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Translation, solving scheme, and implementation of a periodic and optimal impulsive state control problem

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Abstract

The periodic solution of the impulsive state feedback controls (ISFC) has been investigated extensively in the last decades. However, if the ecosystem is exploited in a period mode, what strategies are implemented to optimize the cost function at the minimal cost? Firstly, under the hypothesis that the system has a periodic solution, an optimal problem of ISFC is transformed into a parameter optimization problem in an unspecified time with inequality constraints, and together with the constraint of the first arrival threshold. Secondly, the rescaled time and a constraint violation function are introduced to translate the above optimal problem to a parameter selection problem in a specified time with the unconstraint. Thirdly, gradients of the objective function on all parameters are given to compute the optimal value of the cost function. Finally, three examples involving the marine ecosystem, computer virus, and resource administration are illustrated to confirm the validity of our approaches.

Keywords: Impulsive state feedback control (ISFC); Rescaled time transformation; Constraint violation function; Parameter optimization; Numerical simulation

1 Introduction

The topic about impulsive state feedback controls (abbreviated as ISFC) has been investigated extensively in the last decades due to its potential applications in culturing microorganisms [1–3], pest integrated management [4–6], disease control [7, 8], fish harvesting [9–11], and wildlife management [12, 13]. For example, [1] proposed a bioprocess model with ISFC to acquire an equivalent stable output by the precise feeding. Ref. [4] explored the periodic solution of an entomopathogenic nematode invading the insect model with ISFC. Ref. [7] considered some vaccines into a disease by ISFC, and got the uniqueness of order one periodic solution (OOPS) by geometric method. Ref. [12] formulated a white-headed langur's ISFC model with sparse effect and continuous delay to study the periodic and artificial releasing. On ISFC models, scholars often pay close attention to the qualitative analysis of OOPS. Ref. [11] proposed a phytoplankton–fish model with ISFC and then formulated an optimal control problem (OCP, for short) and strived to find the appropriate harvesting rates to maximize the cost function in an impulsive period. Here, the solvability of the system in one period provides convenience for solving the OCP by Lagrange multiplier. But for the complicated ecosystem whose analytical solution cannot be expressed explicitly, if it is exploited in a period mode, what period and strategies are implemented

to optimize the cost function at the minimal cost? Furthermore, how to translate the OCP of the ISFC to a problem with parameter optimization in one period is interesting. So far, few researchers have paid attention to these tasks which are the focuses of our paper.

The utilizations of optimal control can be found almost in all applied science fields, such as fishery model [14], iatrochemistry [15], switching powers [16], astrovehicle controls [17], undersea vehicles [18], eco-epidemiology [19], and virus therapies [20]. The Pontryagin principle and the Hamilton equation are the main theoretical tools to solve continuous system control [21]. However, the hybrid optimization problems involving the pulse threshold and system parameters are still sufficiently challenging and worth exploring. The control parameter technique offers the feasibility for solving this problem [21]. Teo et al. described the detailed and basic theory of the control parameter method in [22]. Until now, many important results have been achieved in recent years. We will apply these theories together with the constraint transcription technique [23] to resolve the above issues.

The other components of this paper are as follows. In Section 2, an optimal problem of state impulse feedback control is transformed into a parameter optimization problem in an unspecified time with inequality constraint, and together with the constraint of the first arrival threshold. In Section 3 we derive the required gradient formulas and present an algorithm for solving the approximate OCP. In Section 4, we give three examples and numerical simulations. Finally, a conclusion is provided in Section 5.

2 Problem statement and translation

Consider the ISFC system:

$$\frac{d\hat{\mathbf{y}}}{dt} = \mathbf{f}(\hat{\mathbf{y}}, \delta), \quad \phi(\hat{\mathbf{y}}, \delta) \neq 0, \tag{2.1}$$

$$\Delta\hat{\mathbf{y}} = \mathbf{I}(\hat{\mathbf{y}}, \beta), \quad \phi(\hat{\mathbf{y}}, \delta) = 0, \tag{2.2}$$

where $t \in \mathbb{R}$, $\hat{\mathbf{y}} \in \mathbb{R}^n$, $\mathbf{f} \in \mathbb{R}^n$ is a given function, $\delta = (\delta_1, \dots, \delta_m)$ and $\beta = (\beta_1, \dots, \beta_q)$ are the control parameter vectors.

Denote by $\hat{\mathbf{y}}(t; 0, \hat{\mathbf{y}}_0, \delta, \beta)$ the solution of (2.1) and (2.2) satisfying the initial value

$$\hat{\mathbf{y}}(0^+; 0, \hat{\mathbf{y}}_0, \delta, \beta) = \hat{\mathbf{y}}_0.$$

For convenience, $\hat{\mathbf{y}}(t; 0, \hat{\mathbf{y}}_0, \delta, \beta)$ is abbreviated to $\hat{\mathbf{y}}(t)$ or $\hat{\mathbf{y}}$. $\mathbf{I}(\hat{\mathbf{y}}, \beta)$ is the impulsive effect and $\phi(\hat{\mathbf{y}}, \delta) = 0$ is the impulsive set. In detail,

$$\hat{\mathbf{y}} = \begin{pmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_n \end{pmatrix}, \quad \hat{\mathbf{y}}_0 = \begin{pmatrix} \hat{y}_{10} \\ \vdots \\ \hat{y}_{n0} \end{pmatrix},$$

$$\mathbf{f} = \begin{pmatrix} f_1(\hat{y}_1, \dots, \hat{y}_n, \delta) \\ \vdots \\ f_n(\hat{y}_1, \dots, \hat{y}_n, \delta) \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} I_1(\hat{y}_1, \dots, \hat{y}_n, \beta) \\ \vdots \\ I_n(\hat{y}_1, \dots, \hat{y}_n, \beta) \end{pmatrix}.$$

Next, let us introduce the following assumptions.

- (H1) Assume that $f_i, \phi,$ and I are continuously differentiable.
- (H2) Denote the Euclidean norm by $\|\cdot\|$. Suppose that there exists a constraint $k > 0$ meeting $|f_i(\hat{y})| \leq k(1 + \|\hat{y}\|)$ for all \hat{y} .
- (H3) Assume that for fixed $\hat{y}_0,$ (2.1) and (2.2) have a unique OOPS $\Gamma_{A \rightarrow B}$ possessing period $T,$ where A and B are the terminal and initial points of OOPS, respectively. When the impulsive effect take places, the point A is mapped to $B,$ namely

$$A(\hat{y}_1(T) + I_1(\hat{y}(T), \beta), \dots, \hat{y}_n(T) + I_n(\hat{y}(T), \beta)) \rightarrow B(\hat{y}_{10}, \dots, \hat{y}_{n0}).$$

Next, our aim is to formulate an optimal problem on one period under hypothesis (H3), namely (2.1) and (2.2) possess an OOPS $\Gamma_{A \rightarrow B}.$ Then system (2.1) is modified into

$$\frac{d\hat{y}}{dt} = f(\hat{y}, \delta), \quad t \in (0, T). \tag{2.3}$$

By (H3) it is obtained that

$$\hat{y}_{i0} = \hat{y}_i(T) + I_i(\hat{y}(T), \beta), \quad i = 1, \dots, n. \tag{2.4}$$

In order to guarantee that T is the first positive time at which the solution $\hat{y}(t)$ of (2.1) intersects with the surface $\phi(\hat{y}(t), \delta) = 0,$ $\hat{y}(t)$ should be defined as follows:

- (H4) $\hat{y}(t) \in \Omega,$ where

$$\Omega = \{\hat{y}(t) \mid \phi(\hat{y}(t), \delta) \neq 0 \text{ for } t \in (0, T) \text{ and } \phi(\hat{y}(T), \delta) = 0\}. \tag{2.5}$$

Remark If T is not the first positive time at which the solution \hat{y} of (2.1) intersects with the surface $\phi(\hat{y}(t), \delta) = 0,$ then there exists $0 < \check{T} < T$ such that $\phi(\hat{y}(\check{T}), \delta) = 0.$ This appears in contradiction to the definitions of (2.5).

