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Time–space fractional (2 + 1) dimensional nonlinear Schrödinger equation for envelope gravity waves in baroclinic atmosphere and conservation laws as well as exact solutions

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Abstract

In this article, nonlinear propagation of envelope gravity waves is studied in baroclinic atmosphere. The classical (2 + 1) dimensional nonlinear Schrödinger (NLS) equation can be derived by using the multiple-scale, perturbation method. Further, via the semi-inverse method, the Euler–Lagrange equation and Agrawal's method, the time–space fractional (2 + 1) dimensional nonlinear Schrödinger (FNLS) equation is obtained to describe the envelope gravity waves. Furthermore, the conservation laws of time–space FNLS equation are discussed on the basis of Lie group analysis method. Finally, the exact solutions to the equation are given by employing the $\exp(-\phi(\xi))$ method. The results demonstrate that the nonlinear effect caused by the fractional order leads to the change of the propagation characteristics of envelope gravity waves, the construction of fractional model has far-reaching significance for the research of nonlinear propagation of envelope gravity waves in actual atmospheric and ocean movement.

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Keywords: Time–space fractional (2 + 1) dimensional nonlinear Schrödinger equation; Envelope gravity waves; Conservation laws; $\exp(-\phi(\xi))$ method

1 Introduction

It is well known that envelope gravity waves play an important role in atmospheric dynamics [1–5], the troposphere is excited by convection, topography and other excitation processes to transfer energy and momentum from the source (e.g., mountains, thermal forcing) to the middle and upper atmosphere [6–9]. Atmospheric parameters, such as density and temperature, oscillate with fluctuations due to the influence of envelope gravity waves at the middle and upper atmosphere. In addition, the atmosphere is further achieved through the wave process of envelope gravity waves in actual atmospheric movement. Envelope gravity waves closely relate to the changes in the troposphere weather and climate, such as topographic precipitation, deep convection and typhoon rainstorms [10, 11].

The problem of propagation of nonlinear wave in plasma and fluid can be described by differential equations such as the KDV equation, the mKDV equation, the NLS equation, and the Boussinesq equation [12–20]. The NLS equation describes the time–space evolution of slow-changing envelopes, it has high theoretical value in quantum matters and has been extensively used in various branches of physics, such as optics, envelope gravity waves etc. [21, 22]. However, most of these studies are based on the integer-order model and the research on the fractional model is still relatively small. Fractional calculus is one of the best tools to investigate various scientific, engineering and mathematical models [23–25]. Numerous marine processes exhibit fractional dynamics, which are fractional systems. The use of fractional model can better reveal the nature of the phenomenon and behavior. Fractional calculus is the promotion of integral calculus. The study of fractional systems [26–31] has a more universal meaning. Thus we discuss the influence of fractional order for the propagation of envelope gravity waves by constructing the time–space fractional (2 + 1) NLS equation.

Conservation laws are very important for the study of nonlinear physical phenomena, symmetry and conservation laws [32–35] provide much information as regards systems simulated by differential equations. Only a few scholars discuss the conservation laws of fractional partial differential equation, for example, the generalizations of Noether's theorem [36], the new conservation theorem [37], the fractional generalized Noether operator [38], therefore, the conservation laws of the time–space nonlinear FPDES need further research. Moreover, there are many ways to solve differential equations such as the trial function method [39], the subequation method [40], the function variable method [41], the first integral method [42], etc. We can also get many other kinds of solutions, such as lump-soliton solutions [43–46], which are also very important.

The organization of the article is as follows: in Section 2, the integer-order model is derived by using the multiple-scale, perturbation method [47]. In Section 3, the fractional-order model is obtained by employing the semi-inverse method, Euler–Lagrange equation and Agrawal's method [48, 49]. In Section 4, with the help of Lie group analysis method [50–52], the conservation laws of time–space FNLS equation will be discussed. In Section 5, we get the exact solutions to the above equation by using the $\exp(-\phi(\xi))$ method [53]. In Section 6, some brief conclusions can be drawn.

2 Derivation of (2 + 1) dimensional NLS equation

Starting with the basic dynamic equations of atmospheric motion, they can be written in the form

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{g\theta}{\theta_0}, \\ \frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + \sigma w = 0, \\ \frac{\partial \rho_0 u}{\partial x} + \frac{\partial \rho_0 v}{\partial y} + \frac{\partial \rho_0 w}{\partial z} = 0, \end{cases} \tag{1}$$

where ρ_0 denotes the density; θ_0 is the temperature of the environmental flow field; $\sigma = \frac{\partial\theta_0}{\partial z}$.

Introducing the dimensionless quantities

$$\begin{aligned} &t = f^{-1}(t^*), \qquad (x,y) = L(X^*, Y^*), \qquad Z = D(z^*), \\ &(u,v) = U(u^*, v^*), \qquad w = \frac{U}{L}D(w^*), \qquad \rho_0 = \frac{P}{gH}(\rho_s), \\ &\theta = \delta\theta(\theta^*), \qquad \delta p_{x,y} = \frac{P}{gH}fLU(p^*), \qquad \delta p_z = \frac{P}{\theta_0 H}\delta\theta(p^*), \end{aligned} \tag{2}$$

where the quantities with asterisk mean they are dimensionless.

Assuming $D \sim H$, $\frac{U}{fL} \sim o(1)$, $\delta\theta = \frac{\delta UD}{fL}$ and substituting Eqs. (2) into Eqs. (1) yields

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_s}\frac{\partial p}{\partial x} + v, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho_s}\frac{\partial p}{\partial y} - u, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \epsilon^{-1}(-\frac{1}{\rho_s}\frac{\partial p}{\partial z} + \theta), \\ \frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w = 0, \\ \frac{\partial \rho_s u}{\partial x} + \frac{\partial \rho_s v}{\partial y} + \frac{\partial \rho_s w}{\partial z} = 0, \end{cases} \tag{3}$$

where we omit the subscript asterisks for simplicity, $\epsilon = \frac{f^2}{N^2}$ and $N^2 = \frac{g\theta}{\theta_0}$ are the new parameters.

We suppose that the solution of Eq. (3) has the following form:

$$\begin{cases} u = U(y,z) + \epsilon(u_0 + \epsilon u_1 + \epsilon^2 u_2) + \cdots, \\ v = V(y,z) + \epsilon(v_0 + \epsilon v_1 + \epsilon^2 v_2) + \cdots, \\ w = \epsilon(w_0 + \epsilon w_1 + \epsilon^2 w_2) + \cdots, \\ \theta = \Theta(y,z) + \epsilon(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2) + \cdots, \\ p = P(y,z) + \epsilon(p_0 + \epsilon p_1 + \epsilon^2 p_2) + \cdots, \end{cases} \tag{4}$$

where U represents the speed of the basic flow, Θ indicates the temperature field and P denotes the air pressure.

We introduce the slow time and space scales for the purpose of addressing the effects of nonlinearity and amplitude modulation of space,

$$T_1 = \epsilon t, \qquad T_2 = \epsilon^2 t, \qquad X_1 = \epsilon x, \qquad X_2 = \epsilon^2 x, \qquad Y = \epsilon y. \tag{5}$$

We introduce a new set of variables,

$$\rho_s u_n = u_n, \qquad \rho_s v_n = v_n, \qquad \rho_s w_n = w_n, \qquad \rho_s \theta_n = \theta_n, \quad n = 0,1,2. \tag{6}$$

