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Global stability of an SIS epidemic model with feedback mechanism on networks

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Abstract

We study the global stability of endemic equilibrium of an SIS epidemic model with feedback mechanism on networks. The model was proposed by J. Zhang and J. Sun (Physica A 394:24–32, 2014), who obtained the local asymptotic stability of endemic equilibrium. Our main purpose is to show that if the feedback parameter is sufficiently large or if the basic reproductive number belongs to the interval $(1, 2]$, then the endemic equilibrium is globally asymptotically stable. We also present numerical simulations to illustrate the theoretical results.

Keywords: Epidemic model; Complex network; Feedback mechanism; Global stability

1 Introduction

In recent years, the spreading dynamics of epidemic models on complex networks have been extensively studied by many researchers [2–13]. In the famous works [2, 3], Pastor-Satorras and Vespignani established a dynamical mean-field equation based on the SIS model and showed that the finite-size effects induce an epidemic threshold, which approaches zero at increasing sizes, so that if the effective spreading rate is above this threshold, then the infection spreads and becomes endemic. For such significant finding, until recently, a mathematically rigorous proof was given by Wang and Dai [14]. Stability analysis is an important tool to understand the spreading dynamics of epidemic models on complex networks. Recently, there have been many studies concerning the stability and asymptotic behavior of epidemic models on complex networks [15–23].

We note that in most of the models mentioned, the initiative response of people is not considered when epidemic diseases prevail. In fact, as soon as an epidemic outbreaks, people are more cautious and reduce contacts with other people consciously. Clearly, the feedback mechanism can affect the contacts among people. To investigate the efficiency of feedback mechanism, new models based on the SIS model were established [1, 24–26]. Under the assumption that all individuals understand the instant situation of the propagation and can estimate the possibility of contacting with infected individuals, Zhang and Sun [1] proposed the following model with a feedback mechanism:

$$\begin{cases} \frac{dS_k(t)}{dt} = b(S_k(t) + I_k(t)) - dS_k(t) - \lambda k S_k(t)(1 - \alpha \Theta(t))\Theta(t) + \gamma I_k(t), \\ \frac{dI_k(t)}{dt} = \lambda k S_k(t)(1 - \alpha \Theta(t))\Theta(t) - (\gamma + d)I_k(t), \quad k = 1, 2, \dots, n, \end{cases} \quad (1.1)$$

where $S_k(t)$ and $I_k(t)$ represent the relative densities of susceptible and infected nodes with given degree k , respectively, n is the maximum degree of all nodes, $\Theta(t)$ is the probability that any given link pointing to an infected individual and has the form $\Theta(t) = \langle k \rangle^{-1} \sum_{k=1}^n kP(k)I_k(t)$, where $P(k)$ is the probability that a node has degree k with $\langle k \rangle = \sum_{k=1}^n kP(k)$, b is the natural birth rate, d is the death rate, λ is the transmission rate, γ is recovery rate, and the parameter $\alpha > 0$ is called the “fear factor.” The term $\lambda k S_k(t)(1 - \alpha \Theta(t))\Theta(t)$ represents the proportion of individuals that acquire infection and become infected individuals. This means that, on one hand, the higher the fear degree of people to the epidemic disease, the lower the spreading speed; on the other hand, when Θ is larger than a certain threshold, the speed of the propagation decreases based on the feedback, which is consistent with the actual prevalence law of infectious diseases.

Like in [1], we assume that $b = d$ (i.e. the birth rate equals the death rate), and the initial conditions of system (1.1) satisfy

$$0 \leq S_k(0), I_k(0) \leq 1, \quad S_k(0) + I_k(0) = 1, \quad k = 1, 2, \dots, n. \tag{1.2}$$

Thus $S_k(t) + I_k(t) \equiv 1$. This leads to the following system:

$$\begin{cases} \frac{dS_k(t)}{dt} = b - bS_k(t) - \lambda k S_k(t)(1 - \alpha \Theta(t))\Theta(t) + \gamma I_k(t), \\ \frac{dI_k(t)}{dt} = \lambda k S_k(t)(1 - \alpha \Theta(t))\Theta(t) - (\gamma + b)I_k(t), \quad k = 1, 2, \dots, n. \end{cases} \tag{1.3}$$

The second equation of (1.3) can be written as

$$\frac{dI_k(t)}{dt} = \lambda k(1 - I_k(t))(1 - \alpha \Theta(t))\Theta(t) - (\gamma + b)I_k(t), \quad k = 1, 2, \dots, n. \tag{1.4}$$

They obtained the basic reproductive number $R^0 = \frac{\lambda \langle k^2 \rangle}{(\gamma + b) \langle k \rangle}$, where $\langle k^2 \rangle = \sum_{k=1}^n k^2 P(k)$, and proved that if $R^0 < 1$, then the disease-free equilibrium is globally asymptotically stable, and if $R^0 > 1$, then there exists a unique positive equilibrium $E^*(S_1^*, I_1^*, S_2^*, I_2^*, \dots, S_n^*, I_n^*)$ determined by

$$\begin{cases} I_k^* = \frac{\lambda k(1 - \alpha \Theta^*)\Theta^*}{\gamma + b + \lambda k(1 - \alpha \Theta^*)\Theta^*}, \\ S_k^* = \frac{\gamma + b}{\gamma + b + \lambda k(1 - \alpha \Theta^*)\Theta^*}, \quad k = 1, 2, \dots, n, \end{cases} \tag{1.5}$$

where $\Theta^* = \langle k \rangle^{-1} \sum_{j=1}^n jP(j)I_j^*$ is the unique positive root of the equation

$$f(\Theta) := 1 - \frac{1}{\langle k \rangle} \sum_{k=1}^n \frac{\lambda k^2 P(k)(1 - \alpha \Theta)}{b + \gamma + \lambda k(1 - \alpha \Theta)\Theta} = 0. \tag{1.6}$$

Moreover, E^* is locally asymptotically stable. It should be pointed out that in the special case $\alpha = 0$, which corresponds to the original model by Pastor-Satorras and Vespignani [2, 3], the global stability of endemic equilibrium was shown in [27]. As far as we know, however, the global asymptotic behavior of the endemic equilibrium remained unsolved for $\alpha > 0$. In the present paper, we prove that the endemic equilibrium of system (1.3) is globally asymptotically stable if $\alpha \geq \frac{\lambda n}{4(b + \gamma)} (\frac{R^0 - 1}{R^0})^2$ or if $R^0 \in (1, 2]$ and $\alpha > 0$.

