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q-Mittag-Leffler stability and Lyapunov direct method for differential systems with q-fractional order

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Abstract

In this paper, using the theory of q-fractional calculus, we deal with the q-Mittag-Leffler stability of q-fractional differential systems, and based on it, we analyze the direct Lyapunov method of q-fractional differential systems. Several sufficient criteria are established to guarantee the q-Mittag-Leffler stability and asymptotic stability for the differential systems with q-fractional order.

Keywords: q-fractional calculus; q-Mittag-Leffler functions; q-Mittag-Leffler stability; Lyapunov method

1 Introduction

The development of the theory of q-calculus can be dated back to the early 20th century in order to look for a better description of the phenomena having both discrete and continuous behaviors. The q-analog of fractional integrals and derivatives were first studied by Al-Salam [1–3] and then by Agrawal [4]. Recently, the q-fractional calculus has been payed more attention [5–8] because it serves as a bridge between fractional calculus and q-calculus.

In nonlinear systems, Lyapunov's direct method provides an effective way to analyze the stability of a system without explicitly solving the differential equations. Motivated by the application of fractional calculus in nonlinear systems Li,Chen, and Podlubny [9, 10] proposed the Mittag-Leffler stability and Lyapunov direct method, and a considerable number results of stability analysis for fractional systems have been reported; see [11–21] and the references therein. However, to our knowledge, the q-Mittag-Leffler stability of q-fractional dynamic systems has not been studied. In this paper, we propose the q-Mittag-Leffler stability and the q-fractional Lyapunov direct method with a hope to enrich the knowledge of the theory of q-fractional calculus. We also present a simple Lyapunov function to get the q-Mittag-Leffler stability for many q-fractional-order systems and show that q-fractional-order dynamical systems also do not have to decay exponentially for the system to be stable in the Lyapunov sense.

2 Preliminaries

2.1 Definitions and properties of *q*-caculus

This section is devoted to recall some essential definitions and properties of q-calculus [1–4, 8].

If $q \in R, 0 < q < 1$, a subset A of R is called q-geometric if $qx \in A$ whenever $x \in A$. If a subset A of R is q-geometric, then it contains all geometric sequences $\{xq^n\}_{n=0}^{\infty}$, $x \in A$.

Definition 2.1 ([8]) Let $f(x)$ be a real function defined on a q-geometric set A. The q-derivative is defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \in A \setminus \{0\}, \tag{1}$$

and

$$D_q f(x)|_{x=0} = \lim_{n\to\infty} \frac{f(q^n) - f(0)}{q^n}. \tag{2}$$

Setting $q \to 1$, we have $\lim_{q\to 1} D_q f(x) = f'(x)$.

Also, the q-integral is given as

$$\int_0^x f(t)\, d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f\bigl(q^n x\bigr), \quad x \in A, \tag{3}$$

and

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t, \quad a, b \in A. \tag{4}$$

We present here two basic properties concerning q-derivatives.

Property 1 ([7])

$$D_q(f \pm g)(x) = D_q f(x) \pm D_q g(x). \tag{5}$$

Property 2 ([7]) The q-Leibniz product rule is given by

$$D_q\bigl[g(x)f(x)\bigr] = g(qx)D_q f(x) + f(x)D_q g(x), \tag{6}$$

where D_q is the q-derivative.

The q-analogue of exponent $(s-t)^{(k)}$ is

$$(s-t)^{(0)} = 1, \qquad (s-t)^{(k)} = \prod_{j=0}^{k-1}\bigl(x - yq^j\bigr), \quad k \in N, x, y \in R.$$

Definition 2.2 ([7]) A q-analogue of the Riemann–Liouville fractional integral is defined as

$$I_{q,a}^{\alpha} f(x) = \int_0^x \frac{(x-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\, d_q s, \quad \alpha > 0. \tag{7}$$

If we let $q \to 1$, then the q-analogue of Riemann–Liouville fractional integral ${}_qI^{\alpha}_{q,a}f(x) \to I^{\alpha}_{a}f(x)$.