Substituting Eqs. (4), (5) and (6) into Eqs. (1) one acquires the lowest-order approximate equations of ϵ,

$$\varepsilon^0 : \begin{cases} -\frac{1}{\rho_s}\frac{\partial P}{\partial y} - U = 0, \\ -\frac{1}{\rho_s}\frac{\partial P}{\partial z} + \Theta = 0, \end{cases} \tag{7}$$

obviously, we will get

$$\frac{\partial U}{\partial z} = -\frac{\partial \Theta}{\partial y}. \tag{8}$$

Further, the first-order approximate equations of ϵ will be obtained,

$$\varepsilon^1 : \begin{cases} \frac{\partial u_0}{\partial t} + U\frac{\partial u_0}{\partial x} + V\frac{\partial u_0}{\partial y} + (U_y - 1)v_0 + U_z w_0 + \frac{\partial p_0}{\partial x} = 0, \\ \frac{\partial v_0}{\partial t} + U\frac{\partial v_0}{\partial x} + V\frac{\partial v_0}{\partial y} + \frac{\partial p_0}{\partial y} + u_0 = 0, \\ \frac{\partial p_0}{\partial z} - \theta_0 = 0, \\ \frac{\partial \theta_0}{\partial t} + U\frac{\partial \theta_0}{\partial x} + V\frac{\partial \theta_0}{\partial y} + \Theta_y v_0 + w_0 = 0, \\ \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0. \end{cases} \tag{9}$$

Eliminating the other variables except for p_0 in Eqs. (9) yields the following equation:

$$M\left(\frac{\partial p_0}{\partial x}\right) = 0, \tag{10}$$

where

$$\begin{aligned} M &= \frac{\partial^2}{\partial y^2} - (U_y - 1)\frac{\partial^2}{\partial z^2} + 2U_z\frac{\partial^2}{\partial y\,\partial z} + \left[U_{zz} - \frac{\Psi_y}{\Psi} - U_z\frac{\Psi_z}{\Psi}\right]\frac{\partial}{\partial y} \\ &\quad + \left[(U_y - 1)\frac{\Psi_z}{\Psi} - U_z\frac{\Psi_y}{\Psi}\right]\frac{\partial}{\partial z} - \frac{1}{U}\left[U_z z - \frac{\Psi_y}{\Psi} - U_z\frac{\Psi_z}{\Psi}\right], \\ \Psi &= U_y - 1 + U_z^2, \qquad \Psi_y = \frac{\partial \Psi}{\partial y}, \qquad \Psi = \frac{\partial \Psi}{\partial y}. \end{aligned} \tag{11}$$

Suppose Eq. (10) has the following solution in the form of separate variables:

$$p_0 = p_0^*(y,z)A(T_1,T_2,X_1,X_2,Y)\exp\bigl[i(kx - \omega t)\bigr], \tag{12}$$

where k represents the zonal wave number, ω indicates the frequency of the envelope gravity waves and A denotes a slowly varying envelope complex amplitude.

Therefore, other solutions to Eqs. (9) are also given

$$\begin{cases} u_0 = u_0^*(y,z)A(T_1,T_2,X_1,X_2,Y)\exp[i(kx - \omega t)], \\ v_0 = v_0^*(y,z)A(T_1,T_2,X_1,X_2,Y)\exp[i(kx - \omega t)], \\ w_0 = w_0^*(y,z)A(T_1,T_2,X_1,X_2,Y)\exp[i(kx - \omega t)], \\ \theta_0 = \theta_0^*(y,z)A(T_1,T_2,X_1,X_2,Y)\exp[i(kx - \omega t)]. \end{cases} \tag{13}$$

Next, the second-order approximate equations of ϵ will be given,

$$\varepsilon^2: \begin{cases} \frac{\partial u_1}{\partial t} + U\frac{\partial u_1}{\partial x} + V\frac{\partial u_1}{\partial y} + (U_y - 1)v_1 + U_z w_1 + \frac{\partial p_1}{\partial x} \\ \quad = -[\frac{\partial u_0}{\partial T_1} + U\frac{\partial u_0}{\partial X_1} + V\frac{\partial u_0}{\partial Y} + \frac{1}{\rho_s}(u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + w_0\frac{\partial u_0}{\partial z}) + \frac{\partial p_0}{\partial X_1}] \equiv \Psi_{u_1}, \\ \frac{\partial v_1}{\partial t} + U\frac{\partial v_1}{\partial x} + V\frac{\partial v_1}{\partial y} + \frac{\partial p_1}{\partial y} + u_1 + V_y v_1 + V_z w_1 \\ \quad = -[\frac{\partial v_0}{\partial T_1} + U\frac{\partial v_0}{\partial X_1} + V\frac{\partial v_0}{\partial Y} + \frac{1}{\rho_s}(u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + w_0\frac{\partial v_0}{\partial z}) + \frac{\partial p_0}{\partial Y}] \equiv \Psi_{v_1}, \\ \frac{\partial p_1}{\partial z} - \theta_1 = -(\frac{\partial w_0}{\partial t} + U\frac{\partial w_0}{\partial x} + V\frac{\partial w_0}{\partial y}) \equiv \Psi_{w_1}, \\ \frac{\partial \theta_1}{\partial t} + U\frac{\partial \theta_1}{\partial x} + V\frac{\partial \theta_1}{\partial y} + \Theta_y v_1 + w_1 \\ \quad = -[\frac{\partial \theta_0}{\partial T_1} + U\frac{\partial \theta_0}{\partial X_1} + V\frac{\partial \theta_0}{\partial Y} + \frac{1}{\rho_s}(u_0\frac{\partial \theta_0}{\partial x} + v_0\frac{\partial \theta_0}{\partial y})] \equiv \Psi_{\theta_1}, \\ \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = -(\frac{\partial u_0}{\partial X_1} + \frac{\partial v_0}{\partial Y}) \equiv \Psi_{p_1}. \end{cases} \tag{14}$$

Substituting Eqs. (12) and (13) into Eqs. (14) one acquires

$$\begin{cases} \phi_{u1} = -[u_0^* A_{T_1} + (Uu_0^* + p_0^*)A_{X_1} + Vu_0^* A_Y]\exp[i(kx - \omega t)] \\ \qquad - \frac{1}{\rho_s}(ik(u_0^*)^2 + v_0^* u_{0y}^* + w_0^* u_{0z}^*)|A|^2\exp[2i(kx - \omega t)], \\ \phi_{v1} = -[v_0^* A_{T_1} + Uv_0^* A_{X_1} + (Vv_0^* + p_0^*)A_Y]\exp[i(kx - \omega t)] \\ \qquad - \frac{1}{\rho_s}(iku_0^* v_0^* + +v_0^* v_{0y}^* + w_0^* v_{0z}^*)|A|^2\exp[2i(kx - \omega t)], \\ \phi_{w1} = [iw_0^*(\omega - kU) - Vw_{0y}^*]A\exp[i(kx - \omega t)], \\ \phi_{\theta_1} = -[\theta_0^* A_{T_1} + U\theta_0^* A_{X_1} + V\theta_0^* A_Y]\exp[i(kx - \omega t)] \\ \qquad - \frac{1}{\rho_s}(iku_0^*\theta_0^* + v_0^*\theta_{0y}^*)|A|^2\exp[2i(kx - \omega t)], \\ \phi_{p_1} = -(u_0^* A_{X_1} + v_0^* A_Y)\exp[i(kx - \omega t)]. \end{cases} \tag{15}$$

Similarly, eliminating the other variables except for p_1 in Eqs. (14) leads to the following equation:

$$M\left(\frac{\partial p_1}{\partial x}\right) = M_1(\phi_{u_1}) + M_2(\phi_{v_1}) + M_3(\phi_{w_1}) + M_4(\phi_{\theta_1}), \tag{16}$$

where

$$\begin{cases} M_1 = \frac{\partial}{\partial y} + U_{zz} + U_z\frac{\partial}{\partial z} + \frac{1}{U} + \frac{1}{\Psi}(\Psi_y + U_z\Psi_z), \\ M_2 = -\frac{1}{U}[\frac{\partial}{\partial y} + U_z z + U_z\frac{\partial}{\partial z} - \frac{1}{\Psi}(\Psi_y + U_y\Psi_z)], \\ M_3 = \frac{1}{U}[U_z\frac{\partial}{\partial y} - (U_y - 1)\frac{\partial}{\partial z} - \frac{1}{\Psi}(U_y - 1)\Psi_z], \\ M_4 = U_z\frac{\partial}{\partial y} + \frac{U_z}{U} - (U_y - 1)\frac{\partial}{\partial z} - \frac{\Psi_y}{\Psi}U_z - \frac{(U_y-1)}{U}\frac{\Psi_z}{\Psi}. \end{cases} \tag{17}$$