We will give a remark on Θ^* , which plays an important role in proving the following Theorems 3.1.

Remark 1.1 Following the arguments in [1, Section 2], we can see that if $R^0 > 1$, then $\Theta^* < \alpha^{-1}$. In fact, if $\{(S_k^*, I_k^*)\}_{k=1}^n$ is a positive equilibrium of system (1.3), it must satisfy (1.5). It follows from $0 < I_k^* S_k^* = \frac{(\gamma+b)\lambda k(1-\alpha\Theta^*)\Theta^*}{[\gamma+b+\lambda k(1-\alpha\Theta^*)\Theta^*]^2}$ that $\Theta^* < \alpha^{-1}$. On the other hand, since $f(\alpha^{-1}) = 1 > 0$ and $f(0) = 1 - R^0 < 0$, we find that Θ^* is indeed the unique root of the equation $f(\Theta) = 0$ in $(0, \alpha^{-1})$ because $f'(\Theta) > 0$ in $(0, \alpha^{-1})$.

We further claim that $\Theta^* \leq \frac{R^0-1}{\alpha R^0}$ if $R^0 > 1$. By (1.5) we have $(b + \gamma)I_k^* = \lambda k(1 - I_k^*)(1 - \alpha\Theta^*)\Theta^* \leq \lambda k(1 - \alpha\Theta^*)\Theta^*$, so $b + \gamma \leq \frac{\lambda \langle k^2 \rangle}{\langle k \rangle} (1 - \alpha\Theta^*)$, which implies the claim.

This paper is organized as follows. In Section 2, we show that the disease is permanent if $R^0 > 1$. In Section 3, we prove the global asymptotic stability of the endemic equilibrium for sufficiently large α by using a Lyapunov function. In Section 4, we show the global asymptotic stability of the endemic equilibrium for $R^0 \in (1, 2]$ and $\alpha > 0$ by the monotone iterative technique. In Section 5, we present numerical simulations. In Section 6, we give our conclusions.

2 Permanence of disease

In this section, we state our main results as follows.

Theorem 2.1 *Let $R^0 > 1$. Assume that $\{(S_k, I_k)\}_{k=1}^n$ is a nonnegative solution of system (1.3) with the initial conditions (1.2) and $\Theta(0) > 0$. Then $\Theta(t) > 0$ for all $t \geq 0$, and*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \Theta(t) &\geq \frac{(R^0 - 1)}{\alpha R^0 + \lambda n(\gamma + b)^{-1}} =: \theta_\alpha, \\ \limsup_{t \rightarrow \infty} \Theta(t) &\leq \frac{(R^0 - 1)}{\alpha R^0}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \liminf_{t \rightarrow \infty} I_k(t) &\geq \frac{\lambda k \theta_\alpha}{(\gamma + b)R^0 + \lambda k \theta_\alpha}, \\ \limsup_{t \rightarrow \infty} I_k(t) &\leq \frac{\lambda k}{4\alpha(\gamma + b) + \lambda k}. \end{aligned} \tag{2.2}$$

Proof From (1.4) we have

$$\begin{aligned} \frac{d\Theta}{dt} &= \frac{\lambda}{\langle k \rangle} \sum_{j=1}^n j^2 P(j)(1 - I_j)(1 - \alpha\Theta)\Theta - (\gamma + b)\Theta \\ &= \left(\frac{\lambda \langle k^2 \rangle}{\langle k \rangle} - \gamma - b \right) \Theta - \frac{\alpha \lambda \langle k^2 \rangle}{\langle k \rangle} \Theta^2 - \frac{\lambda}{\langle k \rangle} \sum_{j=1}^n j^2 P(j) I_j \Theta + \frac{\alpha \lambda}{\langle k \rangle} \sum_{j=1}^n j^2 P(j) I_j \Theta^2 \\ &=: Y(t)\Theta(t), \end{aligned} \tag{2.3}$$

which gives that $\Theta(t) = \Theta(0) \exp\{\int_0^t Y(s) ds\} > 0$ for all $t \geq 0$. By [1, Theorem 2.1], $0 \leq I_k \leq 1$ for $k = 1, 2, \dots, n$, thus we deduce from (2.3) that

$$\begin{aligned} \frac{d\Theta}{dt} &\geq \left(\frac{\lambda \langle k^2 \rangle}{\langle k \rangle} - \gamma - b\right)\Theta - \frac{\alpha \lambda \langle k^2 \rangle}{\langle k \rangle} \Theta^2 - \frac{\lambda}{\langle k \rangle} \sum_{j=1}^n j^2 P(j) I_j \Theta \\ &\geq \left(\frac{\lambda \langle k^2 \rangle}{\langle k \rangle} - \gamma - b\right)\Theta - \frac{\alpha \lambda \langle k^2 \rangle}{\langle k \rangle} \Theta^2 - \lambda n \Theta^2 \\ &= (\gamma + b) \{ (R^0 - 1)\Theta - [\alpha R^0 + \lambda n(\gamma + b)^{-1}]\Theta^2 \}, \end{aligned}$$

which implies $\liminf_{t \rightarrow \infty} \Theta(t) \geq \theta_\alpha$.