Definition 2.3 ([6]) The Riemann–Liouville type fractional q-derivative of a function $f : (0,\infty) \to R$ is defined by

$$\bigl(D^{\alpha}_{q,a}f\bigr)(x) = \begin{cases} (I^{-\alpha}_{q,a}f)(x), & \alpha \le 0, \\ (D^{[\alpha]}_{q,a}I^{[\alpha]-\alpha}_{q,a}f)(x), & \alpha > 0, \end{cases} \tag{8}$$

where $[\alpha]$ denotes the smallest integer greater than or equal to α.

Definition 2.4 ([6]) The Caputo type fractional q-derivative of a function $f : (0,\infty) \to R$ is define by

$$\bigl({}^{C}D^{\alpha}_{q,a}f\bigr)(x) = \begin{cases} (I^{-\alpha}_{q,a}f)(x), & \alpha \le 0, \\ (I^{[\alpha]-\alpha}_{q,a}D^{[\alpha]}_{q,a}f)(x), & \alpha > 0, \end{cases} \tag{9}$$

where $[\alpha]$ denotes the smallest integer greater or equal to α.

2.2 *q*-Mittag-Leffler function

Similar to the Mittag-Leffler function frequently used in the solutions of fractional-order equations, the functions frequently used in the solutions of q-fractional-order equations are the q-analogues of Mittag-Leffler functions defined as

$$e_{\alpha,\beta}(z,q) = \sum_{n=0}^{\infty} \frac{z^{n\alpha}}{\Gamma_q(n\alpha+\beta)} \quad \bigl(\bigl|z(1-q)^{\alpha}\bigr| < 1\bigr) \tag{10}$$

and

$$E_{\alpha,\beta}(z,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{\alpha n(n-1)}{2}} z^{n\alpha}}{\Gamma_q(n\alpha+\beta)} \quad (z \in C), \tag{11}$$

where $\alpha > 0$ and $\beta \in \mathcal{C}$. When $\beta = 1$, the functions $e_{\alpha,\beta}(z,q)$ and $E_{\alpha,\beta}(z,q)$ are defined by

$$e_{\alpha,1}(z,q) = \sum_{n=0}^{\infty} \frac{z^{n\alpha}}{\Gamma_q(n\alpha+1)} \quad \bigl(\bigl|z(1-q)^{\alpha}\bigr| < 1\bigr) \tag{12}$$

and

$$E_{\alpha,1}(z,q) = \sum_{n=0}^{\infty} \frac{q^{\frac{\alpha n(n-1)}{2}} z^{n\alpha}}{\Gamma_q(n\alpha+1)} \quad (z \in C). \tag{13}$$

2.3 *q*-Laplace transform of fractional *q*-integrals, *q*-derivatives, and *q*-Mittag-Leffler functions

Theorem 2.5 ([6]) *If* $f \in \mathcal{L}^1_q[0,a]$ *and* $\Phi(s) = {}_qL_sf(x)$, *then*

$${}_qL_sI^{\alpha}_{q}f(x) = \frac{(1-q)^{\alpha}}{s^{\alpha}}\Phi(s) \quad \textit{for } \alpha > 0. \tag{14}$$

If $n-1<\alpha\le n$ and $I_q^{n-\alpha}f(x)\in C_1^{(n)}[0,a]$, then let $\Phi(s)={}_qL_sf(x)$. The q-Laplace transform of the Riemann–Liouville fractional and the Caputo fractional q-derivatives are given by

$$ {}_qL_s^C D_q^\alpha f(x)=\frac{s^\alpha}{(1-q)^\alpha}\left(\Phi(s)-\sum_{r=0}^{n-1}D_q^r f\left(0^+\right)\frac{(1-q)^r}{s^{r+1}}\right) \tag{15}$$

and

$$ {}_qL_s D_q^\alpha f(x)=\frac{s^\alpha}{(1-q)^\alpha}\Phi(s)-\sum_{m=1}^{n}D_q^{\alpha-m}f\left(0^+\right)\frac{s^{m-1}}{(1-q)^m}. \tag{16}$$