In addition, for Eq. (16) the following solution in the form of separate variables exists by analysis and assumptions:

$$p_1 = p_1^*(y,z)\big[A\exp\big[i(kx - \omega t)\big] + |A|^2\exp\big[2i(kx - \omega t)\big]\big]. \tag{18}$$

Thus, other solutions to Eqs. (9) are also obtained

$$\begin{cases} u_1 = u_1^*(y,z)[A_{X_1}\exp[i(kx - \omega t)] + |A|^2\exp[2i(kx - \omega t)]], \\ v_1 = v_1^*(y,z)[A_Y\exp[i(kx - \omega t)] + |A|^2\exp[2i(kx - \omega t)]], \\ w_1 = w_1^*(y,z)[A_Y\exp[i(kx - \omega t)] + |A|^2\exp[2i(kx - \omega t)]], \\ \theta_1 = \theta_1^*(y,z)[A_{X_1}\exp[i(kx - \omega t)] + |A|^2\exp[2i(kx - \omega t)]], \end{cases} \tag{19}$$

and the third-order approximate equations of ϵ will be written as

$$\varepsilon^3: \begin{cases} \frac{\partial u_2}{\partial t} + U\frac{\partial u_2}{\partial x} + V\frac{\partial u_2}{\partial y} + (U_y - 1)v_2 + U_z w_2 + \frac{\partial p_2}{\partial x} \\ \quad = -[\frac{\partial u_0}{\partial T_2} + U\frac{\partial u_0}{\partial X_2} + U\frac{\partial u_1}{\partial X_1} + V\frac{\partial u_1}{\partial Y} + \frac{\partial u_1}{\partial T_1} + \frac{1}{\rho_s}(u_0\frac{\partial u_0}{\partial X_1} + v_0\frac{\partial u_0}{\partial Y} + u_1\frac{\partial u_0}{\partial x} \\ \quad\quad + v_1\frac{\partial u_0}{\partial y} + w_1\frac{\partial u_0}{\partial z} + u_0\frac{\partial u_1}{\partial x} + v_0\frac{\partial u_1}{\partial y} + w_0\frac{\partial u_1}{\partial z}) + \frac{\partial p_0}{\partial X_2} + \frac{\partial p_1}{\partial X_1}] \\ \quad \equiv \Psi_{u_2}, \\ \frac{\partial v_2}{\partial t} + U\frac{\partial v_2}{\partial x} + V\frac{\partial v_2}{\partial y} + \frac{\partial p_2}{\partial y} + u_2 + V_y v_2 + V_z w_2 \\ \quad = -[\frac{\partial v_0}{\partial T_2} + U\frac{\partial v_0}{\partial X_2} + U\frac{\partial v_1}{\partial X_1} + V\frac{\partial v_1}{\partial Y} + \frac{\partial v_1}{\partial T_1} + \frac{1}{\rho_s}(u_0\frac{\partial v_0}{\partial X_1} + v_0\frac{\partial v_0}{\partial Y} + u_1\frac{\partial v_0}{\partial x} \\ \quad\quad + v_1\frac{\partial v_0}{\partial y} + w_1\frac{\partial v_0}{\partial z} + u_0\frac{\partial v_1}{\partial x} + v_0\frac{\partial v_1}{\partial y} + w_0\frac{\partial v_1}{\partial z}) + \frac{\partial p_1}{\partial Y}] \\ \quad \equiv \Psi_{v_2}, \\ \frac{\partial p_2}{\partial z} - \theta_2 = -[\frac{\partial w_0}{\partial t_1} + U\frac{\partial w_0}{\partial X_1} + V\frac{\partial w_0}{\partial Y} + U\frac{\partial w_1}{\partial x} + \frac{1}{\rho_s}(u_0\frac{\partial w_1}{\partial x} + v_0\frac{\partial w_1}{\partial y} \\ \quad\quad\quad + w_0\frac{\partial w_1}{\partial z}) + \frac{\partial w_1}{\partial t}] \\ \quad\quad\quad \equiv \Psi_{w_2}, \\ \frac{\partial \theta_2}{\partial t} + U\frac{\partial \theta_2}{\partial x} + V\frac{\partial \theta_2}{\partial y} + \Theta_y v_2 + w_2 \\ \quad = -[\frac{\partial \theta_0}{\partial T_2} + U\frac{\partial \theta_0}{\partial X_2} + U\frac{\partial \theta_1}{\partial X_1} + V\frac{\partial \theta_1}{\partial Y} + \frac{\partial \theta_1}{\partial T_1} + \frac{1}{\rho_s}(u_0\frac{\partial \theta_0}{\partial X_1} + v_0\frac{\partial \theta_0}{\partial Y} + u_1\frac{\partial \theta_0}{\partial x} \\ \quad\quad + v_1\frac{\partial \theta_0}{\partial y} + w_1\frac{\partial \theta_0}{\partial z} + u_0\frac{\partial \theta_1}{\partial x} + v_0\frac{\partial \theta_1}{\partial y} + w_0\frac{\partial \theta_1}{\partial z}) + \frac{\partial p_1}{\partial Y}] \\ \quad \equiv \Psi_{\theta_2}, \\ \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = -(\frac{\partial u_1}{\partial X_1} + \frac{\partial u_0}{\partial X_2} + \frac{\partial v_1}{\partial Y}) \equiv \Psi_{p_2}. \end{cases} \tag{20}$$

Then substitute Eqs. (18) and (19) into Eq. (20) and we can get the following forms by means of the secular-producing terms proportional to $\exp[i(kx - \omega t)]$:

$$\begin{cases} \phi_{u2} = u_0^*(A_{T_2} + UA_{X_2}) + Uu_1^* A_{X_1X_1} \\ \qquad + \frac{1}{\rho_s}(3iku_o^* u_1^* + u_{oy}^* u_{1y}^* + u_{0z}^* w_1^* + v_0^* u_{1y}^* + w_0^* u_{1z}^*)|A|^2 A, \\ \phi_{v2} = v_0^*(A_{T_2} + UA_{X_2}) + Vv_1^* A_{YY} \\ \qquad + \frac{1}{\rho_s}(iku_1^* v_0^* + v_1^* v_{oy}^* + w_1^* v_{0z}^* + 2iku_0^* v_1^* + v_0^* v_{1y}^* + w_0^* v_{1z}^*)|A|^2 A, \\ \phi_{w2} = w_0^*(A_{T_1} + UA_{X_1}) + Vw_1^* A_{YY}, \\ \phi_{\theta_2} = \theta_0^*(A_{T_2} + UA_{X_2}) + U\theta_1^* A_{X_1X_1} + \frac{1}{\rho_s}(iku_1^*\theta_0^* + v_1^*\theta_{0y}^* + 2iku_0^*\theta_1^* + v_0^*\theta_{1y}^*)|A|^2 A, \\ \phi_{p_2} = u_0^* A_{X_2} + u_1^* A_{X_1X_1} + v_1^* A_{YY}. \end{cases} \tag{21}$$

Meanwhile, eliminating the other variables except for p_1 in Eq. (14) one obtains the following equation:

$$M\left(\frac{\partial p_2}{\partial x}\right) = M_1(\phi_{u2}) + M_2(\phi_{v2}) + M_3(\phi_{w2}) + M_4(\phi_{\theta_2}). \tag{22}$$

We adopt the following variable transformations:

$$X = \frac{1}{U}X_1 = \frac{1}{U}X_2, \qquad t = T_1 = T_2, \tag{23}$$

and the (2 + 1) NLS equation will be obtained,

$$i\left(\frac{\partial A}{\partial t} + \frac{\partial A}{\partial X}\right) + a_1\frac{\partial^2 A}{\partial X^2} - a_2\frac{\partial^2 A}{\partial y^2} + a_3|A|^2 A = 0, \tag{24}$$

where the coefficients are expressed as

$$\begin{cases} a_1 = -i(Uu_1^* + u_1^*), \\ a_2 = i(Vv_1^* + Vw_1^* + v_1^*), \\ a3 = \frac{1}{\rho_s}[3ku_0^* u_1^* + ku_1^* v_0^* + 2ku_0^* v_1^* - i(u_{0y}^* v_1^* + u_{0z}^* w_1^* \\ \qquad + v_0^* u_{1y}^* + w_0^* u_{1z}^* + v_{0y}^* v_1^* + w_1^* v_{0z}^* + v_0^* v_{1y}^*)]. \end{cases} \tag{25}$$