We further show that $\limsup_{t \rightarrow \infty} \Theta(t) \leq \frac{(R^0 - 1)}{\alpha R^0}$. Since $0 \leq I_k \leq 1$ for $k = 1, 2, \dots, n$, we can deduce from (2.3) that $\frac{d^2\Theta(t)}{dt^2}$ is bounded in $(0, \infty)$. By Corollary C in the Appendix, there is a sequence $\{t_m\}$, increasing to infinity, such that

$$\theta_\alpha \leq \lim_{m \rightarrow \infty} \Theta(t_m) = \limsup_{t \rightarrow \infty} \Theta(t) =: \theta, \quad \lim_{m \rightarrow \infty} \Theta'(t_m) = 0.$$

On the other hand, since $\{I_k(t_m)\}_{m=1}^\infty \in [0, 1]$ for every k , there is a subsequence of $\{t_m\}_{m=1}^\infty$, denoted by itself, and some constants $i_k \in [0, 1]$ for $k = 1, 2, \dots, n$ such that $I_k(t_m) \rightarrow i_k$ ($m \rightarrow \infty$). Passing to the limit as $m \rightarrow \infty$ in (2.3) at $t = t_m$ yields

$$(\gamma + b)\theta = \frac{\lambda}{\langle k \rangle} \sum_{j=1}^n j^2 P(j) (1 - i_k)(1 - \alpha\theta)\theta \leq \frac{\lambda \langle k^2 \rangle}{\langle k \rangle} (1 - \alpha\theta)\theta. \tag{2.4}$$

This implies the required result.

Finally, we prove (2.2). Note that $\frac{d^2 I_k(t)}{dt^2}$ is bounded in $(0, \infty)$ for every k . Based on Corollary C in the Appendix and (2.1), we can derive by an argument similar to the previous one that, for any fixed $k \in \{1, 2, \dots, n\}$, there are two sequences $\{t_m^{(k)}\}$ and $\{s_m^{(k)}\}$, increasing to infinity, and some constants $\bar{\theta}_k, \underline{\theta}_k \in [\theta_\alpha, \frac{(R^0 - 1)}{\alpha R^0}]$ such that

$$\begin{aligned} \lim_{m \rightarrow \infty} I_k(t_m^{(k)}) &= \limsup_{t \rightarrow \infty} I_k =: \bar{i}_k \in [0, 1], & \lim_{m \rightarrow \infty} I_k'(t_m^{(k)}) &= 0, \\ \lim_{m \rightarrow \infty} I_k(s_m^{(k)}) &= \liminf_{t \rightarrow \infty} I_k =: \underline{i}_k \in [0, 1], & \lim_{m \rightarrow \infty} I_k'(s_m^{(k)}) &= 0, \\ \lim_{m \rightarrow \infty} \Theta(t_m^{(k)}) &= \bar{\theta}_k, & \lim_{m \rightarrow \infty} \Theta(s_m^{(k)}) &= \underline{\theta}_k. \end{aligned} \tag{2.5}$$

Passing to the limits as $m \rightarrow \infty$ in (1.4) at $t = t_m^{(k)}$ and $t = s_m^{(k)}$, respectively, we obtain

$$\begin{aligned} 0 &= \lambda k(1 - \bar{i}_k)(1 - \alpha \bar{\theta}_k) \bar{\theta}_k - (\gamma + b) \bar{i}_k, \\ 0 &= \lambda k(1 - \underline{i}_k)(1 - \alpha \underline{\theta}_k) \underline{\theta}_k - (\gamma + b) \underline{i}_k, \end{aligned}$$

and therefore

$$\bar{i}_k = \frac{\lambda k(1 - \alpha \bar{\theta}_k) \bar{\theta}_k}{\gamma + b + \lambda k(1 - \alpha \bar{\theta}_k) \bar{\theta}_k}, \quad \underline{i}_k = \frac{\lambda k(1 - \alpha \underline{\theta}_k) \underline{\theta}_k}{\gamma + b + \lambda k(1 - \alpha \underline{\theta}_k) \underline{\theta}_k}. \tag{2.6}$$

By the inequality $x(1 - \alpha x) \leq \frac{1}{4\alpha}$ for $x \in \mathbb{R}$ we have

$$\bar{i}_k \leq \frac{\frac{\lambda k}{4\alpha}}{\gamma + b + \frac{\lambda k}{4\alpha}} = \frac{\lambda k}{4\alpha(\gamma + b) + \lambda k}, \quad k = 1, 2, \dots, n.$$

Since $\theta_\alpha \leq \underline{\theta}_k \leq \frac{(R^0-1)}{\alpha R^0}$, we have $\bar{i}_k \geq \frac{\frac{\lambda k \theta_\alpha}{R^0}}{b + \gamma + \frac{\lambda k \theta_\alpha}{R^0}}$. Thus, the proof is completed. □

Remark 2.1 When $R^0 = 1$ (i.e., $\gamma + b = \frac{\lambda \langle k^2 \rangle}{\langle k \rangle}$), we can obtain (2.4) by a similar argument, where $\theta \in [0, 1]$, which implies that $\limsup_{t \rightarrow \infty} \Theta = \theta = 0$, so $\lim_{t \rightarrow \infty} I_k(t) = 0$ for $k = 1, 2, \dots, n$, i.e., the disease-free equilibrium is globally attractive. This can also be seen from (2.1) by letting $R^0 \rightarrow 1^+$. This result is a supplement of [1, Theorem 3.1].

3 Global stability for large α

The main result of this section can be stated as follows.

Theorem 3.1 *Let $R^0 > 1$ and $\Theta(0) > 0$. If $\alpha \geq \frac{\lambda n}{4(b+\gamma)} (\frac{R^0-1}{R^0})^2$, then the endemic equilibrium E^* of system (1.3) is globally asymptotically stable, that is, the disease becomes endemic.*

Proof According to [1, Theorem 3.2], the endemic equilibrium E^* is locally asymptotically stable. To prove the theorem, it suffices to show that E^* is globally attractive.