Theorem 2.6 ([6]) *If* $|\frac{s}{1-q}|>|a|^{\frac{1}{Re(\alpha)}}$, *then*

$$ {}_qL_s\left(x^{\beta-1}e_{\alpha,\beta}(ax;q)\right)=\frac{1}{1-q}\frac{(\frac{s}{1-q})^{\alpha-\beta}}{(\frac{s}{1-q})^\alpha-a}. \tag{17}$$

Taking $\beta=1$, *we have*

$$ {}_qL_s\left(e_{\alpha,1}(ax;q)\right)=\frac{1}{1-q}\frac{(\frac{s}{1-q})^{\alpha-1}}{(\frac{s}{1-q})^\alpha-a}. \tag{18}$$

3 q-Mittag-Leffler stability and Lyapunov direct method for differential systems with q-fractional order

Consider the Caputo fractional nonautonomous system q-Mittag-Leffler stability of solutions of the following system:

$$\begin{cases} {}^C D_q^\alpha x(t)=f(t,x(t)),\\ x(t_0)=x_0, \end{cases} \tag{19}$$

where $t\ge t_0, t, t_0\in A, A=[t_0,t]_q, 0<\alpha<1$, and $f:[t_0,t]\times R\to R$ is a function with $f\in\mathcal{L}_{q,1}[t_0,t]$. Let $f(t,0)=0$, for all $t\in[t_0,t]_q$, so that system (19) admits the trivial solution.

Now we give some definitions that will be used in studying the q-Mittag-Leffler stability of (19).

Definition 3.1 The trivial solution $x(t)=0$ of (19) is said to be asymptotically stable if for all $\epsilon>0$ and $t_0\in A$, there exists $\delta=\delta(t_0,\epsilon)$ such that if $\|x_0\|<\delta$ implies that $\lim_{t\to\infty}\|x(t)\|=0$.

Definition 3.2 (q-Mittag-Leffler stability) The solution of (19) is said to be q-Mittag-Leffler stability if

$$\left\|x(t)\right\|\le\left\{m\left[x(t_0)\right]e_{q,\alpha}\left(-\lambda(t-t_0)^\alpha\right)\right\}^b, \tag{20}$$

where $t_q\in A$ is the initial time, $\alpha\in(0,1), \lambda\ge 0, b>0, m(0)=0, m(x)\ge 0$, and $m(x)$ is locally Lipschitz on $x\in B\subset R$ with Lipschitz constant m_0. We further assume that $t_0=0$.

Theorem 3.3 *Let $x = 0$ be an equilibrium point for system* (19), *and let $D \subset R$ be a domain containing origin. Let $V(t,x(t)) : [0,T] \times D \to R$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\beta_1 \|x(t)\|^a \le V\big(t,x(t)\big) \le \beta_2 \|x(t)\|^{ab}, \tag{21}$$

$$ {}_0^C D_q^\alpha V\big(t,x(t)\big) \le (-\beta_3)\|x(t)\|^{ab}, \tag{22}$$

where $t \in [0,T], t > 0$, $0 < \alpha < 1$, and β_1, β_2, β_3, a, and b are arbitrary positive constants. Then $x = 0$ is q-Mittag-Leffler stable.

Proof It follows from equations (19) and (20) that

$$ {}_0^C D_q^\alpha V\big(t,x(t)\big) \le -\frac{\beta_3}{\beta_2} V\big(t,x(t)\big). \tag{23}$$

There exists a nonnegative function $M(t)$ satisfying

$$ {}_0^C D_q^\alpha V\big(t,x(t)\big) + M(t) = -\frac{\beta_3}{\beta_2} V\big(t,x(t)\big). \tag{24}$$