According to [9], the corresponding transformation can be defined

$$x = X - t, \tag{26}$$

so Eq. (24) is rewritten as follows:

$$i\frac{\partial A}{\partial t} + a_1\frac{\partial^2 A}{\partial x^2} - a_2\frac{\partial^2 A}{\partial y^2} + a_3|A|^2 A = 0. \tag{27}$$

3 Formulation of time–space fractional (2 + 1) dimensional NLS equation

The semi-inverse method and the fractional variational principle are used to get the time–space fractional (2 + 1) dimensional NLS equation as follows:

Defining a potential equation $A(x, y, t) = u(x, y, t) + iv(x, y, t)$, where $u(x, y, t)$ and $v(x, y, t)$ indicate real functions of x, y and t, and the potential equation of the classical (2 + 1) dimensional NLS equation (27) will be given,

$$-v_t + a_1u_{xx} - a_2u_{yy} + a_3u\left(u^2 + v^2\right) + i\left[u_t + a_1v_{xx} - a_2v_{yy} + a_3v\left(u^2 + v^2\right)\right] = 0. \tag{28}$$

Further, the system of two second-order equations can be represented in the following form:

$$-v_t + a_1u_{xx} - a_2u_{yy} + a_3u\left(u^2 + v^2\right) = 0, \tag{29}$$

$$u_t + a_1v_{xx} - a_2v_{yy} + a_3v\left(u^2 + v^2\right) = 0. \tag{30}$$

We will construct a trial-functional with the help of the semi-inverse method for getting a variational principle for systems (29) and (30). Further, the system of two second-order equations can be represented in the following form:

$$J(u,v) = \int \left[u_tv - \frac{a_1}{2}v_x^2 + \frac{a_2}{2}v_y^2 + \frac{a_3}{4}\left(2u^2v^2 + v^4\right) + H(u)\right] d\Omega, \tag{31}$$

where $d\Omega = dx\,dy\,dt$ and $H(u)$ is an unknown function consisting of the derivatives of u and u.

Considering the variation in Eq. (31) for u and we can get the Euler–Lagrange equation

$$-v_t + a_3uv^2 + \frac{\delta H}{\delta u} = 0, \tag{32}$$

where $\frac{\delta H}{\delta u}$ means He's variational differential [48] for u,

$$\begin{aligned}\frac{\delta H}{\delta u} &= \frac{\partial H}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial H}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial u_y}\right) \\ &\quad + \frac{\partial^2}{\partial t^2}\left(\frac{\partial H}{\partial u_{tt}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial H}{\partial u_{xx}}\right) + \cdots.\end{aligned} \tag{33}$$

In order to make equation (32) satisfy equation (29), we set

$$\frac{\delta H}{\delta u} = v_t - a_3uv^2 = a_1u_{xx} - a_2u_{yy} + a_3u^3, \tag{34}$$

and H can also be defined as follows:

$$H = -\frac{a_1}{2}u_x^2 + \frac{a_2}{2}u_y^2 + \frac{a_3}{4}u^4. \tag{35}$$

Further, we will obtain the final variational principle

$$J(u,v) = \int \left[u_tv - \frac{a_1}{2}\left(u_x^2 + v_x^2\right) + \frac{a_2}{2}\left(u_y^2 + v_y^2\right) + \frac{a_3}{4}\left(u^2 + v^2\right)^2\right] d\Omega. \tag{36}$$

Substituting $u = \frac{A+A^*}{2}$, $v = i\frac{A-A^*}{2}$, where A^* expresses the complex conjugate of A and $A^* = u - iv$, and we will obtain the following variational principle:

$$J(A) = \int \left[\frac{i}{4}\left(A^* - A\right)\left(\frac{\partial A}{\partial t} + \frac{\partial A^*}{\partial t}\right) - \frac{a_1}{2}\left(\frac{\partial A}{\partial x}\frac{\partial A^*}{\partial x}\right) + \frac{a_2}{2}\left(\frac{\partial A}{\partial y}\frac{\partial A^*}{\partial y}\right) + \frac{a_3}{4}\left(AA^*\right)^2 \right] d\Omega, \tag{37}$$

from which we can identify the Lagrangian of (2 + 1) dimensional NLS equation,

$$L = \frac{i}{4}\left(A^* - A\right)\left(\frac{\partial A}{\partial t} + \frac{\partial A^*}{\partial t}\right) - \frac{a_1}{2}\left(\frac{\partial A}{\partial x}\frac{\partial A^*}{\partial x}\right) + \frac{a_2}{2}\left(\frac{\partial A}{\partial y}\frac{\partial A^*}{\partial y}\right) + \frac{a_3}{4}\left(AA^*\right)^2. \tag{38}$$

Similarly, the Lagrangian of the time–space fractional (2 + 1) dimensional NLS equation is written as

$$F = \frac{i}{4}\left(A^* - A\right)\left(D_t^\alpha A + D_t^\alpha A^*\right) - \frac{a_1}{2}\left(D_x^\beta A D_x^\beta A^*\right) + \frac{a_2}{2}\left(D_y^\gamma A D_y^\gamma A^*\right) + \frac{a_3}{4}\left(AA^*\right)^2, \tag{39}$$

here the fractional derivative $D_t^\alpha A$ shows the *mRL* fractional derivative defined in [54]

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d_z}\int_a^z d\zeta \frac{[f(\zeta) - f(a)]}{(z-\zeta)^\alpha}, \quad 0 \le \alpha < 1. \tag{40}$$

Next, the functional of the time–space fractional (2 + 1) dimensional NLS equation can take the form

$$J\left(A^*\right) = \int_X (dx)^\beta \int_Y (dy)^\gamma \int_T (dt)^\alpha F\left(A^*, D_t^\alpha A^*, D_x^\beta A^*, D_y^\gamma A^*\right). \tag{41}$$

Moreover, we have the following definition [54, 55]:

$$\int_a^t (dt)^\alpha = \alpha \int_a^t d\tau (t-\tau)^\alpha f(\tau). \tag{42}$$

Integrating by parts according to the above relation [54, 55],

$$\int_a^b (dz)^\alpha f(z) D_z^\alpha g(z) = \Gamma(1+\alpha)\left[g(z)f(z)\right]\Big|_a^b - \int_a^b (dz)^\alpha g(z) D_z^\alpha f(z), \quad f(z), g(z) \in [a,b]. \tag{43}$$

Further, by optimizing the variation of the functional $\delta J_F(A^*) = 0$, the Euler–Lagrange equation of the time–space fractional (2 + 1) dimensional NLS equation can be written

$$\frac{\partial F}{\partial A^*} - D_t^\alpha\left(\frac{\partial F}{\partial D_t^\alpha A^*}\right) - D_x^\beta\left(\frac{\partial F}{\partial D_x^\beta A^*}\right) - D_y^\gamma\left(\frac{\partial F}{\partial D_y^\gamma A^*}\right) = 0. \tag{44}$$

Finally, substituting the Lagrange defined by Eq. (5) into the Euler–Lagrange formula, we can obtain the time–space fractional (2 + 1) dimensional NLS equation

$$iD_t^\alpha A + a_1 D_x^{2\beta} A - a_2 D_y^{2\gamma} A + a_3 A|A|^2 = 0. \tag{45}$$

4 Conservation laws of time–space fractional (2 + 1) dimensional NLS equation

In the section, we present a time–space FPDE with three independent variables,

$$G\big(x,y,t,A,D_t^{\alpha}A,D_x^{2\beta}A,D_y^{2\gamma}A,\ldots\big)=0,\quad \alpha>0,\beta>0,\gamma>0. \tag{46}$$

Introducing an one-parameter Lie group for infinitesimal transformations,

$$\begin{aligned}
&\overline{x}\to x+\epsilon\xi(x,y,t,A)+o\big(\epsilon^2\big),\\
&\overline{y}\to y+\epsilon\zeta(x,y,t,A)+o\big(\epsilon^2\big),\\
&\overline{t}\to t+\epsilon\tau(x,y,t,A)+o\big(\epsilon^2\big),\\
&\overline{A}\to A+\epsilon\eta(x,y,t,A)+o\big(\epsilon^2\big),\\
&D_t^{\alpha}\overline{A}\to D_t^{\alpha}A+\epsilon\eta_t^{\alpha}(x,y,t,A)+o\big(\epsilon^2\big),\\
&D_x^{2\beta}\overline{A}\to D_x^{2\beta}A+\epsilon\eta_x^{2\beta}(x,y,t,A)+o\big(\epsilon^2\big),\\
&D_y^{2\gamma}\overline{A}\to D_y^{2\gamma}A+\epsilon\eta_y^{2\gamma}(x,y,t,A)+o\big(\epsilon^2\big),\\
&\cdots
\end{aligned} \tag{47}$$

where ξ, ζ, τ and η indicate infinitesimals and $\epsilon\ll 1$ means a group parameter.

