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Quadratic symmetry of modified q-Euler polynomials

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Abstract
We use the p-adic q-integral and group action to count the number of the generating functions of modified q-Euler polynomials in a prescribed set. Some generating function yields modified q-Euler polynomials with the isotropy group D_4 and some gives Euler polynomials with the isotropy group V_4.

MSC: 11B83; 42A16

Keywords: p-adic q-integral; modified q-Euler polynomials; group action

1 Introduction

We attribute a substantial study of Bernoulli polynomials and Euler polynomials to Carlitz. He used an inductive method to define these special polynomials (see [1]).

In this paper we use the p-adic q-integral to define the special polynomials. It is Kim who has established the theory of the p-adic q-integral (see [2]). The p-adic q-integral provides fruitful and essential results in studying special polynomials such as Bernoulli, Euler and Catalan polynomials. One of such results is the symmetry of the special polynomials.

Throughout this paper p is a fixed odd prime number. We use the notations $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ to express the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. For $q, x \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$. We define the *$q$-analogue of a number x* to be $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$ and $f \in UD(\mathbb{Z}_p)$. Kim has introduced the fermionic p-adic q-integral $I_{-q}(f)$ on $\mathbb{Z}_p$ (see [1, 3–10]).

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\, du_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.1}$$

The q-Euler polynomials $\mathcal{E}_{n,q}(x)$ are defined by the generating function

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t}\, d\mu_{-q}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x) \frac{t^n}{n!} \quad (\text{see } [2, 10–18]). \tag{1.2}$$

Also the modified q-Euler polynomials $E_{n,q}(x)$ are defined by the generating function

$$\int_{\mathbb{Z}_p} e^{[x+y]_q t}\,d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} \quad \bigl(\text{see [19, 20]}\bigr). \tag{1.3}$$

Note that $\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_{-q}(y)$ and $E_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n\,d\mu_{-1}(y)$.

For v and w odd positive integers, put $p(x,u,v,w) = \sum_{i=0}^{v-1}(-1)^i\int_{\mathbb{Z}_p} e^{[ux+vy+wi]_q t}\,d\mu_{-1}(y)$. Then

$$\begin{aligned} p(x,u,v,w) &= \sum_{i=0}^{v-1}(-1)^i\int_{\mathbb{Z}_p} e^{[ux+vy+wi]_q t}\,d\mu_{-1}(y) \\ &= \sum_{i=0}^{v-1}(-1)^i \lim_{N\to\infty}\sum_{y=0}^{p^N-1} e^{[ux+vy+wi]_q t}(-1)^y \\ &= \sum_{i=0}^{v-1}(-1)^i \lim_{N\to\infty}\sum_{y=0}^{p^N-1}\sum_{k=0}^{w-1} e^{[ux+v(k+wy)+wi]_q t}(-1)^{k+wy} \\ &= \sum_{i=0}^{v-1}\sum_{k=0}^{w-1}(-1)^{i+k} \lim_{N\to\infty}\sum_{y=0}^{p^N-1} e^{[ux+vwy+vk+wi]_q t}(-1)^y. \end{aligned} \tag{1.4}$$

Note that the last line is invariant under the transposition (v,w). So we obtain the 'basic' symmetry of the generating functions

$$p(x,u,v,w) = p(x,u,w,v). \tag{1.5}$$

In this paper we use four weights w_1, w_2, w_3, w_4 instead of u, v, w. Then the symmetric group S_4 of degree 4 naturally acts on the generating functions. We shall describe the group action more generally. Let $p = p(x,w_1,\dots,w_n)$ be generating functions with a variable x and n weights $w_1,\dots,w_n$ involved in the definition of p. For $\sigma \in S_n$, define

$$\sigma\bigl(p(x,w_1,\dots,w_n)\bigr) = p\bigl(x,w_\sigma(1),\dots,w_\sigma(n)\bigr). \tag{1.6}$$

Then it is obvious that (i) $id(p) = p$ and (ii) $(\sigma\tau)(p) = (\sigma(\tau(p)))$ hold. Hence this is a group action naturally.

In this paper we will consider a set X of 315 generating functions, $p = p(x,w_1,\dots,w_4)$, and let S_4 act on this set. There will be 19 orbits in X under the action of S_4, and 4 non-isomorphic subgroups of S_4 as the isotropy groups.

Throughout this paper w_1, w_2, w_3, w_4 are odd positive integers. Consider the generating functions $p(x)$, where

$$p(x) = \sum_{i=0}^{w_j w_k - 1}(-1)^i\int_{\mathbb{Z}_p} e^{[(w_a w_b + w_c w_d)x + w_j w_k y + w_l w_m i]_q t}\,d\mu_{-1}(y), \tag{1.7}$$

where $1 \le a,b,c,d,j,k,l,m \le 4$, $a \ne b$, $c \ne d$, $j \ne k$, $l \ne m$ and $w_a w_b \ne w_c w_d$. That is, we use square-free quadratic forms of $w's$.