Obviously, (2.5) is equivalent to

$$\Omega = \{\hat{y}(t) \mid \phi(\hat{y}(t), \delta) > \phi(\hat{y}(T), \delta) \text{ or } \phi(\hat{y}(t), \delta) < \phi(\hat{y}(T), \delta) \text{ for } t \in (0, T)\}. \tag{2.6}$$

Together with (2.4), the solution which firstly arrives at the surface $\phi(\hat{y}(T), \delta) = 0$ at time T from the initial point $(0, \hat{y}_0)$ (that is, the solution in (2.5) or (2.6)) is renewed by

$$\Omega = \{\Psi(\hat{y}(t), \hat{y}_0) < 0 \text{ and } \Psi^*(\hat{y}(t), \hat{y}(T)) < 0, t \in (0, T)\}. \tag{2.7}$$

Define admissible sets Λ and $\Theta,$ which have respectively p and q dimensions, such that $\delta \in \Lambda, \beta \in \Theta.$ Then, for each $(\delta, \beta) \in \Lambda \times \Theta,$ the boundary condition of the mixed type (2.4) is equivalently expressed as

$$\Phi(\hat{y}_0, \hat{y}(T|\delta, \beta)) = 0, \tag{2.8}$$

where $\Phi = (\Phi_1, \dots, \Phi_n)^T$ is an n -dimensional vector function. Clearly, the terminal time T depends on the vector $(\delta, \beta),$ and hence is a variable. It is assumed that there is $\hat{T} < \infty$ such that it is the upper bound of all admissible $T.$

(H5) Assume that Φ_i, Ψ , and Ψ^* are continuously differentiable.

Next, we give the cost (objective) function:

$$J_0 = \Theta_0(\hat{\mathbf{y}}(\delta)(T), \boldsymbol{\beta}) + \int_0^T L_0(\hat{\mathbf{y}}(\delta)(t)) dt, \tag{2.9}$$

where $\Theta_0 \in \mathbb{R}^n \rightarrow \mathbb{R}$ defines the terminal cost and L_0 defines the running cost. Equation (2.9) also can be called the objective function [21].

Assume that Θ_0 and L_0 satisfy the following conditions.

(H6) Θ_0 is continuously differentiable.

(H7) Functions L_0 are continuously differentiable concerning the component $\hat{\mathbf{y}}$ for each $t \in [0, \hat{T}]$. Additionally, there is a constraint $l > 0$ such that $|L_0(\hat{\mathbf{y}})| \leq l(1 + \|\hat{\mathbf{y}}\|)$ for all $\hat{\mathbf{y}}$.

Now, an OCP is formulated officially as follows:

(P_0) Subject to (2.3), seek a parameter vector $(\delta, \boldsymbol{\beta}) \in \Lambda \times \Theta$ satisfying that the objective function (2.9) is minimized over $\Lambda \times \Theta$. Here T is a period meeting conditions (2.7) and (2.8).

In particular, if the solution $\hat{\mathbf{y}}(t)$ ($t \in (0, T)$) of (2.1) and (2.2) is monotonic, then (2.7) is rewritten as

$$\hat{y}_{i0} < \hat{y}_i(t) < \hat{y}_i(T) \quad \text{or} \quad \hat{y}_i(T) < \hat{y}_i(t) < \hat{y}_{i0}, \quad i = 1, \dots, n, t \in (0, T), \tag{2.10}$$

which ensures that T is the first positive time of the solution $\hat{\mathbf{y}}(t)$ arriving at the surface $\phi(\hat{\mathbf{y}}(t), \delta) = 0$. Correspondingly, combined with (2.4), (2.10) is equivalently adapted by

$$\Psi_i(\hat{y}_i(t), \hat{y}_{i0}) < 0, \quad i = 1, \dots, n, \quad \Psi_j^*(\hat{y}_j(t), \hat{y}_j(T)) < 0, \quad j = 1, \dots, n. \tag{2.11}$$

3 Solving scheme

The variability of jump time increases the difficulty in solving the problem (P_0). To circumvent this difficulty, we choose the time-scaling transformation technology called the control parameter enhancing transform (CPET, for short). Ref. [24] firstly preferred CPET to ascertain optimal switching instants for time-optimal controls. Afterwards, we employ CPET to project the variable jump times to fixed points by an updated time scale, thus an updated optimal issue with the fixed jump times is yielded. For applying this method, we introduce the rescaled time [24, 25]

$$s = t/T. \tag{3.1}$$

Obviously, system (2.3) is rewritten as

$$\frac{d\mathbf{y}}{ds} = \mathbf{h}(\mathbf{y}, \delta, T), \tag{3.2}$$

where $s \in (0, 1)$. Here we refer to T as an organic parameter which is a decision variable. In addition,

$$\begin{aligned} \mathbf{y}(s) &= \hat{\mathbf{y}}(Ts), \\ \mathbf{h}(\mathbf{y}(s), \delta, T) &= T\mathbf{f}(\hat{\mathbf{y}}(Ts), \delta). \end{aligned}$$

Then (2.7) and (2.8) can be respectively expressed as

$$\Psi(\mathbf{y}(s), \mathbf{y}_0) < 0, \quad \Psi^*(\mathbf{y}(s), \mathbf{y}(1)) < 0, \quad s \in (0, 1), \tag{3.3}$$

and

$$\Phi(\mathbf{y}_0, \mathbf{y}(1), \boldsymbol{\beta}) = 0. \tag{3.4}$$

And the cost function (2.9) is equivalent to

$$J_1 = \Theta_0(\mathbf{y}(1), \boldsymbol{\beta}) + \int_0^1 L_0(\mathbf{y}(s), \boldsymbol{\delta}, T) ds, \tag{3.5}$$

with

$$\Theta_0(\mathbf{y}(1), \boldsymbol{\beta}) = \Theta_0(\hat{\mathbf{y}}(\boldsymbol{\delta})(T), \boldsymbol{\beta}), \quad L_0(\mathbf{y}(s), \boldsymbol{\delta}, T) = TL_0(\hat{\mathbf{y}}(\boldsymbol{\delta})(t)).$$

Thus, we can change the problem (P_0) into the following problem:

(P_1) Given system (3.2), find a combined parameter vector $(\boldsymbol{\delta}, \boldsymbol{\beta}, T) \in \Lambda \times \Theta \times (0, \hat{T})$ to minimize the objective functional (3.5) and meanwhile satisfy (3.3) and (3.4).

By Theorems in [25] and [26], the next result is valid.

Lemma 3.1 *The OCP (P_0) \Leftrightarrow the problem (P_1).*

Next, we recommend an exact penalty method to overcome the remaining difficulty that the constraints (3.3) define a disjoint feasible region. Such constraints are referred to as functional inequality or path constraints. The essential dilemma about these constraints lies in the innumerable restriction on the state variables in the time scale [21].

Constraint (3.3) is a non-standard “open” state constraint. So we can approximate it as follows:

$$\Psi(\mathbf{y}(s), \mathbf{y}_0) \leq \bar{\varepsilon}, \quad \Psi^*(\mathbf{y}(s), \mathbf{y}(1)) \leq \bar{\varepsilon}, \quad s \in [\delta, 1 - \delta], \tag{3.6}$$

where $\bar{\varepsilon} > 0$ and $\delta \in (0, \frac{1}{2})$ are adjustable parameters.

Then, we define a constraint violation function as

$$\begin{aligned} \Xi(\boldsymbol{\delta}, \boldsymbol{\beta}, T) &= [\Phi(\mathbf{y}_0, \mathbf{y}(1), \boldsymbol{\beta})]^2 \\ &+ T \int_0^1 \{ [\max\{\bar{\varepsilon}, \Psi(\mathbf{y}(s), \mathbf{y}_0)\}]^2 + [\max\{\bar{\varepsilon}, \Psi^*(\mathbf{y}(s), \mathbf{y}(1))\}]^2 \} ds. \end{aligned} \tag{3.7}$$

Note that $\Xi(\boldsymbol{\delta}, \boldsymbol{\beta}, T) = 0$ if and only if (3.3) and (3.4) hold. By the strategy presented in [27–30], one sets up an exact penalty function

$$J_2(\boldsymbol{\delta}, \boldsymbol{\beta}, T, \varepsilon) = \begin{cases} J_1(\boldsymbol{\delta}, \boldsymbol{\beta}, T), & \text{if } \varepsilon = 0 \text{ and } \Xi(\boldsymbol{\delta}, \boldsymbol{\beta}, T) = 0, \\ J_1(\boldsymbol{\delta}, \boldsymbol{\beta}, T) + \varepsilon^{-\alpha} \Xi(\boldsymbol{\delta}, \boldsymbol{\beta}, T) + \sigma \varepsilon^\gamma, & \text{if } \varepsilon > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\sigma > 0$ is a positive penalty parameter. $\alpha > 0$ and $\gamma > 0$ are constants meeting $1 \leq \gamma \leq \alpha$. The new decision variable ε satisfies

$$0 \leq \varepsilon \leq \varepsilon_1, \tag{3.8}$$

where ε_1 is a small positive number. This method was first mentioned in [31].