The extended infinitesimals [56] can be written

$$\begin{aligned}
\eta_t^{\alpha}={}&D_t^{\alpha}(\eta)+\xi D_t^{\alpha}(A_x)-D_t^{\alpha}(\xi A_x)+\zeta D_t^{\alpha}(A_y)-D_t^{\alpha}(\zeta A_y)+D_t^{\alpha}\big(D_t(\tau)A\big)\\
&-D_t^{\alpha+1}(\tau A)+\tau D_t^{\alpha+1}(A),\\
\eta_x^{2\beta}={}&D_x^{2\beta}(\eta)+D_x^{\beta}\big(\tau D_x^{\beta}(A_t)\big)-D_x^{2\beta}(\tau A_t)+D_x^{\beta}\big(\zeta D_x^{\beta}(A_y)\big)\\
&-D_x^{2\beta}(\zeta A_y)+D_x^{2\beta}\big(D_x(\xi)A\big)-D_x^{2\beta+1}(\xi A)+D_x^{\beta}\big(\xi D_x^{\beta+1}(A)\big)\\
&+\xi D_x^{2\beta}(A_x)+\tau D_x^{2\beta}(A_t)+\zeta D_x^{2\beta}(A_y),\\
\eta_y^{2\gamma}={}&D_y^{2\gamma}(\eta)+D_y^{\gamma}\big(\tau D_y^{\gamma}(A_t)\big)-D_y^{2\gamma}(\tau A_t)+D_y^{\gamma}\big(\xi D_y^{\gamma}(A_x)\big)\\
&-D_y^{2\gamma}(\xi A_x)+D_y^{2\gamma}\big(D_y(\zeta)A\big)-D_y^{2\gamma+1}(\zeta A)+D_y^{\gamma}\big(\zeta D_y^{\gamma+1}(A)\big)\\
&+\xi D_y^{2\gamma}(A_x)+\tau D_y^{2\gamma}(A_t)+\zeta D_y^{2\gamma}(A_y),
\end{aligned} \tag{48}$$

where D_t, D_x and D_y represent the total derivative operator defined in the following form:

$$D_{x_j}=\frac{\partial}{\partial x_j}+A_j\frac{\partial}{\partial A}+A_{jk}\frac{\partial}{\partial A_k}+\cdots,\quad j,k=1,2,3,\ldots. \tag{49}$$

Obviously, $x_1=t$, $x_2=x$, $x_3=y$, $A_j=\frac{\partial A}{\partial x_j}$ and $A_{jk}=\frac{\partial^2}{\partial x_j\,\partial x_k}$.

On the basis of the generalized Leibnitz rule [57] and the generalization of the chain rule [58], we can get the expression of the extended symmetry operator η_t^{α},

$$\begin{aligned}
\eta_t^{\alpha}={}&D_t^{\alpha}\eta+\big(\eta_A-\alpha D_t(\tau)\big)D_t^{\alpha}A-AD_t^{\alpha}\eta_A+\mu_l\\
&+\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}D_t^{\alpha}\eta_A-\binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(A)\\
&-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^{n}(\xi)D_t^{\alpha-n}A_x-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^{n}(\zeta)D_t^{\alpha-n}A_y,
\end{aligned} \tag{50}$$

where

$$\mu_l = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{l}{n}\binom{n}{m}\binom{k}{r} \times \frac{1}{r!}\frac{t^{n-l}}{\Gamma(n+1-l)}[-A]^r\frac{\partial^m}{\partial t^m}\bigl[A^{k-r}\bigr]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\,\partial A^k}, \quad l=\alpha,\beta,\gamma. \tag{51}$$

In the same way, the extended infinitesimal $\eta_x^{2\beta}$ and $\eta_y^{2\gamma}$ can also be defined as

$$\begin{aligned}\eta_x^{2\beta} &= D_x^{2\beta}\eta + \bigl(\eta_A - \beta D_x(\xi)\bigr)D_x^{2\beta}A - AD_x^{2\beta}\eta_A + \mu_\beta \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{2\beta}{n}D_x^n\eta_A - \binom{2\beta}{n+1}D_x^{n+1}(\xi) + \binom{\beta}{n+1}D_x^n(\xi)\right]D_x^{2\beta-n}A \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{\beta}{n} - \binom{2\beta}{n}\right]D_x^n(\tau)D_x^{2\beta-n}(A_t) + \sum_{n=1}^{\infty}\left[\binom{\beta}{n} - \binom{2\beta}{n}\right]D_x^n(\zeta)D_x^{2\beta-n}(A_y) \\ &\quad + \xi D_x^{2\beta}(A_x) + \tau D_x^{2\beta}(A_t) + \zeta D_x^{2\beta}(A_y), \end{aligned} \tag{52}$$

$$\begin{aligned}\eta_y^{2\gamma} &= D_y^{2\gamma}\eta + \bigl(\eta_A - \gamma D_y(\zeta)\bigr)D_y^{2\gamma}A - AD_y^{2\gamma}\eta_A + \mu_\gamma \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{2\gamma}{n}D_y^n\eta_A - \binom{2\gamma}{n+1}D_y^{n+1}(\zeta) + \binom{\gamma}{n+1}D_y^n(\zeta)\right]D_y^{2\gamma-n}A \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{\gamma}{n} - \binom{2\gamma}{n}\right]D_y^n(\tau)D_y^{2\gamma-n}(A_t) + \sum_{n=1}^{\infty}\left[\binom{\gamma}{n} - \binom{2\gamma}{n}\right]D_y^n(\xi)D_y^{2\gamma-n}(A_x) \\ &\quad + \xi D_y^{2\gamma}(A_x) + \tau D_y^{2\gamma}(A_t) + \zeta D_y^{2\gamma}(A_y). \end{aligned} \tag{53}$$

By the Lie symmetry theory, we will obtain the infinitesimal generator

$$X = \xi(x,y,t,A)\frac{\partial}{\partial x} + \zeta(x,y,t,A)\frac{\partial}{\partial y} + \tau(x,y,t,A)\frac{\partial}{\partial t} + \eta(x,y,t,A)\frac{\partial}{\partial A}. \tag{54}$$

The invariance criterion of system (46) can be given as follows:

$$P_r^{(n)}V(\Delta)\big|_{\Delta=0} = 0, \quad n = 1,2,\ldots, \tag{55}$$

where $\Delta = G(x,y,t,A,D_t^\alpha A, D_x^{2\beta}A, D_y^\gamma A,\ldots)$.

The invariance condition leads to

$$\tau(x,y,t,A)\big|_{t=0} = 0. \tag{56}$$

Using the second prolongation to Eq. (45), the following invariance criterion is obtained:

$$i\eta_t^\alpha + a_1\eta_x^{2\beta} - a_2\eta_y^{2\gamma} + 2a_3A^*A\eta = 0. \tag{57}$$

Substituting (50), (52) and (53) into Eq. (57) yields the determining equations, from which we identify

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t}, \\ &X_4 = \beta y\frac{\partial}{\partial x} - \gamma x\frac{\partial}{\partial y}, \qquad X_5 = \frac{x}{\beta}\frac{\partial}{\partial x} + \frac{y}{\gamma}\frac{\partial}{\partial y} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - A\frac{\partial}{\partial A}. \end{aligned} \tag{58}$$

With the help of Lie group analysis, we acquire the conservation laws of Eq. (45) defined by the following form:

$$D_t(C_t) + D_x(C_x) + D_y(C_y) = 0, \tag{59}$$

where C_t, C_x and C_y denote the conserved vectors.