Let us consider a nonnegative solution $\{(S_k, I_k)\}_{k=1}^n$ and define

$$V(t) = \frac{1}{2} \sum_{k=1}^n a_k (S_k(t) - S_k^*)^2 + \Theta(t) - \Theta^* - \Theta^* \ln \frac{\Theta(t)}{\Theta^*},$$

where $a_k = \frac{kP(k)}{\langle k \rangle S_k^*}$. Calculating the derivative of $V(t)$ along the solution, we have

$$\frac{dV}{dt} = \sum_{k=1}^n a_k (S_k - S_k^*) \frac{dS_k}{dt} + \frac{\Theta - \Theta^*}{\Theta} \frac{d\Theta}{dt} =: V_1 + V_2.$$

Using the first equation of (1.3) and the identity $\gamma + b = (\gamma + b)S_k^* + \lambda k S_k^* (1 - \alpha \Theta^*) \Theta^*$, we have

$$V_1 = \sum_{k=1}^n a_k (S_k - S_k^*) \{ (\gamma + b)(S_k^* - S_k) + \lambda k [S_k^* (1 - \alpha \Theta^*) \Theta^* - S_k (1 - \alpha \Theta) \Theta] \}. \quad (3.1)$$

Note that

$$\begin{aligned} & S_k^* (1 - \alpha \Theta^*) \Theta^* - S_k (1 - \alpha \Theta) \Theta \\ &= (1 - \alpha \Theta) \Theta (S_k^* - S_k) - S_k^* (1 - 2\alpha \Theta^*) (\Theta - \Theta^*) + \alpha S_k^* (\Theta - \Theta^*)^2. \end{aligned}$$

Substituting this into (3.1) yields

$$\begin{aligned} V_1 = & - \sum_{k=1}^n (\gamma + b) a_k (S_k - S_k^*)^2 - (1 - \alpha \Theta) \Theta \sum_{k=1}^n \lambda k a_k (S_k - S_k^*)^2 \\ & - \sum_{k=1}^n \lambda k a_k S_k^* (1 - 2\alpha \Theta^*) (S_k - S_k^*) (\Theta - \Theta^*) + \sum_{k=1}^n \alpha \lambda k a_k S_k^* (S_k - S_k^*) (\Theta^* - \Theta)^2 \end{aligned}$$

$$\begin{aligned}
 &= -\sum_{k=1}^n (\gamma + b)a_k(S_k - S_k^*)^2 - (1 - \alpha\Theta)\Theta \sum_{k=1}^n \lambda k a_k(S_k - S_k^*)^2 \\
 &\quad + \sum_{k=1}^n \alpha \lambda k a_k S_k^* \Theta^* (S_k - S_k^*)(\Theta - \Theta^*) - \sum_{k=1}^n \alpha \lambda k a_k (S_k^*)^2 (\Theta - \Theta^*)^2 \\
 &\quad - \sum_{k=1}^n \lambda k a_k S_k^* [(1 - \alpha\Theta^*)(S_k - S_k^*)(\Theta - \Theta^*) - \alpha S_k (\Theta - \Theta^*)^2]. \tag{3.2}
 \end{aligned}$$

Using (2.3) and the identity $\gamma + b = \frac{\lambda}{\langle k \rangle} \sum_{k=1}^n k^2 P(k) S_k^* (1 - \alpha\Theta^*)$, we have

$$\begin{aligned}
 V_2 &= (\Theta - \Theta^*) \left[-(\gamma + b) + \frac{\lambda}{\langle k \rangle} \sum_{k=1}^n k^2 P(k) S_k (1 - \alpha\Theta) \right] \\
 &= (\Theta - \Theta^*) \frac{\lambda}{\langle k \rangle} \sum_{k=1}^n k^2 P(k) [S_k (1 - \alpha\Theta) - S_k^* (1 - \alpha\Theta^*)] \\
 &= \frac{\lambda}{\langle k \rangle} \sum_{k=1}^n k^2 P(k) [(1 - \alpha\Theta^*)(S_k - S_k^*)(\Theta - \Theta^*) - \alpha S_k (\Theta - \Theta^*)^2]. \tag{3.3}
 \end{aligned}$$

Adding (3.2) and (3.3) and noticing that $a_k = \frac{kP(k)}{\langle k \rangle S_k^*}$, we obtain

$$\begin{aligned}
 \frac{dV}{dt} &= -\sum_{k=1}^n (\gamma + b)a_k(S_k - S_k^*)^2 - (1 - \alpha\Theta)\Theta \sum_{k=1}^n \lambda k a_k(S_k - S_k^*)^2 \\
 &\quad + \sum_{k=1}^n \alpha \lambda k a_k S_k^* \Theta^* (S_k - S_k^*)(\Theta - \Theta^*) - \sum_{k=1}^n \alpha \lambda k a_k (S_k^*)^2 (\Theta - \Theta^*)^2 \\
 &= -(1 - \alpha\Theta)\Theta \sum_{k=1}^n \lambda k a_k(S_k - S_k^*)^2 \\
 &\quad - \sum_{k=1}^n a_k [(\gamma + b)Z_k^2 - \alpha \lambda k S_k^* \Theta^* Z_k X + \alpha \lambda k (S_k^*)^2 X^2], \tag{3.4}
 \end{aligned}$$

where $Z_k = S_k - S_k^*$ and $X = \Theta - \Theta^*$. Note that $\Theta^* \leq \frac{R^0 - 1}{\alpha R^0}$ (see Remark 1.1). Then if $\alpha \geq \frac{\lambda n}{4(b + \gamma)} (\frac{R^0 - 1}{R^0})^2$, then we have

$$\begin{aligned}
 \Delta &:= (\alpha \lambda k S_k^* \Theta^*)^2 - 4(\gamma + b)\alpha \lambda k (S_k^*)^2 \\
 &\leq (\lambda k S_k^*)^2 \left(\frac{R^0 - 1}{R^0} \right)^2 - 4(\gamma + b)\alpha \lambda k (S_k^*)^2 \\
 &\leq \lambda k (S_k^*)^2 \left[\lambda n \left(\frac{R^0 - 1}{R^0} \right)^2 - 4\alpha(\gamma + b) \right] \leq 0.
 \end{aligned}$$