Now we give the unconstrained control problem:

(P_2) Optimize a combined parameter vector $(\delta, \beta, T) \in \Lambda \times \Theta \times (0, \hat{T})$ and the new decision variable $\varepsilon \in [0, \varepsilon_1]$ to minimize the transformed equivalent cost function $J_2(\delta, \beta, T, \varepsilon)$ subject to the dynamics given by (3.1) and (3.2) in the interval $(0, 1)$.

According to the main convergence result of [29], when $\bar{\varepsilon}$ and δ approach zero, the cost of J_2 approaches the optimal cost J_1 of problem (P_2).

Theorem 3.1 *Let $\iota > 0$ be an arbitrary fixed number. For any enough small $\delta > 0$, we can find a corresponding positive point $\bar{\varepsilon}_1(\delta) > 0$ such that*

$$|J_2 - J_1| < \iota, \quad \bar{\varepsilon} \in (0, \bar{\varepsilon}_1].$$

The above approximate problem is a nonlinear optimization one. For minimizing the objective function which subjects to a group of constraints, the narrow decision variables are selected. And for the decision vector, the cost and objective functions are implicit in problem (P_2). Then we can develop their gradients to produce search directions which guide profitability for the search space [21]. For implementing these algorithms, it is essential to compute the partial derivatives of the final cost function. A method for computing gradients is the so-called costate method. From Theorem 4.1 in [32] and Section 5.2 in [33], we define the corresponding Hamiltonian function

$$\begin{aligned} H(s, \mathbf{y}(s), \mathbf{y}(1), \lambda(s), \delta, T, \varepsilon) &= L_0(\mathbf{y}(s), \delta, \beta, T) + \lambda^T \mathbf{h}(\mathbf{y}, \delta, T) \\ &+ \varepsilon^{-\nu} T \{ [\max\{\bar{\varepsilon}, \Psi(\mathbf{y}(s), \mathbf{y}_0)\}]^2 \\ &+ [\max\{\bar{\varepsilon}, \Psi^*(\mathbf{y}(s), \mathbf{y}(1))\}]^2 \}, \end{aligned} \tag{3.9}$$

where $\lambda^T(s) = (\lambda_1(s), \dots, \lambda_n(s))$ and $\lambda_i(s)$ is the corresponding costate for $i = 1, 2, \dots, n$. Furthermore, $\lambda(s)$ is determined by the following differential equations:

$$\frac{d\lambda}{ds} = - \left(\frac{\partial H(s, \mathbf{y}(s), \mathbf{y}(1), \lambda(s), \delta, T, \varepsilon)}{\partial \mathbf{y}} \right)^T, \tag{3.10}$$

$$\lambda^T(1) = \frac{\partial \Theta_0}{\partial \mathbf{y}(1)} + 2\varepsilon^{-\alpha} \Phi(\mathbf{y}_0, \mathbf{y}(1), \beta)^2 \frac{\partial \Phi}{\partial \mathbf{y}(1)} + \int_0^1 \frac{\partial H}{\partial \mathbf{y}(1)} ds. \tag{3.11}$$

Theorem 3.2 *The gradients of J_2 concerning T, δ, β as well as ε are awarded by*

$$\frac{\partial J_2}{\partial T} = \int_0^1 \frac{\partial H}{\partial T} ds, \tag{3.12}$$

$$\frac{\partial J_2}{\partial \delta} = \int_0^1 \frac{\partial H}{\partial \delta} ds, \tag{3.13}$$

$$\frac{\partial J_2}{\partial \beta} = \frac{\partial \Theta_0}{\partial \beta} + 2\varepsilon^{-\alpha} \Phi(\mathbf{y}_0, \mathbf{y}(1), \beta) \frac{\partial \Phi}{\partial \beta}, \tag{3.14}$$

$$\frac{\partial J_2}{\partial \varepsilon} = -\alpha \varepsilon^{-\alpha-1} \Xi(\delta, \beta, T) + \gamma \sigma \varepsilon^{\gamma-1}. \tag{3.15}$$

Note that, instead of an initial condition, the costate systems (3.10) and (3.11) involve a terminal value. So, we must integrate them from $s = 1$ to $s = 0$. Furthermore, in view of equations (3.12)–(3.15), we address the algorithm about calculating J_2 and its gradients as follows.

Algorithm 1 Input a group $(\delta, \beta, T) \in \Lambda \times \Theta \times (0, \hat{T})$,

- (i) Solve systems (3.2), (3.10), and (3.11) to obtain $\mathbf{y}(s)$ and λ .
- (ii) Use $\mathbf{y}(s)$ to compute J_2 .
- (iii) Use $\mathbf{y}(s)$ and λ to compute $\frac{\partial J_2}{\partial T}$, $\frac{\partial J_2}{\partial \delta}$, and $\frac{\partial J_2}{\partial \beta}$ according to equations (3.12), (3.13), and (3.14).

In the above, the methodology proposed involves transforming the periodic optimal control problem into a standard optimal control problem, after which standard computational techniques can be applied. Similarly, the case of the first positive time T assured by (2.11) can be derived.

4 Application

In this section, three examples are given to implement the above theories and approaches; and furthermore, to verify the validity of our algorithm.

Example 4.1 (Phytoplankton–fish system) Consider the following impulsive system [11]:

$$\left\{ \begin{array}{l} \frac{dp}{dt} = (r - az)p, \\ \frac{dz}{dt} = (bp - u - \frac{dp}{\gamma + p})z, \end{array} \right\} \quad z < H, \tag{4.1}$$

$$\left\{ \begin{array}{l} \Delta p = p(T^+) - p(T) = -e_1 p, \\ \Delta z = z(T^+) - z(T) = -e_2 z, \end{array} \right\} \quad z = H,$$

$$p(0) = p_0 \geq 0, \quad z(0) = z_0 \geq 0.$$

Here p_0 and z_0 present respectively the initial levels of phytoplankton and fish. From Theorems 3.1 and 4.3 in [11], it is obtained that for fixed (p_0, z_0) system (4.1) has an OOPS $\Gamma_{A \rightarrow B}$ from $A((1 - e_1)p_1, (1 - e_2)H)$ to $B(e_1, H)$. Then Zhao et al. [11] formulated an OCP and strived to seek the appropriate harvesting rates e_1^* and e_2^* to maximize the cost function $J(e_1, e_2) = C_1 e_1 p_1 + C_2 e_2 H$ in an impulsive period. C_1 and C_2 describe the prices per unit biomass of the phytoplankton and fish, respectively.

In our paper, based on the periodic solution theory in [11], we know that the resources are exploited in a period mode. Then, what strategies are implemented to optimize the cost function at the minimal cost? For this, we take the harvesting rates e_1 , e_2 and the harvest period T as control parameters to achieve the maximal revenue, namely

$$\min_{e_1, e_2, T} \{J_1(e_1, e_2)\} = \min_{e_1, e_2, T} \{-C_1 e_1 p_1 - C_2 e_2 H\}.$$

Combined with the periodicity of harvesting, the solution $(p(t), z(t))$ of system (4.1) on $(0, T]$ meets the following conditions:

$$\begin{cases} \frac{dp}{dt} = (r - az)p, \\ \frac{dz}{dt} = (bp - u - \frac{dp}{\gamma+p})z, \end{cases} \quad t \in (0, T), \tag{4.2}$$

$$p(T) = \frac{p_0}{1 - e_1}, \quad z(T) = H = \frac{z_0}{1 - e_2}. \tag{4.3}$$

Furthermore, together with the monotonicity of $z(t)$, it is obtained

$$z(0) < z(t) < z(T), \tag{4.4}$$

where T also is the first positive time such that (4.3) and (4.4) hold.