A formal Lagrangian [59] for Eq. (45) is written as

$$\mathcal{L} = \omega(x,y,t)\bigl(iD_t^{\alpha}A + a_1 D_x^{2\beta}A - a_2 D_y^{2\gamma}A + a_3 A^2 A^*\bigr), \tag{60}$$

where $\omega(x,y,t)$ indicates a new dependent variable.

The adjoint equation of Eq. (45) is given by

$$Q^* = \frac{\delta\mathcal{L}}{\delta A}, \tag{61}$$

where the expression of the Euler–Lagrange operator $\frac{\delta}{\delta A}$ is defined as

$$\begin{aligned} \frac{\delta}{\delta A} = {}& \frac{\partial}{\partial A} + \bigl(D_t^{\alpha}\bigr)^*\frac{\partial}{\partial(D_t^{\alpha}A)} + \bigl(D_x^{\beta}\bigr)^*\frac{\partial}{\partial(D_x^{\beta}A)} + \bigl(D_y^{\gamma}\bigr)^*\frac{\partial}{\partial(D_y^{\gamma}A)} \\ &+ \sum_{k=1}^{\infty}(-1)^k D_{i1}D_{i2}\cdots D_{ik}\frac{\partial}{\partial A_{i1i2\cdots ik}}. \end{aligned} \tag{62}$$

Here $(D_t^{\alpha})^*$ means the adjoint operator of D_t^{α}.

In addition, $W = \eta - \tau A_t - \xi A_x - \zeta A_y$ is the Lie characteristic functions, we get

$$\begin{aligned} &W_1 = -A_x, \qquad W_2 = -A_y, \qquad W_3 = -A_t, \\ &W_4 = -\beta y A_x + \gamma x A_y, \qquad W_5 = -A - \frac{x}{\beta}A_x - \frac{y}{\gamma}A_y - \frac{2t}{\alpha}A_t. \end{aligned} \tag{63}$$

According to the fractional generalizations of the Noether operators [38], the component of the conserved vector has the form

$$C_{\psi} = \xi_{\psi}\mathcal{L} + \sum_{k=0}^{n-1}(-1)^k D_{\psi}^{l-1-k}(W)D_{\psi}^k\frac{\partial\mathcal{L}}{\partial D_{\psi}^l A} - (-1)^n J\left(W, D_{\psi}^n\frac{\partial\mathcal{L}}{\partial D_{\psi}^l A}\right), \tag{64}$$

where $J(\cdot)$ expresses the integral

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(\tau,x,y)g(\mu,x,y)}{(\mu-\tau)^{\alpha+1-n}}\,d\mu\,d\tau. \tag{65}$$

Let us take W_5 as an example to calculate the conservation laws of Eq. (45) by using the preceding formula,

$$\begin{aligned} C_t &= \tau\mathcal{L} + D_t^{\alpha-1}(W_5)\frac{\partial\mathcal{L}}{\partial D_t^\alpha A} + J\left(W, D_t\frac{\partial\mathcal{L}}{\partial D_t^\alpha A}\right) \\ &= i\omega D_t^{\alpha-1}\left(-A - \frac{x}{\beta}A_x - \frac{y}{\gamma}A_y - \frac{2t}{\alpha}A_t\right) + J\left[\left(-A - \frac{x}{\beta}A_x - \frac{y}{\gamma}A_y - \frac{2t}{\alpha}A_t\right), i\omega_t\right], \end{aligned} \tag{66}$$

$$\begin{aligned} C_x &= \xi\mathcal{L} + D_x^{2\beta-1}(W_5)\frac{\partial\mathcal{L}}{\partial D_x^{2\beta} A} + J\left(w, D_x\frac{\partial\mathcal{L}}{\partial D_x^{2\beta} A}\right) \\ &= a_1\omega D_x^{2\beta-1}\left(-A - \frac{x}{\beta}A_x - \frac{y}{\gamma}A_y - \frac{2t}{\alpha}A_t\right) \\ &\quad + J\left[\left(-A - \frac{x}{\beta}A_x - \frac{y}{\gamma}A_y - \frac{2t}{\alpha}A_t\right), a_1\omega_x\right], \end{aligned} \tag{67}$$

$$\begin{aligned} C_y &= \zeta\mathcal{L} + D_y^{2\gamma-1}(W_5)\frac{\partial\mathcal{L}}{\partial D_y^{2\gamma} A} + J\left(w, D_y\frac{\partial\mathcal{L}}{\partial D_y^{2\gamma} A}\right) \\ &= -a_2\omega D_y^{2\gamma-1}\left(-A - \frac{x}{\beta}A_x - \frac{y}{\gamma}A_y - \frac{2t}{\alpha}A_t\right) \\ &\quad + J\left[\left(-A - \frac{x}{\beta}A_x - \frac{y}{\gamma}A_y - \frac{2t}{\alpha}A_t\right), -a_2\omega_y\right]. \end{aligned} \tag{68}$$

5 Exact solutions of time–space fractional (2 + 1) dimensional NLS equation

In this part, the $\exp(-\phi(\xi))$ method will be applied to obtain the exact solutions to Eq. (45), as follows.

To begin with, consider the following fractional complex transformation:

$$A(x,y,t) = A(\xi), \quad \xi = \frac{b_1x^\beta}{\Gamma(\beta+1)} + \frac{b_2y^\gamma}{\Gamma(\gamma+1)} + \frac{b_3t^\alpha}{\Gamma(\alpha+1)}, \tag{69}$$

where b_1, b_2 and b_3 indicate constants defined later.

Thus, Eq. (45) can be transformed into the following differential equation [60]:

$$ib_3A_\xi + b_1^2a_1A_{\xi\xi} - b_2^2a_2A_{\xi\xi} + a_3A|A|^2 = 0, \tag{70}$$

and the complex function A can be given,

$$A(\xi) = \frac{ib_3}{\exp[2(a_2b_2^2 - a_1b_1^2)]}\xi U(\xi), \tag{71}$$

where $U(\xi)$ denotes the real function, we can get the following equation:

$$c_1U + c_2U^3 + c_3U_{\xi\xi} = 0, \tag{72}$$

here $c_1 = b_3^2$, $c_2 = 4a_3(a_1b_1^2 - a_2b_2^2)$, $c_3 = -4(a_1b_1^2 - a_2b_2^2)$.

For the purpose of balancing the highest-order derivative term and the nonlinear term of Eq. (45), it can be determined that $n = 1$, and the solutions of Eq. (72) are written

$$U(\xi) = d_0 + d_1\exp(-\phi(\xi)), \tag{73}$$

where d_1 and d_2 are constants and $\exp(-\phi(\xi))$ satisfies the following equation:

$$\phi'(\xi) = \exp\bigl(-\phi(\xi)\bigr) + \mu \exp\bigl(\phi(\xi)\bigr) + \lambda. \tag{74}$$

Substituting Eq. (73) and Eq. (74) into Eq. (70) and extracting all the same power terms of $[\exp(-\phi(\xi))]^j$, $j = -3, \dots, 0$, dividing the coefficients into zero one acquires a set of algebraic equations,

$$\begin{aligned}
&\phi^{3\xi} : d_0 c_1 + d_0^3 c_2 + c_3 d_1 \mu\lambda = 0,\\
&\phi^{2\xi} : d_1 c_1 + 3d_0^2 d_1 c_2 + 2d_1 c_3 \mu + d_1 \lambda^2 c_3 = 0,\\
&\phi^{1\xi} : 3c_2 d_0 d_1^2 + 3d_1 c_3 \lambda = 0,\\
&\phi^{0\xi} : d_1^3 c_2 + 2d_1 c_3 = 0.
\end{aligned} \tag{75}$$

Further solving the above algebraic equations (75), we will get

$$d_0 = \pm i\frac{\sqrt{c_3}\lambda}{\sqrt{2c_2}}, \qquad d_1 = \pm i\frac{\sqrt{2c_3}}{\sqrt{(c_2)}}, \qquad c_1 = \frac{1}{2}\bigl(c_3\lambda^2 + 4c_3\mu\bigr). \tag{76}$$

Substituting Eq. (76) into Eq. (73) leads to

$$U(\xi) = \pm i\frac{\sqrt{c_3}\lambda}{\sqrt{2c_2}} + \pm i\frac{\sqrt{2c_3}}{\sqrt{c_2}} \exp\bigl(-\phi(\xi)\bigr). \tag{77}$$

Finally, based on the above results, we can discuss solutions of different cases.