This implies that

$$(\gamma + b)Z_k^2 - \alpha \lambda k S_k^* \Theta^* Z_k X + \alpha \lambda k (S_k^*)^2 X^2 \geq 0, \quad k = 1, 2, \dots, n.$$

It follows from (3.4) that

$$\frac{dV}{dt} \leq -(1 - \alpha\Theta)\Theta \sum_{k=1}^n \lambda k a_k (S_k - S_k^*)^2, \quad \forall t > 0.$$

By (2.1), there exists some $T > 0$ such that $\Theta(t) < \alpha^{-1}$ for all $t \geq T$. Consequently,

$$\frac{dV}{dt} \leq -(1 - \alpha\Theta)\Theta \sum_{k=1}^n \lambda k a_k (S_k - S_k^*)^2 \leq 0, \quad \forall t \geq T.$$

Therefore, the only invariant set in $\{\frac{dV}{dt} = 0, t \geq T\}$ is the singleton $\{E^*\}$. According to the LaSalle invariant principle [28], E^* is globally attractive, so E^* is globally asymptotically stable. The proof is completed. \square

4 Global stability for $R^0 \in (1, 2]$

Theorem 4.1 *Let $R^0 > 1$ and $\Theta(0) > 0$. If $R^0 \leq 2$, then the endemic equilibrium E^* of system (1.3) is globally asymptotically stable for any $\alpha > 0$.*

Proof According to [1, Theorem 3.2], the endemic equilibrium E^* is locally asymptotically stable. To prove the theorem, it suffices to show that

$$\lim_{t \rightarrow \infty} I_k(t) = I_k^*, \quad k = 1, 2, \dots, n.$$

To this end, we will use an iterative technique and define the sequence $\{u_k^{(m)}\}_{m=1}^\infty$ for $k = 1, 2, \dots, n$ by

$$\begin{cases} u_k^{(1)} = \frac{\lambda k}{4\alpha(\gamma+b)+\lambda k} < \frac{\lambda k}{4\alpha(\gamma+b)}, \\ u_k^{(m+1)} = g_k(U_m), \quad U_m = \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j^{(m)}, \quad m = 1, 2, \dots, \end{cases}$$

where

$$g_k(x) = \begin{cases} \frac{\lambda k(1-\alpha x)}{\gamma+b+\lambda k(1-\alpha x)}, & \forall x \in [0, \alpha^{-1}], \\ 0, & \forall x \in (-\infty, 0) \cup (\alpha^{-1}, +\infty). \end{cases}$$

Note that, for every $k = 1, 2, \dots, n$,

$$g_k(x) \text{ is increasing in } x \text{ on } \left[0, \frac{1}{2\alpha}\right]. \tag{4.1}$$

Since $1 < R^0 \leq 2$, we obtain

$$0 < U_1 = \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j^{(1)} < \frac{\lambda \langle k^2 \rangle}{4\alpha(\gamma+b)\langle k \rangle} = \frac{R^0}{4\alpha} \leq \frac{1}{2\alpha}.$$

By (4.1) we have

$$\begin{aligned}
 0 < u_k^{(2)} = g_k(U_1) &= \frac{\lambda k(1 - \alpha U_1)U_1}{\gamma + b + \lambda k(1 - \alpha U_1)U_1} \\
 &\leq \frac{\lambda k}{4\alpha(\gamma + b) + \lambda k} = u_k^{(1)} \\
 &< \frac{\lambda k}{4\alpha(\gamma + b)}.
 \end{aligned}
 \tag{4.2}$$

This leads to

$$0 < U_2 = \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j^{(2)} < \frac{\lambda \langle k^2 \rangle}{4\alpha(\gamma + b)\langle k \rangle} = \frac{R^0}{4\alpha} \leq \frac{1}{2\alpha},$$

and therefore $0 < U_2 \leq U_1 < \frac{1}{2\alpha}$ by (4.2). Thus, we have, by (4.1),

$$\begin{aligned}
 0 < u_k^{(3)} = g_k(U_2) &= \frac{\lambda k(1 - \alpha U_2)U_2}{\gamma + b + \lambda k(1 - \alpha U_2)U_2} \\
 &\leq \frac{\lambda k(1 - \alpha U_1)U_1}{\gamma + b + \lambda k(1 - \alpha U_1)U_1} = u_k^{(2)} < \frac{\lambda k}{4\alpha(\gamma + b)}.
 \end{aligned}$$

By induction we find that

$$0 < U_m < \frac{1}{2\alpha}, \quad 0 < u_k^{(m+1)} \leq u_k^{(m)} < \frac{\lambda k}{4\alpha(\gamma + b)}, \quad m = 1, 2, \dots,
 \tag{4.3}$$

and

$$u_k^{(m+1)} = g_k(U_m) = \frac{\lambda k(1 - \alpha U_m)U_m}{\gamma + b + \lambda k(1 - \alpha U_m)U_m}, \quad m = 1, 2, \dots
 \tag{4.4}$$

Using (2.2) yields $\limsup_{t \rightarrow \infty} I_k(t) \leq u_k^{(1)}$. Thus, applying Lemma D in the Appendix repeatedly, we have

$$\limsup_{t \rightarrow \infty} I_k(t) \leq u_k^{(m)}, \quad k = 1, 2, \dots, n, m = 1, 2, \dots
 \tag{4.5}$$