After the rescaled time transformation, (4.2), (4.3), and (4.4) can be rewritten as follows:

$$\begin{cases} \frac{dp}{ds} = T(r - az)p \doteq f_1, \\ \frac{dz}{ds} = T(bp - u - \frac{dp}{\gamma+p})z \doteq f_2, \end{cases} \quad \text{for } s \in (0, 1), \tag{4.5}$$

$$p(1) = \frac{p_0}{1 - e_1}, \quad z(1) = H = \frac{z_0}{1 - e_2}, \tag{4.6}$$

$$z(0) < z(s) < z(1). \tag{4.7}$$

Our cost function can be expressed as follows: subject to (4.5)–(4.7),

$$\min_{e_1, e_2, T} \{J_2(e_1, e_2)\} = \min_{e_1, e_2, T} \{-C_1 e_1 p(1) - C_2 e_2 z(1)\}.$$

For system (4.5)–(4.7), define the violent function by

$$\begin{aligned} \Xi(e_1, e_2, T) &= [(1 - e_1)p(1) - p_0]^2 + [(1 - e_2)z(1) - z_0]^2 \\ &\quad + T \int_0^1 \{[\max\{\bar{\varepsilon}, z(0) - z(s)\}]^2 + [\max\{\bar{\varepsilon}, z(s) - z(1)\}]^2\}. \end{aligned} \tag{4.8}$$

Noting that $\Xi(e_1, e_2, T) = 0$ if and only if constraints (4.6) and (4.7) are satisfied, then our cost function $J_2(e_1, e_2)$ turns into

$$J_3(e_1, e_2) = -C_1 e_1 p(1) - C_2 e_2 z(1) + \varepsilon^{-\nu} \Xi(e_1, e_2, T) + \sigma \varepsilon^w. \tag{4.9}$$

After that, the corresponding Hamiltonian function is

$$\begin{aligned} H(s, p(s), z(s), \lambda_1(s), \lambda_2(s), T, \varepsilon) &= \lambda_1 f_1 + \lambda_2 f_2 \\ &\quad + T [\max\{\bar{\varepsilon}, z(0) - z(s)\}]^2 \\ &\quad + T [\max\{\bar{\varepsilon}, z(s) - z(1)\}]^2, \end{aligned} \tag{4.10}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are determined by the auxiliary system:

$$\begin{cases} \dot{\lambda}_1(s) = -T[\lambda_1(r - az) + \lambda_2z(b - \frac{dy}{(\gamma+p)^2})], \\ \dot{\lambda}_2(s) = -T[-\lambda_1ap + \lambda_2(bp - u - \frac{dp}{\gamma+p}) \\ \quad - 2\varepsilon^{-\nu}T \max\{\bar{\varepsilon}, z(0) - z(s)\} + 2\varepsilon^{-\nu}T \max\{\bar{\varepsilon}, z(s) - z(1)\}], \\ \lambda_1(1) = -C_1e_1 + 2\varepsilon^{-\nu}((1 - e_1)p(1) - p_0)(1 - e_1), \\ \lambda_2(1) = -C_2e_2 + 2\varepsilon^{-\nu}((1 - e_2)z(1) - z_0)(1 - e_2) \\ \quad - \int_0^1 2\varepsilon^{-\nu}T \max\{\bar{\varepsilon}, z(s) - z(1)\} ds. \end{cases} \tag{4.11}$$

The gradients of (4.9) with respect to T , e_1 , e_2 , and ε are addressed as follows:

$$\begin{aligned} \frac{\partial J_3}{\partial T} = \int_0^1 \left\{ \lambda_1(r - az)p + \lambda_2 \left(bp - u - \frac{dp}{\gamma + p} \right) z \right. \\ \left. + \varepsilon^{-\nu} [\max\{\bar{\varepsilon}, z(0) - z(s)\}]^2 + \varepsilon^{-\nu} [\max\{\bar{\varepsilon}, z(s) - z(1)\}]^2 \right\} ds, \end{aligned} \tag{4.12}$$

$$\frac{\partial J_3}{\partial e_1} = -C_1p_1 - 2p_1\varepsilon^{-\nu}[(1 - e_1)p_1 - p_0], \tag{4.13}$$

$$\frac{\partial J_3}{\partial e_2} = -C_2H - 2z_1\varepsilon^{-\nu}[(1 - e_2)z_1 - z_0], \tag{4.14}$$

$$\frac{\partial J_3}{\partial \varepsilon} = -\nu\varepsilon^{-\nu-1} \Xi(e_1, e_2, T) + w\sigma\varepsilon^{w-1}. \tag{4.15}$$

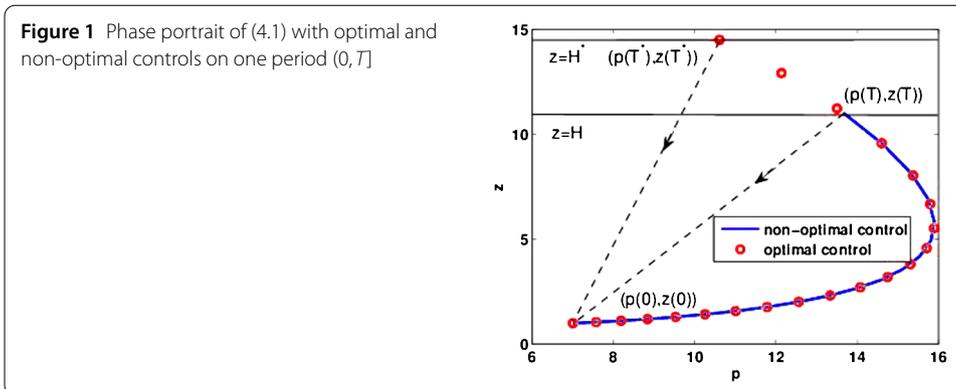
Next, we give the simulation of Example 4.1. Take e_1 , e_2 , T , and ε as control parameters. The parameters are chosen as

$$\begin{aligned} r = 1.144, \quad a = 0.2, \quad b = 0.2, \quad d = 0.5, \quad \gamma = 2, \quad u = 0.5, \\ \nu = 2, \quad w = 1.55, \quad \bar{\varepsilon} = -1e-8, \quad \sigma = 100, \quad C_1 = 3, \quad C_2 = 2, \end{aligned} \tag{4.16}$$

with the initial values $p_0 = 7$, $z_0 = 1$. For the initial guesses $T_0 = 3$, $e_{10} = 0.7$, $e_{20} = 0.9$, and $\varepsilon_0 = 0.1$, we obtain that the cost function $J_0 = 1634.15$ and the threshold $h_0 = S(T_0) = 13.64$. Furthermore, by the costate equations and transversality conditions in (4.5) as well as the gradients (4.12)–(4.15), starting with the above initial values, we recover the optimal control scheme showed in Table 1. Apparently, the harvest period T is extended, harvesting rates e_1 and e_2 are enlarged, and our final gains are also increased after our optimal control. Additionally, we adapt this set of data to plot the phase diagram of food and species according to the optimal and initial controls, respectively (see Figure 1). The red circle

Table 1 Results of simulation 1

Set	Non-optimal control	Optimal control
1	$T_0 = 1.5, e_{10} = 0.7$ $e_{20} = 0.9, \varepsilon_0 = 0.1$ $J_3 = 7931.6, H_0 = 11.23$	$T^* = 1.6789, e_1^* = 0.6823$ $e_2^* = 0.9075, \varepsilon^* = 0.1$ $J_3^* = 8140.94, H^* = 14.49$
2	$T_0 = 1.5, e_{10} = 0.4$ $e_{20} = 0.8, \varepsilon_0 = 0.1$ $J_3 = 6291.7, H_0 = 11$	$T^* = 1.8038, e_1^* = 0.4285$ $e_2^* = 0.8429, \varepsilon^* = 0.1$ $J_3^* = 7592.98, H^* = 16.44$
3	$T_0 = 1, e_{10} = 0.7$ $e_{20} = 0.9, \varepsilon_0 = 0.1$ $J_3 = 5333.51, H_0 = 13.64$	$T^* = 1.8364, e_1^* = 0.7127$ $e_2^* = 0.9665, \varepsilon^* = 0.1$ $J_3^* = 8041.65, H^* = 16.84$



represents the trajectories of populations on the basis of the optimal scheme, whereas the blue solid line represents the trajectory under the initial scheme. From Figure 1, we also notice that the threshold of plankton also increases, which is consistent with the results in Table 1. Summarily, our optimal tactics not only delay the harvest but also magnify the harvesting threshold and benefit, which are desirable for human exploitation.

Example 4.2 (Computer virus propagation under media coverage) In this section, a new computer virus model with state impulsive control [8, 34, 35] is utilized to exemplify our algorithm and approach:

$$\begin{cases} \left. \begin{aligned} \frac{ds}{dt} &= g - ds + ni - (b_1 - b_2 \frac{i}{m+i})si, \\ \frac{di}{dt} &= (b_1 - b_2 \frac{i}{m+i})si - ni - di, \end{aligned} \right\} & i < H, \\ \left. \begin{aligned} \Delta s &= -e_1s, \\ \Delta i &= -e_2i, \end{aligned} \right\} & i = H, \end{cases} \tag{4.17}$$

where n denotes the recovery rate from an infected computer to a susceptible one due to the application of antivirus software. b_1 denotes the contact rate at which the susceptible computer gets infected before media alert, b_2 is the maximum reduction of contact rate through media coverage, and $m > 0$ represents the effect of media coverage.