Taking the q-Laplace transform of (24) gives

$$\frac{s^\alpha}{(1-q)^\alpha}\Big(V(s) - \frac{1}{s} V\big(0,x(0)\big) + M(s) = -\frac{\beta_3}{\beta_2} V(s), \tag{25}$$

where $V(s) =_q L_s\{V(t,x(t))\}$. It then follows that

$$\begin{aligned} V(s) &= V(0,x(0)) \frac{\frac{s^{\alpha-1}}{(1-q)^\alpha}}{\frac{s^\alpha}{(1-q)^\alpha} + \frac{\beta_3}{\beta_2}} - \frac{M(s)}{\frac{s^\alpha}{(1-q)^\alpha} + \frac{\beta_3}{\beta_2}} \\ &= V(0,x(0)) \frac{1}{1-q} \frac{(\frac{s}{1-q})^{\alpha-1}}{(\frac{s}{1-q})^\alpha + \frac{\beta_3}{\beta_2}} - (1-q)M(s) \frac{1}{1-q} \frac{1}{\frac{s^\alpha}{(1-q)^\alpha} + \frac{\beta_3}{\beta_2}}. \end{aligned} \tag{26}$$

It follows from the inverse Laplace transform that the unique solution of (24) is

$$V(t) = V\big(0,x(0)\big) e_{\alpha,1}\left(-\frac{\beta_3}{\beta_2} t; q\right) - \int_0^t M(\tau)(t-q\tau)^{\alpha-1} e_{\alpha,\alpha}\left(-\frac{\beta_3}{\beta_2}(t-q\tau)^\alpha; q\right) d\tau. \tag{27}$$

Since $0 < q < 1$, $M(t) \ge 0$, and $e_{\alpha,\alpha}(-\frac{\beta_3}{\beta_2}(t-q\tau)^\alpha; q)$ are nonnegative functions, we get

$$V(t) \le V\big(0,x(0)\big) e_{\alpha,1}\left(-\frac{\beta_3}{\beta_2} t; q\right). \tag{28}$$

Substitution of (28) into (21) yields

$$\|x(t)\| \le \left[\frac{V(0,x(0))}{\beta_1} e_{\alpha,1}\left(-\frac{\beta_3}{\beta_2} t; q\right)\right]^{\frac{1}{a}}, \tag{29}$$

where $\frac{V(0,x(0))}{\beta_1} > 0$ for $x(0) \ne 0$.

Let $m = \frac{V(0,x(0))}{\beta_1} \geq 0$. Then we have

$$\|x(t)\| \leq \left[m e_{\alpha,1}\left(-\frac{\beta_3}{\beta_2}t;q\right)\right]^{\frac{1}{a}}, \tag{30}$$

where $m = 0$ if and only if $x(0) = 0$. Because $V(t,x)$ is locally Lipschitz with respect to x and $V(0,x(0)) = 0$ if and only if $x(0)$, it follows that m is also Lipschitz with respect to $x(0)$ and $m(0)$, which implies the q-Mittag-Leffler stability.

In [8], an identity relation between the Caputo fractional q-derivative and the Riemann–Liouville fractional q-derivative is introduced:

$$f(t) = {}_{t_0}D_q^{\alpha} f(t) - {}_{t_0}D_q^{\alpha}\left(\sum_{k=0}^{n-1}\frac{D_q^k f(0^+)}{\Gamma_q(k+1)}x^k\right), \tag{31}$$

where $\alpha > 0$ and $n = [\alpha] + 1$. When $0 < \alpha < 1$, we have

$${}_{t_0}^{C}D_q^{\alpha} f(t) = {}_{t_0}D_q^{\alpha} f(t) - \frac{(t-t_0)_q^{\alpha}}{\Gamma_q(1-\alpha)} f(t_0). \tag{32}$$

□

Theorem 3.4 *If the assumptions in Theorem* 3.3 *are satisfied except replacing* ${}_{t_0}^{C}D_q^{\alpha}$ *by* ${}_{t_0}D_q^{\alpha}$, *then the trivial solution of* (19) *is q-Mittag-Leffler stable.*

Proof From (32) we have

$${}_{0}^{C}D_q^{\alpha} V\bigl(t,x(t)\bigr) = {}_{0}D_q^{\alpha} V\bigl(t,x(t)\bigr) - \frac{t_q^{\alpha}}{\Gamma_q(1-\alpha)} V\bigl(0,x(0)\bigr) \quad \text{for } t \in [0,T], \tag{33}$$

and since $V(0,x(0)) \geq 0$ and $\frac{t_q^{\alpha}}{\Gamma_q(1-\alpha)} \geq 0$, we obtain the result.