By the abuse of notation, we denote $p(x)$ by $p(ab+cd,jk,lm)$. That is,

$$p(ab+cd,jk,lm)=\sum_{i=0}^{w_jw_k-1}(-1)^i\int_{\mathbb{Z}_p}e^{[(w_aw_b+w_cw_d)x+w_jw_ky+w_lw_mi]_qt}\,d\mu_{-1}(y). \tag{1.8}$$

Then, due to the 'basic' symmetry in (1.5), $p(ab+cd,jk,lm)=p(ab+cd,lm,jk)$. Let X be the set of all $p(ab+cd,jk,lm)$. Then there are 15×6 generating functions of the form $p(ab+cd,jk,jk)$ and 15×15 generating functions of the form $p(ab+cd,jk,lm)$, where $w_jw_k\neq w_lw_m$. So X is a set of 315 generating functions.

Let the symmetric group S_4 act on X naturally. That is, for any $\sigma\in S_4$,

$$\sigma\big(p(ab+cd,jk,lm)\big)=p\big(\sigma(a)\sigma(b)+\sigma(c)\sigma(d),\sigma(j)\sigma(k),\sigma(l)\sigma(m)\big). \tag{1.9}$$

We will find each orbit and the isotropy subgroup of S_4 of an element in the orbit. Note that we have the basic counting of group action such that the product of the cardinality of an orbit and that of the corresponding isotropy subgroup is $|S_4|=24$.

2 Quadratic symmetry of the modified q-Euler polynomials with the isotropy group D_4

We use the notation in (1.8) and put $f_1=p(12+34,12,34)$, $f_2=p(13+24,13,24)$ and $f_3=p(14+24,14,23)$. Then, for $\tau=(2,3,4)$ and $\tau^2=(2,4,3)$,

$$\begin{aligned}\tau(f_1)&=\tau\big(p(12+34,12,34)\big)=p\big(\tau(1)\tau(2)+\tau(3)\tau(4),\tau(1)\tau(2),\tau(3)\tau(4)\big)\\&=p(13+42,13,42)=p(13+24,13,24)=f_2,\\\tau^2(f_1)&=\tau^2\big(p(12+34,12,34)\big)=p(14+23,14,23)=f_3.\end{aligned} \tag{2.1}$$

Hence f_1,f_2,f_3 are in an orbit under the action of S_4 on X.

Now consider a 4-cycle $\sigma_1=(1,3,2,4)$ and a 2-cycle $\sigma_2=(1,2)$. It is obvious that σ_2 fixes f_1. Note that $w_1w_2+w_3w_4=w_3w_4+w_1w_2$. By the basic symmetry in (1.5), we obtain

$$\begin{aligned}\sigma_1\big(p(12+34,12,34)\big)&=p\big(\sigma_1(1)\sigma_1(2)+\sigma_1(3)\sigma_1(4),\sigma_1(1)\sigma_1(2),\sigma_1(3)\sigma_1(4)\big)\\&=p(34+12,34,12)=p(12+34,12,34).\end{aligned} \tag{2.2}$$

That is, σ_1 fixes f_1 also. Hence the dihedral group $D_4=\langle(1,3,2,4),(1,2)\rangle$ is a subgroup of the isotropy subgroup of f_1. However, we find three elements f_1,f_2,f_3 in the orbit of f_1 and $|D_4|=8$. Thus $X_1=\{f_1,f_2,f_3\}$ is the orbit of f_1 and $D_4=\langle(1,3,2,4),(1,2)\rangle$ is the isotropy subgroup of S_4 fixing f_1.

Further, we can check that the cyclic group $\langle(2,3,4)\rangle$ acts transitively on X_1. That is,

$$\begin{aligned}(2,3,4)(f_1)&=(2,3,4)\big(p(12+34,12,34)\big)=p(13+42,13,42)=f_2,\\(2,4,3)(f_1)&=(2,4,3)\big(p(12+34,12,34)\big)=p(14+23,14,23)=f_3.\end{aligned} \tag{2.3}$$

Theorem 2.1 *Let X be the set of all generating functions of the form*

$$p(ab+cd,jk,lm)=\sum_{i=0}^{w_jw_k-1}(-1)^i\int_{\mathbb{Z}_p}e^{[(w_aw_b+w_cw_d)x+w_jw_ky+w_lw_mi]_qt}\,d\mu_{-1}(y)$$

and S_4 act on X by permuting w_1, w_2, w_3, w_4. Take $f_1=p(12+34,12,34)\in X$. Then the isotropy subgroup of f_1 is the dihedral group $D_4=\langle(1,3,2,4),(1,2)\rangle$ and the orbit of f_1 is $X_1=\{f_1,f_2=p(13+24,13,24),f_3=p(14+23,14,23)\}$. Moreover, the cyclic group $\langle(2,3,4)\rangle$ acts transitively on X_1.