Since by (4.3) the sequence $\{u_k^{(m)}\}_{m=1}^\infty$ is nonincreasing for every k , its limit exists and is denoted by $\lim_{m \rightarrow \infty} u_k^{(m)} = u_k$. Letting $m \rightarrow \infty$ in (4.4) and (4.5), respectively, we obtain

$$\begin{aligned}
 u_k &= \frac{\lambda k(1 - \alpha \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j) \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j}{\gamma + b + \lambda k(1 - \alpha \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j) \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j}, \\
 \limsup_{t \rightarrow \infty} I_k(t) &\leq u_k, \quad k = 1, 2, \dots, n.
 \end{aligned}
 \tag{4.6}$$

On the other hand, we consider the function

$$f(x) = \frac{1}{\langle k \rangle} \sum_{j=1}^n \frac{\lambda j^2 P(j)x(1 - \alpha x)}{\gamma + b + \lambda jx(1 - \alpha x)} - x, \quad \forall x \in [0, \alpha^{-1}].$$

A simple calculation gives $f'(0) = R^0 - 1 > 0$, so $f(x) > 0$ for each small $x > 0$. Due to (2.2), we can choose $l_k^{(1)}$ for $k = 1, 2, \dots, n$ such that

$$0 < l_k^{(1)} < \liminf_{t \rightarrow \infty} I_k(t) < \frac{1}{\alpha}, \quad f\left(\langle k \rangle^{-1} \sum_{j=1}^n jP(j)l_j^{(1)}\right) > 0. \tag{4.7}$$

We now define the sequence $\{l_k^{(m)}\}_{m=1}^\infty$ for $k = 1, 2, \dots, n$ by

$$l_k^{(m+1)} = g_k(L_m), \quad L_m = \langle k \rangle^{-1} \sum_{j=1}^n jP(j)l_j^{(m)}, \quad m = 1, 2, \dots$$

Since $1 < R^0 \leq 2$, a similar argument gives

$$0 < l_k^{(m)} < \frac{\lambda k}{4\alpha(\gamma + b)}, \quad 0 < L_m < \frac{1}{2\alpha}, \quad m = 1, 2, \dots$$

It follows from the definition of g_k that

$$l_k^{(m+1)} = \frac{\lambda k(1 - \alpha L_m)L_m}{\gamma + b + \lambda k(1 - \alpha L_m)L_m}, \quad m = 1, 2, \dots \tag{4.8}$$

Using (4.7) and applying Lemma D in the Appendix repeatedly, we have

$$\liminf_{t \rightarrow \infty} I_k(t) \geq l_k^{(m)}, \quad k = 1, 2, \dots, n, m = 1, 2, \dots \tag{4.9}$$

Since $\frac{1}{2\alpha} > L_2 > L_1 > 0$ by (4.7) and (4.8), we have, by (4.1),

$$l_k^{(3)} = \frac{\lambda k(1 - \alpha L_2)L_2}{\gamma + b + \lambda k(1 - \alpha L_2)L_2} \geq \frac{\lambda k(1 - \alpha L_1)L_1}{\gamma + b + \lambda k(1 - \alpha L_1)L_1} = l_k^{(2)}.$$

So $\frac{1}{2\alpha} > L_3 \geq L_2 > 0$, and thus

$$l_k^{(4)} = \frac{\lambda k(1 - \alpha L_3)L_3}{\gamma + b + \lambda k(1 - \alpha L_3)L_3} \geq \frac{\lambda k(1 - \alpha L_2)L_2}{\gamma + b + \lambda k(1 - \alpha L_2)L_2} = l_k^{(3)}.$$

By induction we find that the sequence $\{l_k^{(m)}\}_{m=2}^\infty$ is nondecreasing for every k , so its limit exists and is denoted by $\lim_{m \rightarrow \infty} l_k^{(m)} = l_k$. Letting $m \rightarrow \infty$ in (4.8) and (4.9), respectively, we obtain

$$l_k = \frac{\lambda k(1 - \alpha \langle k \rangle^{-1} \sum_{j=1}^n jP(j)l_j) \langle k \rangle^{-1} \sum_{j=1}^n jP(j)l_j}{\gamma + b + \lambda k(1 - \alpha \langle k \rangle^{-1} \sum_{j=1}^n jP(j)l_j) \langle k \rangle^{-1} \sum_{j=1}^n jP(j)l_j},$$

$$\liminf_{t \rightarrow \infty} I_k(t) \geq l_k, \quad k = 1, 2, \dots, n.$$

Finally, it follows from the uniqueness of positive solutions of (1.6) that $u_k = l_k = I_k^*$, so

$$\liminf_{t \rightarrow \infty} I_k(t) = \limsup_{t \rightarrow \infty} I_k(t) = I_k^*.$$

