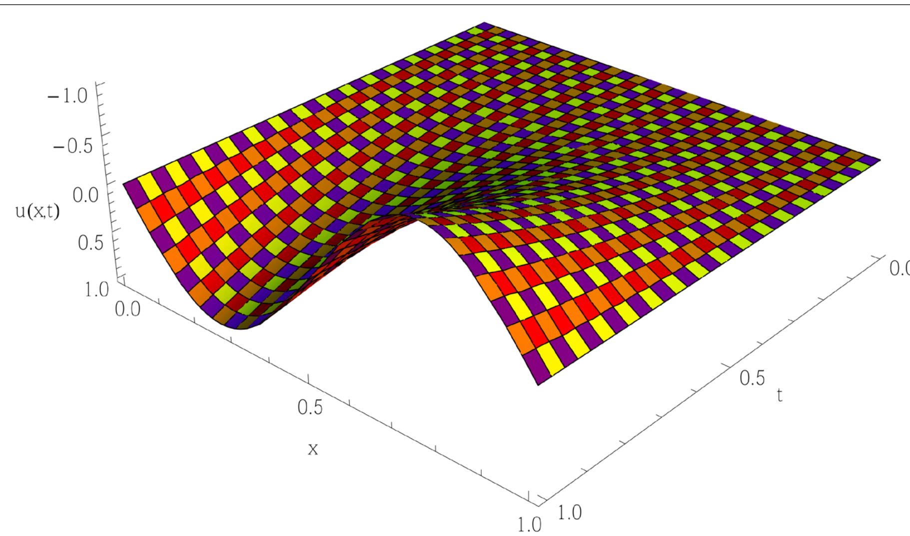

Figure 3 3D plot for $N = 50$, $\gamma = 0.3$, $T = 1$ of problem-2

7.3 Conclusion

A fully implicit finite difference scheme based on extended cubic B-spline has been formulated to solve the time fractional advection–diffusion equation. The proposed technique was examined and found to be unconditionally stable and convergent with $O(\tau + h^2)$. This technique was tested on two test problems, and the results indicated that the method is feasible and accurate.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Center for Research (CFR), University of Islamabad (UoI), Islamabad, Pakistan. [2]Faculty of Engineering & Applied Sciences, Riphah International University, Islamabad, Pakistan. [3]School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia. [4]Department of Mathematics, University of Sargodha, Sargodha, Pakistan.