(I) When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, the trigonometric function solutions are written

$$\begin{aligned}
A_2(x,y,t) = {}&\pm\frac{\sqrt{\lambda^2 + 4\mu}c_3}{\exp[2(a_2 b_2^2 - a_1 b_1^2)]}\\
&\times \xi\left(\frac{\lambda}{2\sqrt{c_2}} + \frac{1}{\sqrt{c_2}}\frac{2\mu}{\sqrt{4\mu - \lambda^2}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}(\xi + C)) - \lambda}\right).
\end{aligned} \tag{78}$$

(II) When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, the hyperbolic function solutions are written

$$\begin{aligned}
A_1(x,y,t) = {}&\pm\frac{\sqrt{\lambda^2 + 4\mu}c_3}{\exp[2(a_2 b_2^2 - a_1 b_1^2)]}\\
&\times \xi\left(\frac{\lambda}{2\sqrt{c_2}} - \frac{1}{\sqrt{c_2}}\frac{2\mu}{\sqrt{\lambda^2 - 4\mu}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C)) + \lambda}\right).
\end{aligned} \tag{79}$$

(III) When $\lambda^2 - 4\mu > 0$ and $\mu = 0$, $\lambda \neq 0$, the hyperbolic function solutions are written

$$\begin{aligned}
A_3(x,y,t) = {}&\pm\frac{\sqrt{\lambda^2 + 4\mu}c_3}{\exp[2(a_2 b_2^2 - a_1 b_1^2)]}\\
&\times \xi\left(\frac{\lambda}{2\sqrt{c_2}} + \frac{1}{\sqrt{c_2}}\frac{\lambda}{\cosh(\lambda(\xi + C)) + \sinh(\lambda(\xi + C)) - 1}\right).
\end{aligned} \tag{80}$$

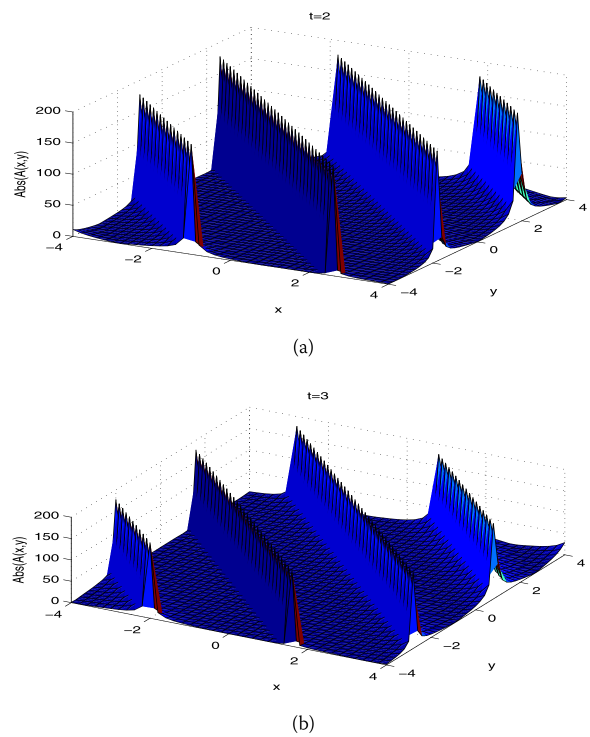


Figure 1 (**a**) The exact solution represented by Eq. (78), $t = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$. (**b**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$

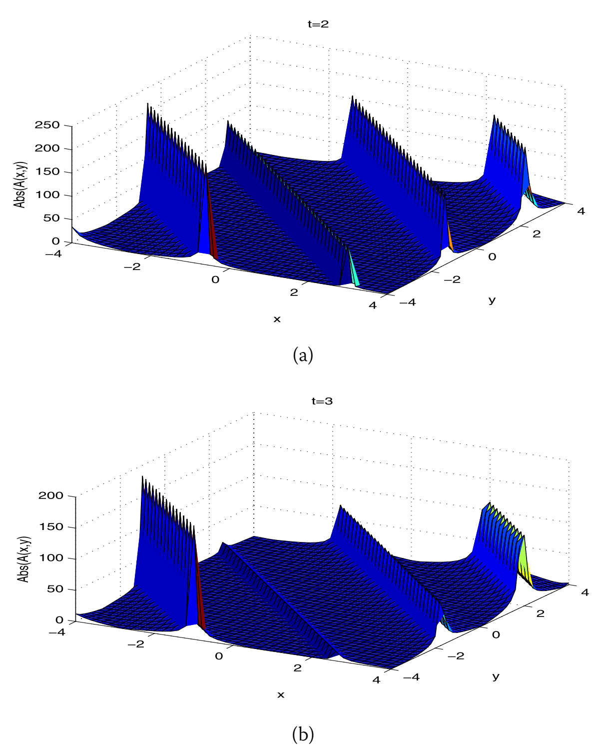
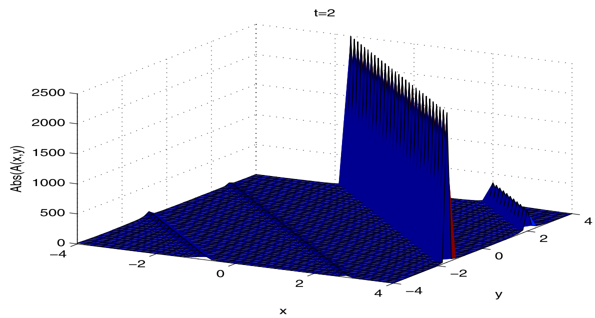
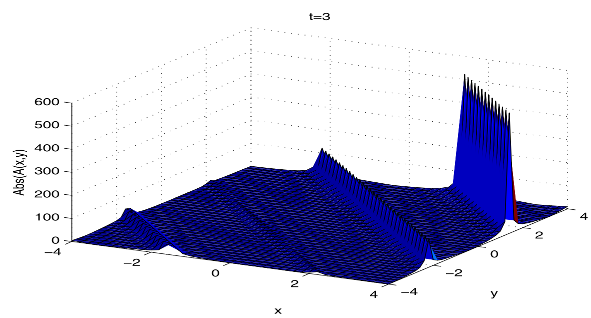


Figure 2 (**a**) The exact solution represented by Eq. (78), $t = 2$, $\alpha = 0.5$, $\beta = 1$, $\gamma = 1$. (**b**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 0.5$, $\beta = 1$, $\gamma = 1$

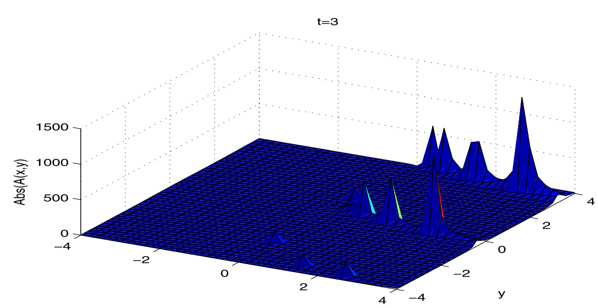


(a)