The proof is completed. □

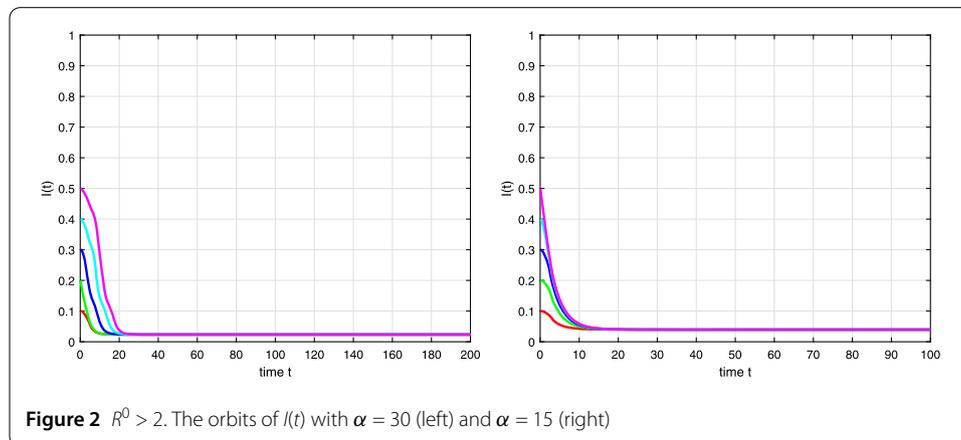
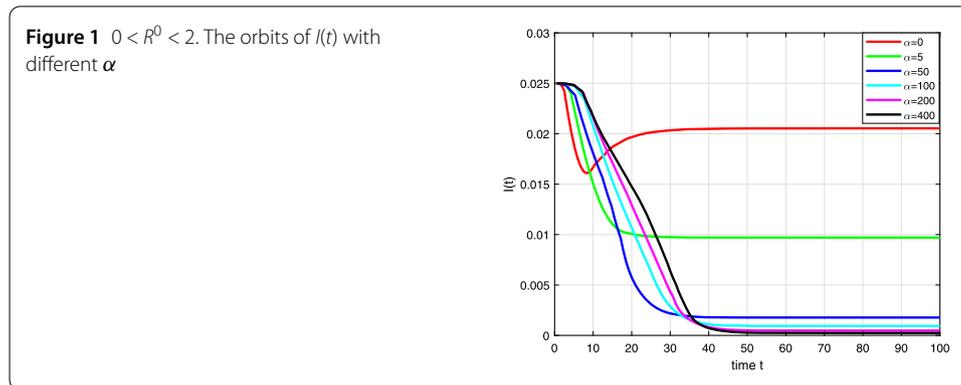
Remark 4.1 According to Theorems 3.1 and 4.1, we have that if $R^0 \in (1, 2]$ and $\alpha > 0$, or if $R^0 > 2$ and $\alpha \geq \alpha_n := \frac{\lambda n}{4(b+\gamma)} (\frac{R^0-1}{R^0})^2$, then the endemic equilibrium of (1.1) is globally asymptotically stable. On the other hand, if we consider the scale-free network (i.e., $P(k) \sim k^{-c}$ ($2 < c \leq 3$)) and fix the parameters λ, b , and γ , then $R^0 \rightarrow +\infty$ and $\alpha_n \rightarrow +\infty$ as $n \rightarrow +\infty$. This means that when the maximum degree n is sufficiently large, we have $R^0 > 2$, so the endemic equilibrium of (1.1) is globally asymptotically stable for sufficiently large α by Theorem 3.1.

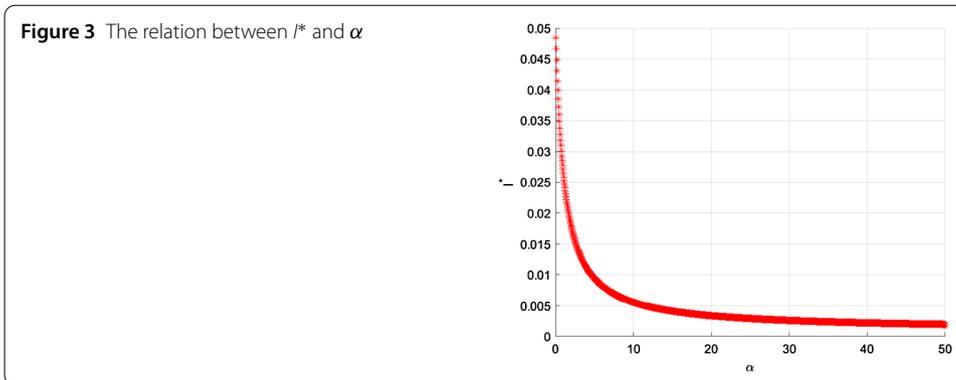
5 Numerical simulations

In this section, we present numerical simulations to illustrate the theoretical results. We consider (1.4) on a finite scale-free network with $n = 500$ and $P(k) = ak^{-3}$, where the constant a satisfies $\sum_{k=1}^{500} P(k) = 1$. Recall that $\alpha_n = \frac{\lambda n}{4(b+\gamma)} (\frac{R^0-1}{R^0})^2$.

In Figure 1, we fix the parameters $b = 0.31, \gamma = 0.41$, and $\lambda = 0.26$ to plot the time evolution of $I(t)$ with different α , where $I(t) = \sum_{k=1}^{500} P(k)I_k(t)$ is the density of infected nodes in the whole network. We can verify that $R^0 = 1.4930 < 2$. Obviously, the orbits converge to stationary levels, which illustrates the correctness of Theorem 4.1. Moreover, we can observe from the simulation that the larger the α , the weaker the disease, which is consistent with the theoretical result (2.2).

In Figure 2, we fix the parameters $b = 0.4, \gamma = 0.3$, and $\lambda = 0.34$ to plot the time evolution of $I(t)$ with five different initial values. We can verify that $R^0 = 2.0082 > 2$ and $\alpha_{500} = 15.3028$. We choose $\alpha = 30$ in Figure 2 (left) and $\alpha = 15$ in Figure 2 (right). The





α value in the left figure satisfies the assumption of Theorem 3.1. whereas the α value in the right figure does not, and despite of this, the endemic equilibrium is globally stable.

In Figure 3, the parameters are chosen as follows: $b = 0.25$, $\gamma = 0.35$, and $\lambda = 0.3$. We can verify that $R^0 = 2.0673 > 1$. The simulation clearly indicates the relation between I^* and α , where $I^* = \sum_{k=1}^{500} P(k)I_k^*$.

We can see from the numerical simulations that the when the disease is endemic, the endemic level decreases significantly as α is increased. The biological meaning is that when an epidemic outbreaks, people consciously reduce the number of contacts with others, and the larger the probability one may contact with infected individuals, the more careful the people are, which is consistent with our real experience.