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References

1. Barkai, E., Metzler, R., Klafter, J.: From continuous time random walks to the fractional Fokker–Planck equation. Phys. Rev. E **61**, 132–138 (2000)
2. Beinum, W., Meeussen, J., Edwards, A., Riemsdijk, W.: Transport of ions in physically heterogeneous systems, convection and diffusion in a column filled with alginate gel beads, predicted by a two-region model. Water Res. **7**, 2043–2050 (2000)
3. Diethelm, K., Freed, A.D.: On solution of nonlinear fractional order differential equations used in modelling of viscoplasticity. In: Scientific Computing in Chemical Engineering II. Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, pp. 217–224. Springer, Heidelberg (1999)
4. Mainardi, F.: Fractals and Fractional Calculus in Continuum Mechanics. Springer, Berlin (1997)
5. Podlubny, I.: Fractional Differential Equations. Academic Press, London (1999)
6. Shlesinger, M.F., West, B.J., Klafter, J.: Lévy dynamics of enhanced diffusion, application to turbulence. Phys. Rev. Lett. **58**(11), 1100–1103 (1987)
7. Zaslavsky, G.M., Stevens, D., Weitzner, H.: Self-similar transport in incomplete chaos. Phys. Rev. E **48**(3), 1683–1694 (1993)
8. Zheng, Y., Li, C., Zhao, Z.: A note on the finite element method for the space-fractional advection-diffusion equation. Comput. Math. Appl. **59**, 1718–1726 (2010)
9. Wang, K., Wang, H.: A fast characteristic finite difference method for fractional advection-diffusion equations. Adv. Water Resour. **34**(7), 810–816 (2011)
10. Shen, S., Liu, F., Anh, V.: Numerical approximations and solution techniques for the space–time Riesz–Caputo fractional advection diffusion equation. Numer. Algorithms **56**, 383–403 (2011)
11. Jiang, H., Liu, F., Turner, I., Burrage, K.: Analytical solutions for the multi-term time–space Caputo–Riesz fractional advection–diffusion equations on a finite domain. J. Math. Anal. Appl. **389**(2), 1117–1127 (2012)
12. Liu, F., Zhuang, P., Turner, I., Burrage, K., Anha, V.: A new fractional finite volume method for solving the fractional diffusion equation. Appl. Math. Model. **38**(15–16), 3871–3878 (2014)
13. Bu, W., Liu, X., Tang, Y., Yang, J.: Finite element multigrid method for multi-term time fractional advection-diffusion equations. Inter. J. Model. Sim. Sci. Comput. **6**(1), 1540001 (2015). ©World Scientific Publishing Company. https://doi.org/10.1142/S1793962315400012
14. Parvizi, M., Eslahchi, M.R., Dehghan, M.: Numerical solution of fractional advection-diffusion equation with a nonlinear source term. Numer. Algorithms **68**, 601–629 (2015)
15. Rubab, Q., Mirza, I.A., Qureshi, M.Z.A.: Analytical solutions to the fractional advection-diffusion equation with time dependent pulses on the boundary. AIP Adv. **6**, 075318 (2016)
16. Povstenko, Y., Kyrylych, T.: Two approaches to obtaining the space–time fractional advection–diffusion equation. Entropy **2017**(19), 297 (2017)
17. Goh, J.: B-splines for initial and boundary value problems. Doctoral dissertation (2013). Retrieved from Universiti Sains Malaysia
18. Han, X.L., Liu, S.J.: An extension of the cubic uniform B-spline curves. J. Comput.-Aided Des. Comput. Graph. **15**(5), 576–578 (2003)
19. Tasbozan, O., Esen, A., Yagmurlu, N.M., Ucar, Y.: A numerical solution to fractional diffusion equation for force-free case. Abstr. Appl. Anal. **2013**, Article ID 187383 (2013)
20. Esen, A., Tasbozan, O., Ucar, Y., Yagmurlu, N.M.: A B-spline collocation method for solving fractional diffusion and fractional diffusion-wave equations. Tbilisi Math. J. **8**(2), 181–193 (2015)

21. Sayevand, K., Yazdani, A., Arjang, F.: Cubic B-spline collocation method and its application for anomalous fractional diffusion equations in transport dynamic systems. J. Vib. Control **22**(9), 2173–2186 (2016)
22. Yaseen, M., Abbas, M., Ismail, A.I., Nazir, T.: A cubic trigonometric B-spline collocation approach for the fractional sub-diffusion equations. Appl. Math. Comput. **293**, 311–319 (2017)
23. Zhu, X.G., Nie, Y.F.: On a collocation method for the time-fractional convection-diffusion equation with variable coefficients. arXiv:1604.02112v2 [math.NA], 26 September (2016)
24. Yaseen, M., Abbas, M., Nazir, T., Baleanu, D.: A finite difference scheme based on cubic trigonometric B-splines for time fractional diffusion-wave equation. Adv. Differ. Equ. **2017**, 274 (2017)
25. Zhu, X.G., Nie, Y.F., Zhang, W.W.: An efficient differential quadrature method for fractional advection–diffusion equation. Nonlinear Dyn. **90**(3), 1807–1827 (2017). https://doi.org/10.1007/s11071-017-3765-x
26. Yuan, Q., Chen, H.: An expanded mixed finite element simulation for two-sided time-dependent fractional diffusion problem. Adv. Differ. Equ. **2018**, 34 (2018)
27. Boyce, W.E., Diprima, R.C.: Elementary Differential Equations and Boundary Value Problems. **9**. Wiley, New York (1969)
28. Cui, M.: A high-order compact exponential scheme for the fractional convection-diffusion equation. J. Comput. Appl. Math. **255**, 404–416 (2014)
29. Kadalbajoo, M.K., Arora, P.: B-spline collocation method for the singular-perturbation problem using artificial viscosity. Comput. Math. Appl. **57**, 650–663 (2009)
30. Hall, C.A.: On error bounds for spline interpolation. J. Approx. Theory **1**, 209–218 (1968)
31. Boor, C.D.: On the convergence of odd degree spline interpolation. J. Approx. Theory **1**, 452–463 (1968)
32. Abbas, M., Majid, A.A., Ismail, A.I., Rashid, A.: The application of cubic trigonometric B-spline to the numerical solution of the hyperbolic problems. Appl. Math. Comput. **239**, 74–88 (2014)