Furthermore, if we extend the Lyapunov direct method to the case of q-fractional-order systems, then the asymptotic stability of the corresponding systems can be obtained. The following properties of the q-Mittag-Leffler function and the class-K functions are applied to analysis of the q-fractional Lyapunov direct method. □

Remark 3.5 Since

$$D_q e_{\alpha,1}\bigl((-\lambda t;q)\bigr) = -\lambda t^{\alpha-1} e_{\alpha,\alpha-1}(-\lambda t;q), \tag{34}$$

where $t > 0$, $0 < \alpha < 1$, $\lambda > 0$, the q-Mittag-Leffler function $e_{\alpha,1}(((-\lambda t)^{\alpha};q))$ is decreasing, so the q-Mittag-Leffler stability implies the asymptotic stability.

4 *q*-Mittag-Leffler stability of linear systems with *q*-fractional order

In this section, we present a new result that allows us to find Lyapunov candidate functions for demonstrating the q-Mittag-Leffler of many fractional-order systems using the results of the Lyapunov direct method in Theorem 3.3.

Theorem 4.1 *Let $x(t) \in R$ be defined in a suitable q-geometric set $A = [0, a]_q$, $D_q x(t) \in C_q[0, q]$ (where $C_q[0, a]$ is the space of all continuous functions on the interval $[0, a]$). Then, for any time $t > 0$, $t \in A$,*

$$ {}_0^C D_q^\alpha x^2(t) \leq \bigl(x(t) + x(tq)\bigr) {}_0^C D_q^\alpha x(t), \quad 0 < \alpha < 1. \tag{35} $$

Proof Proving expression (35) is equivalent to proving that

$$ \bigl(x(t) + x(tq)\bigr) {}_0^C D_q^\alpha x(t) - {}_0^C D_q^\alpha x^2(t) \geq 0. \tag{36} $$

Using Definition 2.2 and Definition 2.4, $(x(t) + x(tq)) {}_0^C D_q^\alpha x(t)$ and ${}_0^C D_q^\alpha x^2(t)$ can be written as

$$ \bigl(x(t) + x(tq)\bigr) {}_0^C D_q^\alpha x(t) = \bigl(x(t) + x(tq)\bigr) \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha} D_q x(s) \, d_q s \tag{37} $$

and

$$ {}_0^C D_q^\alpha x^2(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha} \bigl(x(s) + x(qs)\bigr) D_q x(s) \, d_q s. \tag{38} $$

So, the left side of expression (36) can be written as

$$ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha} \bigl[\bigl(x(t) - x(s)\bigr) + \bigl(x(tq) - x(sq)\bigr)\bigr] D_q x(s) \, d_q s. \tag{39} $$

Now, let us define the axillary variable $y(s) = x(t) - x(s)$, which implies that

$$ \begin{aligned} D_q y^2(s) &= \bigl(y(s) + y(sq)\bigr) D_q y(s) \\ &= -\bigl[\bigl(x(t) - x(s)\bigr) + \bigl(x(tq) - x(sq)\bigr)\bigr] D_q x(s). \end{aligned} \tag{40} $$

In this way, expression (39) can be written as

$$ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha} \, d_q y^2(s) = -\frac{1}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha} \bigl[y(s) + y(sq)\bigr] D_q y(s) \, d_q s. \tag{41} $$

Since $x(t)$ is regular at zero, using the rule of q-integration by parts, expression (41) becomes

$$ \begin{aligned} \int_{t_0}^t (t - qs)^{-\alpha} \, d_q y^2(s) &= y^2(t)(t - qt)^{-\alpha} - \Gamma(1-\alpha) y^2(0) t^{-\alpha} \\ &\quad - \alpha q \int_0 (t - qs)^{-\alpha-1} y^2(qs) \, d_q(s). \end{aligned} \tag{42} $$