For (4.17), the existence together with stability of OPS have been showed in [8]. Then, in the following, we analogously formulate an optimal state pulse control problem of system (4.17) on a period $(0, T]$:

$$\begin{cases} \left. \begin{aligned} \frac{ds}{dt} &= g - ds + ni - (b_1 - b_2 \frac{i}{m+i})si, \\ \frac{di}{dt} &= (b_1 - b_2 \frac{i}{m+i})si - ni - di, \end{aligned} \right\} & t \in (0, T), \end{cases} \tag{4.18}$$

where $s(t)$ and $i(t)$ satisfy the inequality constraint

$$i(t) < i(T), \tag{4.19}$$

and equality constraints

$$s(T) = \frac{s_0}{1 - e_1}, \quad i(T) = \frac{i_0}{1 - e_2}. \tag{4.20}$$

Here T is also the first positive time such that (4.20) holds.

Next we build the cost function. Namely, subject to (4.18)–(4.20), find the appropriate $b_2, T, e_1,$ and e_2 to minimize the objective function

$$J_0(b_2, T, e_1, e_2) = p_1 e_1 s(T) + p_2 e_2 i(T) - \omega T + \int_0^T i(t) dt, \tag{4.21}$$

where p_1 and p_2 denote the cost of susceptible and infected computers and ω is the weight factor.

Utilizing the time scale transformation and the violent violation function, the optimal solution of the above control problem is determined by the following system:

$$\left\{ \begin{aligned} \frac{ds}{dx} &= T[g - ds + ni - (b_1 - b_2 \frac{i}{m+i})si], \\ \frac{di}{dx} &= T[(b_1 - b_2 \frac{i}{m+i})si - ni - di], \end{aligned} \right\} \quad x \in (0, 1), \tag{4.22}$$

$$\left\{ \begin{aligned} \dot{\lambda}_1(x) &= -T[\lambda_1(-d - (b_1 - b_2 \frac{i}{m+i})i) + \lambda_2(b_1 - b_2 \frac{i}{m+i})i], \\ \dot{\lambda}_2(x) &= -2\varepsilon^{-\nu} T \max\{\bar{\varepsilon}, i(s) - i(1)\} - T[\lambda_1(n - (b_1 - b_2 \frac{i}{m+i}))s - si(-\frac{m}{(m+i)^2})] \\ &\quad - \lambda_2 T[(b_1 - b_2 \frac{i}{m+i})s + si(-\frac{m}{(m+i)^2}) - n - d] - T, \\ \lambda_1(1) &= p_1 e_1 + 2\varepsilon^{-\nu}((1 - e_1)s(1) - s_0)(1 - e_1), \\ \lambda_2(1) &= p_2 e_2 + 2\varepsilon^{-\nu}((1 - e_2)i(1) - i_0)(1 - e_2) - 2 \int_0^1 \varepsilon^{-\nu} T \max\{\bar{\varepsilon}, i(x) - i(1)\} dx, \end{aligned} \right. \tag{4.23}$$

and the derivatives of J_1 on $T, b_2, e_1, e_2,$ and ε are administrated by

$$\begin{aligned} \frac{\partial J_1}{\partial T} &= \int_0^1 \left\{ \lambda_1 \left(g - ds + ni - \left(b_1 - b_2 \frac{i}{m+i} \right) \right) + \lambda_2 \left(b_1 - b_2 \frac{i}{m+i} \right) si - ni - di \right. \\ &\quad \left. + \varepsilon^{-\nu} [\max\{\bar{\varepsilon}, i(x) - i(1)\}]^2 + (I - \omega) \right\} dx, \end{aligned} \tag{4.24}$$

$$\frac{\partial J_1}{\partial b_2} = \int_0^1 \left[(\lambda_1 - \lambda_2) T \frac{si^2}{m+i} \right] dx, \tag{4.25}$$

$$\frac{\partial J_1}{\partial e_1} = -p_1 s(1) - 2s(1)\varepsilon^{-\nu}[(1 - e_1)s(1) - s_0], \tag{4.26}$$

$$\frac{\partial J_1}{\partial e_2} = -p_2 H - 2i(1)\varepsilon^{-\nu}[(1 - e_2)i(1) - i_0], \tag{4.27}$$

$$\frac{\partial J_1}{\partial \varepsilon} = -\nu \varepsilon^{-\nu-1} \Xi(e_1, e_2, T) + w \sigma \varepsilon^{w-1}, \tag{4.28}$$

where

$$\begin{aligned} \Xi(e_1, e_2, T) &= ((1 - e_1)s(1) - s(0))^2 + ((1 - e_2)i(1) - i(0))^2 \\ &\quad + T \int_0^1 [\max\{\bar{\varepsilon}, i(x) - i(1)\}]^2 dx, \end{aligned} \tag{4.29}$$

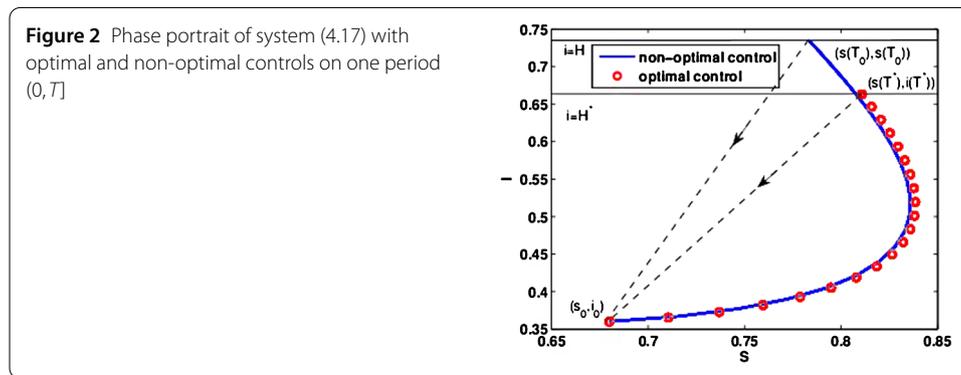
$$J_1 = p_1 e_1 s(1) + p_2 e_2 i(1) + \int_0^1 (-\omega T + Ti(x)) dx + \varepsilon^{-\nu} \Xi(e_1, e_2, T) + \sigma \varepsilon^w. \tag{4.30}$$

Next, we give the simulation of Example 4.2. In system (4.17), a set of parameter values is chosen as

$$\begin{aligned} g &= 0.3, & d &= 0.18, & n &= 0.18, & b_1 &= 0.65, & m &= 3, & \nu &= 2, \\ w &= 1.55, & \bar{\varepsilon} &= -1e-8, & \sigma &= 100, & p_1 &= 1, & p_2 &= 2, & \omega &= 0.5 \end{aligned} \tag{4.31}$$

Table 2 Results of simulation 2

Set	Non-optimal control	Optimal control
1	$T = 8, b_2 = 0.6, \epsilon_0 = 0.1$ $e_1 = 0.8, e_2 = 0.9$ $J_1 = 69.4267, H = 0.7347$	$T^* = 6.5499, b_2^* = 0.6214, \epsilon^* = 4.521e-5$ $e_1^* = 0.1613, e_2^* = 0.4572$ $J^* = 1.36176, H^* = 0.6632$
2	$T = 7, b_2 = 0.55, \epsilon_0 = 0.1$ $e_1 = 0.7, e_2 = 0.6$ $J_1 = 53.4043, H = 0.7005$	$T^* = 6.1392, b_2^* = 0.6345, \epsilon^* = 1.61e-6$ $e_1^* = 0.1703, e_2^* = 0.4371$ $J^* = 0.7111, H^* = 0.6396$
3	$T = 6, b_2 = 0.5, \epsilon_0 = 0.1$ $e_1 = 0.2, e_2 = 0.4$ $J_1 = 32.5521, H = 0.6578$	$T^* = 5.9094, b_2^* = 0.5386, \epsilon^* = 0.441e-7$ $e_1^* = 0.1553, e_2^* = 0.4422$ $J^* = 0.6984, H^* = 0.6454$



with the initial value $s_0 = 0.68$ and $i_0 = 0.36$. In Table 2, we give three sets of parameter initial values to compute the minimal cost function J^* and to find the appropriate parameters. From Table 2, we find that after optimal control the contact rate b_2 through media coverage is increased, while the period T is shortened, the number of the infected computers is maintained at a low level H^* and the cost is reduced. That is, increasing the influence of media and taking regular anti-virus measures for a computer will reduce the cost of prevention and control of computer viruses. Furthermore, we display the dynamic behavior of susceptible and infected computers and the optimal threshold H^* in Figure 2 with the first set of Table 2. All of the red cycles illustrate the behaviors of susceptible and infected computers when optimal control action is taken, while all of the blue solid lines present the habit of susceptible and infected computers when non-optimal control is taken.