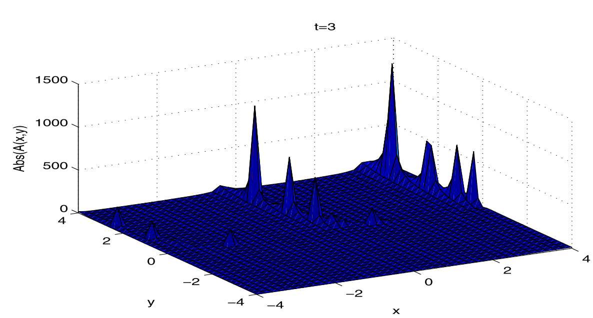


(b)

Figure 3 (**a**) The exact solution represented by Eq. (78), $t = 2$, $\alpha = 0.75$, $\beta = 1$, $\gamma = 1$. (**b**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 0.75$, $\beta = 1$, $\gamma = 1$

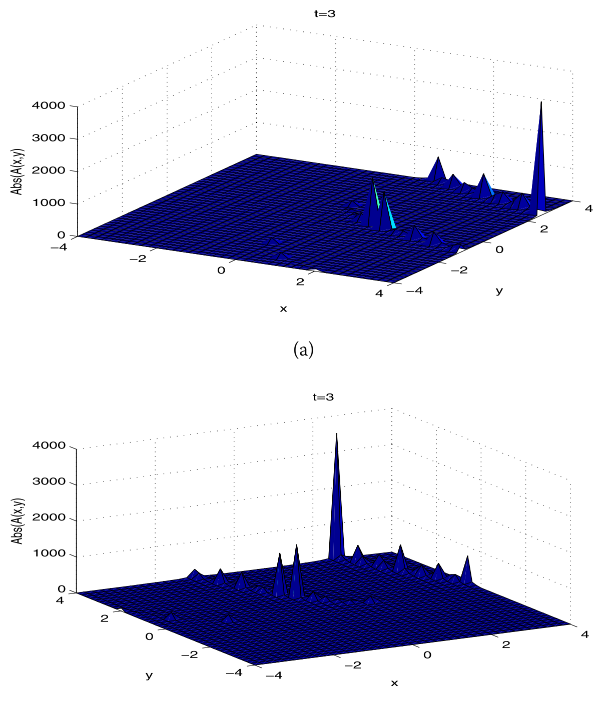


(a)



(b)

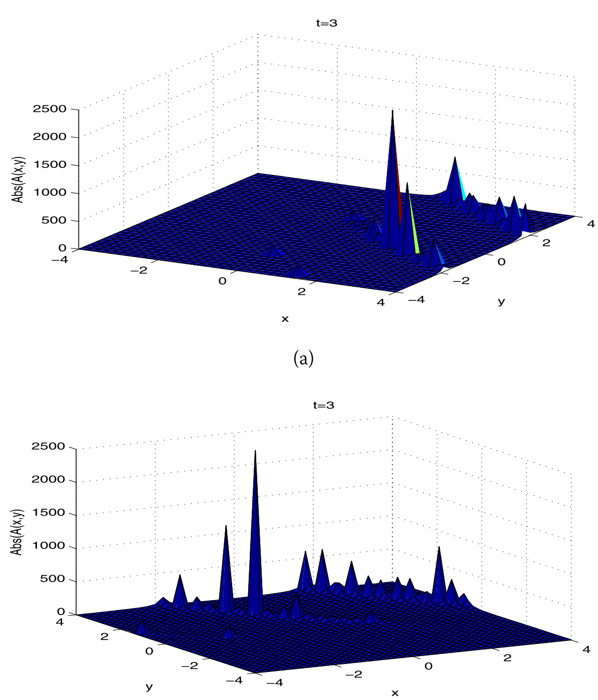
Figure 4 (**a**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 1$, $\beta = 0.25$, $\gamma = 1$. (**b**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 1$, $\beta = 1$, $\gamma = 0.25$



(a)

(b)

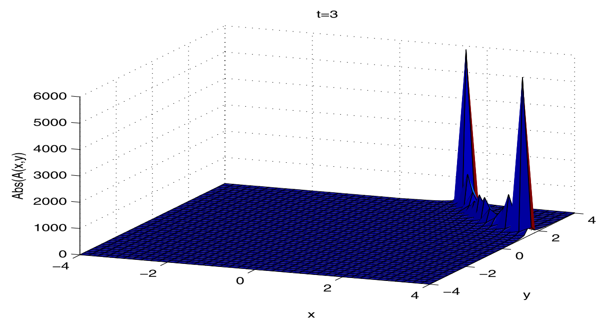
Figure 5 (**a**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 1$, $\beta = 0.5$, $\gamma = 1$. (**b**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 1$, $\beta = 1$, $\gamma = 0.5$



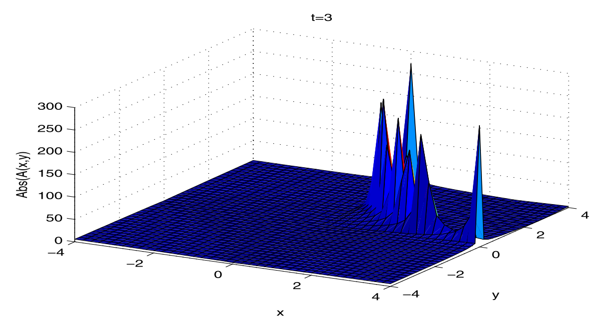
(a)

(b)

Figure 6 (**a**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 1$, $\beta = 0.75$, $\gamma = 1$. (**b**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 1$, $\beta = 1$, $\gamma = 0.75$



(a)



(b)

Figure 7 (**a**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 0.25$, $\beta = 0.25$, $\gamma = 0.25$. (**b**) The exact solution represented by Eq. (78), $t = 3$, $\alpha = 0.5$, $\beta = 0.5$, $\gamma = 0.5$

(IV) When $\lambda^2 - 4\mu = 0$ and $\mu \neq 0$, $\lambda \neq 0$, the rational function solutions are written

$$A_4(x,y,t) = \pm\frac{\sqrt{\lambda^2 + 4\mu}c_3}{\exp[2(a_2b_2^2 - a_1b_1^2)]}\xi\left(\frac{\lambda}{2\sqrt{c_2}} - \frac{1}{\sqrt{c_2}}\frac{\lambda^2(\xi + C)}{2(\lambda(\xi + C) + 2)}\right). \tag{81}$$

(V) When $\lambda^2 - 4\mu = 0$ and $\mu = 0$, $\lambda = 0$, the function solutions are written

$$A_5(x,y,t) = \pm\frac{\sqrt{\lambda^2 + 4\mu}c_3}{\exp[2(a_2b_2^2 - a_1b_1^2)]}\xi\left(-\frac{1}{\sqrt{c_2}}\frac{1}{\xi + C}\right). \tag{82}$$

Adopting the $\exp(-\phi(\xi))$ method, we get the exact solutions to the time–space fractional $(2 + 1)$ dimensional NLS equation. Based on Eq. (78), the following graphics are given (see Figures 1–7).

In the present numerical simulation, we draw some three-dimensional graphs of the exact solution (78). First we compare the effect of the fractional order on the solution at two different moments, $t = 2$ and $t = 3$, and At a certain moment $t = 3$, the change of solution is considered in the case of changing the fractional order of time and space. The behavior shows that the change of fractional order changes the nature of the solution and has a huge influence on the nonlinear propagation of the envelope gravity waves.

6 Conclusion

In this work, we first obtain the $(2 + 1)$ dimensional NLS equation for envelope gravity waves by using multiple-scale, perturbation method. The semi-inverse method is applied to get the Lagrangian of the (2 + 1) dimensional NLS equation. Then the fractional Euler–Lagrange equation with the fractional variational principle is derived. Based on the modified Riemann–Liouville fractional derivative, we can have the time–space fractional (2 + 1) dimensional NLS equation, which can better reflect the propagation characteristics of the envelope gravity wave and capture the nonlinear phenomenon in actual atmospheric movement. Using the Lie group analysis method, the conservation laws of the time–space fractional (2 + 1) dimensional NLS equation are discussed in depth. By adopting the $\exp(-\phi(\xi))$ method, we list the exact solutions to different cases. Taking one of the solutions as an example, we compare the effects of fractional order for the solution. After further research, we find that the fractional model is more practical and is a pioneering effort to find help for the study of actual atmospheric and ocean movement.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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