Since $y^2(t) = (x(t) - x(s))^2 = 0$, it follows that

$$ \begin{aligned} &{}_0^C D_q^\alpha x(t) - {}_0^C D_q^\alpha x^2(t) \\ &\quad = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t - qs)^{-\alpha} [(x(t) - x(s)) + (x(tq) - x(sq))] D_q x(s) \, d_q s \end{aligned} $$

$$\begin{aligned}&= -\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-qs)^{-\alpha}\,d_q y^2(s)\\&= \frac{1}{\Gamma(1-\alpha)}y^2(0)t^{-\alpha} + \frac{\alpha q}{\Gamma(1-\alpha)}\int_0^t (t-qs)^{-\alpha-1}y^2(s)\,d_q(s)\\&\geq 0.\end{aligned} \tag{43}$$

This concludes the proof. □

Corollary 4.2 *For the q-fractional-order system*

$${}_0^C D_q^\alpha x(t) = f\bigl(t, x(t)\bigr), \tag{44}$$

where $\alpha \in (0,1)$, $x = 0$ *is the equilibrium point, and* $D_q x(t) \in C_q[0,a], f(t,x(t)) \in \$_q^1[0,a]$. *If*

$$\bigl(x(t) + x(tq)\bigr) f\bigl(t, x(t)\bigr) \leq 0, \quad \forall x \in A, \tag{45}$$

then the origin of system (44) *is q-Mittag-Leffler stable.*

Proof Let us propose the following Lyapunov candidate function:

$$V\bigl(t, x(t)\bigr) = x^2. \tag{46}$$

Applying Theorem 4.1 results in

$${}_0^C D_q^\alpha V\bigl(t, x(t)\bigr) \leq \bigl(x(t) + x(tq)\bigr) {}_0^C D_q^\alpha x(t) \leq \bigl(x(t) + x(tq)\bigr) f\bigl(t, x(t)\bigr) \leq 0, \tag{47}$$

and thus the origin of system (44) is q-Mittag-Leffler stable. □

Proposition 4.3 *For the system*

$${}_0^C D_q^\alpha x(t) = -x(t) - x(tq), \tag{48}$$

where $0 < \alpha < 1$ *and* $D_q x(t) \in C_q[0,a]$, *the origin of system* (44) *is q-Mittag-Leffler stable.*

Proof Let $V(x(t)) = x^2(t)$. Then

$$\begin{aligned}{}_0^C D_q^\alpha x^2(t) &\leq \bigl(x(t) + x(tq)\bigr) {}_0^C D_q^\alpha x(t)\\&= -\bigl(x(t) + x(tq)\bigr)^2 \leq -\bigl\|x(t)\bigr\|^2.\end{aligned} \tag{49}$$

So we can conclude that the trivial solution of system (48) is asymptotically stable.

Furthermore, from the expression of exact solution for (48) using two q-analogues of the Mittag-Leffler functions defined by (12) and (13),

$$x(t) = c_1 e_{(\alpha,1)}(-x, q) + c_2 E_{(\alpha,1)}(-x, q), \tag{50}$$

and the properties of these two functions the asymptotical stability can also be derived. □

5 Conclusions

In this paper, we studied the stability of systems with q-fractional order. We proposed the definition of q-Mittag-Leffler stability, presented sufficient criteria of q-Mittag-Leffler stability and the q-fractional Lyapunov direct method of nonlinear systems with q-fractional order. Meanwhile, the q-fractional Lyapunov candidate functions for demonstrating the q-Mittag-Leffler stability of many q-fractional-order systems were discussed. With the rapid development of advanced applied science, we believe that many other study subjects of the q-fractional calculus and q-fractional dynamical systems will attract more attention of researchers. In our following study, we will still focus on the stability problem of q-fractional differential equations in a variety of different forms.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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