Example 4.3 (Species-food system) After our little sojourn in the simple examples, it is time to return to a complex example. Consider the following system in [36]:

$$\left\{ \begin{array}{l} \frac{dx}{dt} = -\gamma xy, \\ \frac{dy}{dt} = -y(\epsilon - \delta x), \\ \Delta x(\tau_k) = \lambda, \end{array} \right\} \quad \text{if } x \neq x_1, \tag{4.32}$$

$$\left\{ \begin{array}{l} \Delta y(\tau_k) = \begin{cases} 0, & \text{if } k \text{ is not divisible by } n, \\ -\alpha y(\tau_k), & \text{if } k \text{ is divisible by } n, \end{cases} \end{array} \right\} \quad \text{if } x = x_1,$$

$x(t)$ and $y(t)$ denote the absolute or relative quantities of the food A and the species B at moment t . Let $n > 0$ be an integer and assume that the quantity of food increases by λ unit each impulse effect, while the population of the species decreases by jumps only at

the moments of the impulse effect τ_k whose ordinal number k is a multiple of n . We also assumed that $\lambda > 0$ and $0 < \alpha < 1$.

Based on the research of the period solution in [36], we propose our optimal problem on the condition that (4.32) admits a periodic solution with period T . According to the theory in Section 2, we can rewrite system (4.32) on one period $(0, T)$ as follows:

$$\left\{ \begin{array}{l} \frac{dx}{dt} = -\gamma xy, \\ \frac{dy}{dt} = -y(\epsilon - \delta x), \end{array} \right\} \quad \text{if } t \neq \tau_k, k = 1, 2, \dots, n, k \in \mathbb{Z}_+, \tau_k \in (0, T],$$

$$\left\{ \begin{array}{l} \Delta x = \lambda, \\ \Delta y = 0, \end{array} \right\} \quad \text{if } t = \tau_k, k = 1, 2, \dots, n - 1,$$

$$\left\{ \begin{array}{l} \Delta x = \lambda, \\ \Delta y = -\alpha y, \end{array} \right\} \quad \text{if } t = \tau_n,$$
(4.33)

where n is a positive integer representing the number of releasing food. Furthermore, the populations of the food and the species meet the following restraints on $(0, T]$:

$$\left\{ \begin{array}{l} x(\tau_{k-1}^+) > x(t) > x(\tau_k), \quad k = 1, 2, \dots, n, \\ y(\tau_{k-1}^+) < y(t), \quad k = 1, 2, \dots, n, \end{array} \right.$$
(4.34)

$$x(\tau_k) + \lambda = x_0, \quad k = 1, 2, \dots, n, \quad y(T) = y(\tau_n) = \frac{y_0}{1 - \alpha},$$
(4.35)

where τ_n is also the first positive time such that (4.35) holds. Thus, our OCP can be written as follows.

Problem (P_0) : In order to minimize the cost of the total releasing food and maximize the quality of species at the terminal time, we would like to find λ and α to minimize the cost function

$$J_0^n = p_1 n \lambda - p_2 \alpha y(T),$$

where p_1 and p_2 denote the market prices of the food and species.

First, the time scaling transformation is utilized to project the impulsive moments $t = 0, \tau_1, \tau_2, \dots, \tau_n$ into $s = 0, 1, 2, \dots, n$ in a new time horizon. Define the required transformation:

$$\frac{dt(s)}{ds} = v(s)$$
(4.36)

with $t(0) = 0$. $v(s)$ is the time scaling control function and is discontinuous at $s = 1, 2, \dots, n$. That is,

$$v(s) = \sum_{k=1}^n \bar{\tau}_k \chi_{(k-1, k)}(s),$$
(4.37)

where $\bar{\tau}_k = \tau_k - \tau_{k-1}$ is the duration which satisfies the following condition:

$$\sum_{k=1}^n \bar{\tau}_k = T.$$
(4.38)

Define the indicator function of I by

$$\chi_I(s) = \begin{cases} 1, & \text{if } s \in I, \\ 0, & \text{otherwise.} \end{cases} \tag{4.39}$$

By time scaling transformation, systems (4.33)–(4.35) turn into

$$\left\{ \begin{array}{l} \frac{dx}{ds} = v(s)(-\gamma xy), \\ \frac{dy}{ds} = v(s)[-y(\epsilon - \delta x)], \end{array} \right\} \quad s \in (0, n),$$

$$\left\{ \begin{array}{l} \Delta x = \lambda, \\ \Delta y = 0, \end{array} \right\} \quad s = 1, 2, \dots, n - 1, \tag{4.40}$$

$$\left\{ \begin{array}{l} \Delta x = \lambda, \\ \Delta y = -\alpha y, \end{array} \right\} \quad s = n,$$

$$\begin{cases} x((k - 1)^+) > x(s) > x(k), & k = 1, 2, \dots, n, \\ y((k - 1)^+) < y(s), & k = 1, 2, \dots, n, \end{cases} \tag{4.41}$$

$$x(k) + \lambda = x_0, \quad k = 1, 2, \dots, n, \quad y(n) = y(T) = \frac{y_0}{1 - \alpha}. \tag{4.42}$$

Thus we can change problem (P_0) into the next problem.

Problem (P_1) : Minimize the transformed cost function

$$J_1^n = p_1 n \lambda - p_2 \alpha y(n)$$

by selecting λ and α .

This optimal problem has multi-jump times, which differs from the first two examples and is difficult to solve. For this, time translation transformation will be introduced [37].

Define

$$x_i(s) = x(s + i - 1), \quad y_i(s) = y(s + i - 1), \quad \tau_i(s) = t(s + i - 1), \quad s = 1, 2, \dots, n. \tag{4.43}$$

Then (4.40) becomes

$$\left\{ \begin{array}{l} \frac{dx_i}{ds} = \bar{\tau}_i(-\gamma x_i y_i) \doteq f_1^i, \\ \frac{dy_i}{ds} = \bar{\tau}_i[-y_i(\epsilon - \delta x_i)] \doteq f_2^i, \\ \frac{d\tau_i(s)}{ds} = \bar{\tau}_i. \end{array} \right. \quad s \in (0, 1), \text{ and } i = 1, 2, \dots, n, \tag{4.44}$$

with the initial condition $x_1(0) = x_0$ and $y_1(0) = y_0$. Then constraints (4.41) and (4.42) are rewritten by

$$\begin{cases} x_i(0) > x_i(s) > x_i(1), & i = 1, 2, \dots, n, s \in (0, 1), \\ y_i(0) < y_i(s), & i = 1, 2, \dots, n, \end{cases} \tag{4.45}$$

$$x_i(1) + \lambda = x_0, \quad k = 1, 2, \dots, n, \quad y_n(1) = \frac{y_0}{1 - \alpha}. \tag{4.46}$$

Further, the cost function J_1^n is rewritten as follows:

$$J_2^n = p_1 n \lambda - p_2 \alpha y_n(1).$$

In view of (3.7), we define a constraint violation function

$$\begin{aligned} \Xi(\lambda, \alpha, \bar{\tau}) = & \sum_{i=1}^n [x_i(1) - x_0 + \lambda]^2 + [(1 - \alpha)y_n(1) - y_0]^2 \\ & + \int_0^1 \{ [\max\{x_i(s) - x_0, \bar{\varepsilon}\}]^2 + [\max\{x_i(1) - x_i(s), \bar{\varepsilon}\}]^2 \\ & + [\max\{y_i(s) - y_i(0), \bar{\varepsilon}\}]^2 \} ds. \end{aligned} \tag{4.47}$$

Denote $\bar{\tau} = (\bar{\tau}_1, \bar{\tau}_2, \dots, \bar{\tau}_n)$. Then problem (P_2) is given as follows:

Optimize parameters λ, α , the vector $\bar{\tau}$ together with the decision variable $\varepsilon \in [0, \varepsilon_1]$ to minimize the transformed equivalent cost function

$$J_3^n = p_1 n \lambda - p_2 \alpha y_n(1) + \varepsilon^{-a} \Xi + \sigma \varepsilon^b$$

subject to (4.44)–(4.46) as well as the restraints $0 < \alpha < 1$ and $\lambda > 0$.