6 Conclusions

In this paper, we prove that the disease is permanent if $R^0 > 1$ and that the endemic equilibrium of (1.1) is globally asymptotically stable if $R^0 \in (1, 2]$ and $\alpha > 0$, or if $R^0 > 2$ and $\alpha \geq \alpha_n = \frac{\lambda n}{4(b+\gamma)} (\frac{R^0-1}{R^0})^2$. Moreover, we have performed numerical experiments to illustrate the theoretical results. It is observed from the numerical results that the endemic equilibrium is globally asymptotically stable for any $\alpha > 0$ whenever $R^0 > 1$; however, a mathematical proof remains to be very difficult.

Although the feedback parameter α does not affect the threshold R^0 , it plays a role in weakening the spreading of disease, for which a theoretical analysis can be seen from the second inequality of (2.2). Numerical experiments done in [1] and the present paper indicate that the endemic level decreases significantly as the fear factor α increases. The biological meaning is that as soon as an epidemic outbreaks, people are more cautious and reduce contacts with other people consciously, which is beneficial to preventing epidemic spreading.

Appendix

The following result is due to Barbălat [29, Lemma 1.2.3].

Lemma A *Let $a \in (-\infty, +\infty)$, and let $f : [a, \infty) \rightarrow \mathbb{R}$ be a differentiable function. If $\lim_{t \rightarrow \infty} f(t)$ exists (finite) and $f'(t)$ is uniformly continuous in (a, ∞) , then $\lim_{t \rightarrow \infty} f'(t) = 0$.*

The following result is due to Hirsch, Hanisch, and Gabriel [30, Lemma 4.2]; see also [31, Lemma 1.1].

Lemma B *Suppose that $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a differentiable function. If $\liminf_{t \rightarrow \infty} f < \limsup_{t \rightarrow \infty} f$, then there are two sequences $\{\tau_k\}$ and $\{\sigma_k\}$, increasing to infinity, such that*

$$\begin{aligned} \lim_{k \rightarrow \infty} f(\tau_k) &= \limsup_{t \rightarrow \infty} f, & f'(\tau_k) &= 0, \\ \lim_{k \rightarrow \infty} f(\sigma_k) &= \liminf_{t \rightarrow \infty} f, & f'(\sigma_k) &= 0. \end{aligned}$$

The following result is a combination of Lemmas A and B and will be used to show Theorem 2.1.

Corollary C *Suppose that $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a bounded twice differentiable function with $f''(t)$ bounded in \mathbb{R}^+ . Then there are two sequences $\{\tau_k\}$ and $\{\sigma_k\}$, increasing to infinity, such that*

$$\begin{aligned} \lim_{k \rightarrow \infty} f(\tau_k) &= \limsup_{t \rightarrow \infty} f, & \lim_{k \rightarrow \infty} f'(\tau_k) &= 0, \\ \lim_{k \rightarrow \infty} f(\sigma_k) &= \liminf_{t \rightarrow \infty} f, & \lim_{k \rightarrow \infty} f'(\sigma_k) &= 0. \end{aligned}$$

Proof Since $f''(t)$ is bounded in \mathbb{R}^+ , $f'(t)$ is uniformly continuous in \mathbb{R}^+ . Consequently, if $\lim_{t \rightarrow \infty} f(t)$ exists, then by Lemma A all previous conclusions hold for any increasing to infinity sequence $\{x_k\}$. Otherwise, $\liminf_{t \rightarrow \infty} f < \limsup_{t \rightarrow \infty} f$. Then the conclusions follow by Lemma B. The proof is completed. □

The following result will be used to prove Theorem 4.1.

Lemma D *Let $R^0 > 1$ and $\Theta(0) > 0$. Assume that $\{(S_k, I_k)\}_{k=1}^n$ is a nonnegative solution of system (1.3) with initial conditions (1.2). Then*

(i) *If there are constants $u_k > 0$ for $k = 1, 2, \dots, n$ such that*

$$U := \langle k \rangle^{-1} \sum_{j=1}^n jP(j)u_j \leq \frac{1}{2\alpha}, \quad \limsup_{t \rightarrow \infty} I_k(t) \leq u_k,$$

then

$$\limsup_{t \rightarrow \infty} I_k(t) \leq \frac{\lambda k(1 - \alpha U)U}{\gamma + b + \lambda k(1 - \alpha U)U}.$$

(ii) *If there are constants $l_k > 0$ for $k = 1, 2, \dots, n$ such that*

$$L := \langle k \rangle^{-1} \sum_{j=1}^n jP(j)l_j \leq \frac{1}{2\alpha}, \quad \liminf_{t \rightarrow \infty} I_k(t) \geq l_k,$$

then

$$\liminf_{t \rightarrow \infty} I_k(t) \geq \frac{\lambda k(1 - \alpha L)L}{\gamma + b + \lambda k(1 - \alpha L)L}.$$

Proof We only prove (i) since (ii) can be proved similarly.

Indeed, by the same argument as that yielding (2.5) and (2.6) we have

$$\limsup_{t \rightarrow \infty} I_k(t) = \bar{i}_k = \frac{\lambda k(1 - \alpha \bar{\theta}_k) \bar{\theta}_k}{\gamma + b + \lambda k(1 - \alpha \bar{\theta}_k) \bar{\theta}_k}, \quad k = 1, 2, \dots, n.$$

Note that $0 < \bar{\theta}_k \leq \limsup_{t \rightarrow \infty} \Theta(t) \leq \frac{1}{\langle k \rangle} \sum_{k=1}^n kP(k) \limsup_{t \rightarrow \infty} I_k(t) \leq U \leq \frac{1}{2\alpha}$. This, together with (4.1), yields (i). Thus, the proof is completed. \square

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to writing this paper. All authors read and approved the final manuscript.

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