Next, according to Theorem 4.1 in [38] and [32], we define the corresponding Hamiltonian function

$$\begin{aligned} H_i = & \varepsilon^{-a} \bar{\tau}_i \{ [\max\{x_i(s) - x_i(0), \bar{\varepsilon}\}]^2 + [\max\{x_i(1) - x_i(s), \bar{\varepsilon}\}]^2 \\ & + [\max\{y_i(0) - y_i(s), \bar{\varepsilon}\}]^2 \} + l_1^i f_1^i + l_2^i f_2^i \quad \text{for } i = 1, 2, \dots, n, \end{aligned} \tag{4.48}$$

where l_1^i and l_2^i are determined by the auxiliary system:

$$\left\{ \begin{aligned} \left. \begin{aligned} \frac{dl_1^i}{ds} &= -\bar{\tau}_i \{ 2\varepsilon^{-a} [\max\{x_i(s) - x_i(0), \bar{\varepsilon}\} - \max\{x_i(1) - x_i(s), \bar{\varepsilon}\}] \\ &\quad - l_1^i \gamma y_i + l_2^i \delta y_i \}, \\ \frac{dl_2^i}{ds} &= -\bar{\tau}_i \{ -l_1^i \gamma x_i - l_2^i \varepsilon + l_2^i \delta x_i \\ &\quad - 2\varepsilon^{-a} [\max\{y_i(0) - y_i(s), \bar{\varepsilon}\}]^2 \}, \end{aligned} \right\} & i = 1, \dots, n, \\ \left. \begin{aligned} l_1^i(1) &= 2\varepsilon^{-a} \{ (x_i(1) - x_0 + \lambda) + \int_0^1 \bar{\tau}_i \max\{x_i(1) - x_i(s), \bar{\varepsilon}\} ds \} \\ &\quad + l_1^{i+1}(0), \\ l_2^i(1) &= l_2^{i+1}(0), \end{aligned} \right\} & i = 1, \dots, n - 1, \\ \left. \begin{aligned} l_1^n(1) &= 2\varepsilon^{-a} \{ (x_n(1) - x_0 + \lambda) + \int_0^1 \bar{\tau}_n \max\{x_n(1) - x_n(s), \bar{\varepsilon}\} ds \}, \\ l_2^n(1) &= -p_2 \alpha + 2\varepsilon^{-a} (1 - \alpha) ((1 - \alpha)y_n(1) - y_0), \end{aligned} \right\} & i = n. \end{aligned} \right. \tag{4.49}$$

Now we give the derivatives of J_3^n on $\bar{\tau}, \lambda, \alpha$ as well as ε by

$$\begin{aligned} \nabla_{\bar{\tau}} J_3^n = & \sum_{i=1}^n \int_0^1 \{ \varepsilon^{-a} \{ [\max\{x_i(s) - x_0, \bar{\varepsilon}\}]^2 + [\max\{x_i(1) - x_i(s), \bar{\varepsilon}\}]^2 \\ & + [\max\{y_i(0) - y_i(s), \bar{\varepsilon}\}]^2 \} - l_1^i \gamma x_i y_i - l_2^i [y_i(\varepsilon - \delta x_i)] \} ds, \end{aligned} \tag{4.50}$$

$$\nabla_{\lambda} J_3^n = p_1 n + 2 \sum_{i=1}^n \varepsilon^{-a} (x_i(1) - x_0 + \lambda) + \sum_{i=2}^n l_1^i(0), \tag{4.51}$$

$$\nabla_{\alpha} J_3^n = -p_2 y_n(1) - 2\varepsilon^{-a} y_n(1) ((1 - \alpha)y_n(1) - y_0), \tag{4.52}$$

$$\nabla_{\varepsilon} J_3^n = -a \varepsilon^{-a-1} \Xi(\lambda, \alpha, \bar{\tau}) + b \sigma \varepsilon^{b-1}. \tag{4.53}$$

Table 3 Results of simulation for $n = 5$

Set	Non-optimal control	Optimal control
1	$\tau_1 = 3.23, \tau_2 = 1.98, \tau_3 = 1.44$ $\tau_4 = 1.13, \tau_5 = 0.94$ $\lambda = 4.51, \alpha = 0.8773, \varepsilon_0 = 0.1$ $J_3^5 = 18.2467, x_1 = 0.49$	$\tau_1^* = 2.99, \tau_2^* = 1.77, \tau_3^* = 1.26$ $\tau_4^* = 0.98, \tau_5^* = 0.81$ $\lambda^* = 4.4, \alpha^* = 0.89, \varepsilon^* = 6.581e-5$ $J_3^{5*} = 14.1559, x_1^* = 0.6$
2	$\tau_1 = 2.35, \tau_2 = 1.27, \tau_3 = 0.88$ $\tau_4 = 0.67, \tau_5 = 0.54$ $\lambda = 4, \alpha = 0.9028, \varepsilon_0 = 0.1$ $J_3^5 = 13.5705, x_1 = 1$	$\tau_1^* = 2.34, \tau_2^* = 1.26, \tau_3^* = 0.876$ $\tau_4^* = 0.67, \tau_5^* = 0.54$ $\lambda^* = 3.99, \alpha^* = 0.9029, \varepsilon^* = 71e-5$ $J_3^{5*} = 10.6311, x_1^* = 4$
3	$\tau_1 = 4.77, \tau_2 = 3.82, \tau_3 = 3.2$ $\tau_4 = 2.76, \tau_5 = 2.44$ $\lambda = 4.8, \alpha = 0.7724, \varepsilon_0 = 0.1$ $J_3^5 = 24.2191, x_1 = 0.2$	$\tau_1^* = 4.74, \tau_2^* = 3.80, \tau_3^* = 3.17$ $\tau_4^* = 2.73, \tau_5^* = 2.40$ $\lambda^* = 4.79, \alpha^* = 0.7724, \varepsilon^* = 21e-4$ $J_3^{5*} = 21.30982, x_1^* = 0.21$

Next, the simulation of Example 4.3 is given. Choose $(x_0, y_0) = (5, 0.5)$. The parameter values are taken as

$$\begin{aligned}
 \gamma &= 0.6, & \epsilon &= 0.4, & \delta &= 0.3, & a &= 2, & b &= 1.55, \\
 \bar{\varepsilon} &= -1e-8, & \sigma &= 100, & p_1 &= 1, & p_2 &= 2, & \lambda &= 4.5, & \alpha &= 0.8773.
 \end{aligned}
 \tag{4.54}$$

The optimal problem is solved by using a Matlab program and the above computational approach.

Take $n = 5$ which implies that the food is released five times and the species is harvested one time on one period $(0, T]$. Then the simulation results of system (4.32) are listed in Table 3 for various parameters. Here, the optimal time intervals $\tau_1^*, \tau_2^*, \tau_3^*, \tau_4^*$, and τ_5^* are shortened, which consequently gives rise to the shortness of period T . Also the optimal increment of food λ^* is decreased slightly, meanwhile the optimal harvesting rate α^* keeps invariable and the cost value J_3^{5*} is reduced. Also Figure 3(c) visualizes the optimal control strategy that less food is released at every impulsive time and more amount of species is acquired at the terminal time. We plot the time evolutions of food and species according to the optimal and non-optimal control laws with the first set of Table 3 (see Figures 3(a) and (b)). The black line means the dynamic behavior of non-optimal control and the red line means the dynamic behavior of optimal control. Obviously, the optimal tactics partly promotes the level of the species, which is desirable from the protection population.

Next, we consider the case of $n = 1$, which implies that the food is released and the species is harvested one time on one period $(0, T]$, respectively. The results obtained are shown in Table 4 for three sets of parameters. For given initial interval T , initial recruitment rate α , our goal is to compute the minimal cost function J_3^{1*} with time interval T^* , release amount λ^* and the capture rate α^* . We explore that control policy not only drops the level of the cost function, but also boosts the numbers of species at the terminal time. Meanwhile, the time interval is shortened too. Graphical output in Figure 4(c) directly displays our optimal control strategy expressed by (4.32) with $n = 1$. All of the black lines illustrate the behaviors of non-optimal control while all of the red lines mean the dynamic behavior of optimal control. When optimal control strategy is implemented, the trajectories of food and species are drawn in Figures 4(a) and (b). It shows that the level of species is higher in an optimal mode than in a non-optimal one at the terminal time.

In order to compare the two optimal control modes (namely $n = 1$ and $n = 5$), we compute $5J_3^{1*}$ in view of Table 4, which represents that the mode $n = 1$ is performed five times.

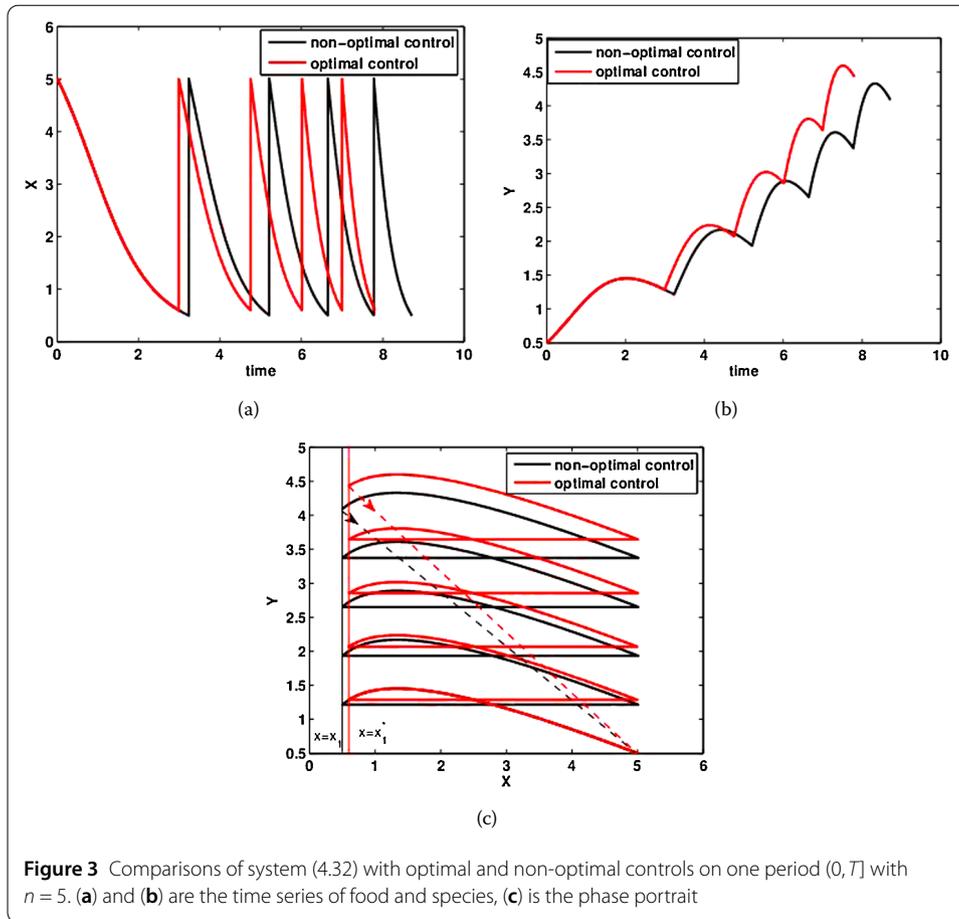


Table 4 Results of simulation for $n = 1$

Set	Non-optimal control	Optimal control
1	$T = 3.23, \lambda = 4.51$ $\alpha = 0.5881, \varepsilon = 0.1$ $J_3^1 = 3.65, x_1 = 0.49$	$T^* = 2.93, \lambda^* = 4.37$ $\alpha^* = 0.62, \varepsilon^* = 1.57e-4$ $J_3^{1*} = 2.77, x_1^* = 0.63$
2	$T = 2.35, \lambda = 4$ $\alpha = 0.6501, \varepsilon = 0.1$ $J_3^1 = 2.71, x_1 = 1$	$T^* = 2.31, \lambda^* = 3.96$ $\alpha^* = 0.65, \varepsilon^* = 9e-5$ $J_3^{1*} = 2.01, x_1^* = 1.04$
3	$T = 4.77, \lambda = 4.8$ $\alpha = 0.2952, \varepsilon = 0.1$ $J_3^1 = 4.94, x_1 = 0.2$	$T^* = 4.42, \lambda^* = 4.76$ $\alpha^* = 0.41, \varepsilon^* = 2.64e-4$ $J_3^{1*} = 4.05, x_1^* = 0.24$

Compared with J_3^{5*} , which represents that the mode $n = 5$ is executed one time, it indicates that the cost function $5J_3^{1*}$ is slightly less than J_3^{5*} . This result implies that the control mode of frequent releasing food and frequent harvest species is superior to that of frequent releasing food and infrequent harvest species.

5 Discussions

The topic about ISFC has been investigated extensively in the last decades due to its potential applications. Many authors have made every endeavor to explore the periodic solution of various systems including population, ecology, chemostat, epidemic, and so on.

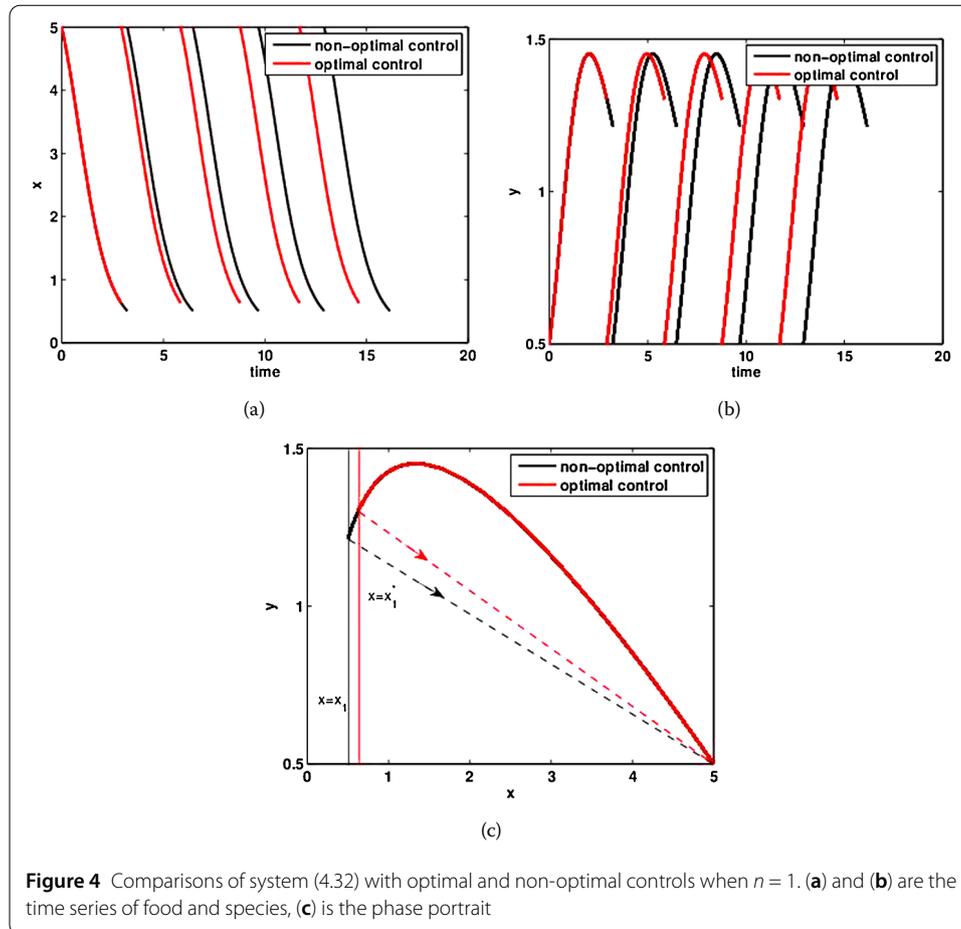


Figure 4 Comparisons of system (4.32) with optimal and non-optimal controls when $n = 1$. (a) and (b) are the time series of food and species, (c) is the phase portrait

However, if the system is exploited in a period mode, what strategies are implemented to achieve the objective of optimal management? So far, few researchers keep a watchful eye on this task which has been our focus in the above sections. In summary, our approaches are concluded by the following three procedures. (1) Under the hypothesis that the ISFC system has a periodic solution, an optimal problem of ISFC is transformed into a parameter optimization problem in an unspecified time with inequality constraint, and together with the constraint of the first arrival threshold. (2) The rescaled time and a constraint violation function are introduced to translate the above optimal problem to a parameter selection problem in a specified time with the unconstraint. (3) The gradients of the objective function on all of parameters are given to compute the optimal value of the cost function. Finally, three examples involving the marine ecosystem, computer virus control, and resource administration are illustrated to confirm the validity of our approaches. In these examples, the parameters of the impulse and system on continuous systems and a hybrid system are optimized respectively.

Despite some endeavors in this paper, it may be beneficial to investigate a wider variety of topics in the future: (1) The actual data is necessary to achieve effective state feedback impulsive control. (2) Exploring other means to solve the OCP of the state dependent impulsive systems. (3) Applying this method to other fields. (4) For the optimal problems of the hybrid system in Example 4.3, the number of impulsive effect happens is worth exploring.

Acknowledgements

The authors thank the referees for their careful reading of the original manuscript and many valuable comments and suggestions that greatly improved the presentation of this paper.

Funding

This work was supported by the National Natural Science Foundation of China (11471243, 11501409).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final draft.

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Received: 1 November 2017 Accepted: 6 February 2018 Published online: 15 March 2018

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