Remark Note that the isotropy subgroup of $f_2=p(13+24,13,24)$ is $D_4^1=\langle(1,2,3,4),(1,3)\rangle$ and the isotropy subgroup of $f_3=p(14+23,14,23)$ is $D_4^2=\langle(1,2,4,3),(1,4)\rangle$.

Now we need to transform the q-analogue in the definition of $f_1=p(12+34,12,34)$ to express it in the form of Euler polynomials.

$$\begin{aligned}
&\big[(w_1w_2+w_3w_4)x+w_1w_2y+w_3w_4i\big]_q\\
&=\frac{1-q^{(w_1w_2+w_3w_4)x+w_1w_2y+w_3w_4i}}{1-q}\\
&=\left(\frac{1-q^{w_1w_2}}{1-q}\right)\left(\frac{1-q^{(w_1w_2+w_3w_4)x+w_1w_2y+w_3w_4i}}{1-q^{w_1w_2}}\right)\\
&=\left(\frac{1-q^{w_1w_2}}{1-q}\right)\left(\frac{1-q^{w_1w_2\{(1+\frac{w_3w_4}{w_1w_2})x+y+\frac{w_3w_4}{w_1w_2}i\}}}{1-q^{w_1w_2}}\right)\\
&=[w_1w_2]_q\left[\left(1+\frac{w_3w_4}{w_1w_2}\right)x+y+\frac{w_3w_4}{w_1w_2}i\right]_{q^{w_1w_2}}.
\end{aligned}\tag{2.4}$$

Hence

$$\begin{aligned}
f_1&=p(12+34,12,34)\\
&=\sum_{i=0}^{w_1w_2-1}(-1)^i\int_{\mathbb{Z}_p}e^{[(w_1w_2+w_3w_4)x+w_1w_2y+w_3w_4i]_qt}\,d\mu_{-1}(y)\\
&=\sum_{n=0}^{\infty}\sum_{i=0}^{w_1w_2-1}(-1)^i[w_1w_2]_q^nE_{n,q^{w_1w_2}}\left(\left(1+\frac{w_3w_4}{w_1w_2}\right)x+\frac{w_3w_4}{w_1w_2}i\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.5}$$

Theorem 2.2 *For $q\in\mathbb{C}_p$ with $|q-1|_p<p^{-\frac{1}{p-1}}$ and $n\geq0$, the modified q-Euler polynomials*

$$\sum_{i=0}^{w_1w_2-1}(-1)^i[w_1w_2]_q^nE_{n,q^{w_1w_2}}\left(\left(1+\frac{w_3w_4}{w_1w_2}\right)x+\frac{w_3w_4}{w_1w_2}i\right)\tag{2.6}$$

are invariant under any permutation in the dihedral group $D_4=\langle(1,3,2,4),(1,2)\rangle$.

3 Quadratic symmetry of the modified q-Euler polynomials with the isotropy group V_4

In this section we investigate two orbits each consisting of six generating functions.

Let X_2 be the set of six generating functions $g_1 = p(13+24,12,34)$, $g_2 = p(14+23,12,34)$, $g_3 = p(12+34,13,24)$, $g_4 = p(14+24,13,24)$, $g_5 = p(12+34,14,23)$ and $g_6 = p(13+24,14,23)$ as in the notation of (1.8). That is, each g_i is a generating function of the form

$$p(ab+cd,ac,bd) = \sum_{i=0}^{w_a w_d - 1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a w_b + w_c w_d)x + w_a w_c y + w_b w_d i]_q t} \, d\mu_{-1}(y), \tag{3.1}$$

where $1 \leq a,b,c,d \leq 4$ and a, b, c, d are distinct.

Note that the action of $S_3 = \langle (1,2),(1,2,3) \rangle$ on $X_2 = \{g_1,\dots,g_6\}$ is well defined.

$$\begin{aligned}
(1,2)(g_1) &= (1,2)\bigl(p(13+24,12,34)\bigr) = p(23+14,21,34) = g_2,\\
(1,3)(g_1) &= (1,3)\bigl(p(13+24,12,34)\bigr) = p(31+24,32,14) = g_6,\\
(2,3)(g_1) &= (2,3)\bigl(p(13+24,12,34)\bigr) = p(12+34,13,24) = g_3,\\
(1,2,3)(g_1) &= (1,2,3)\bigl(p(13+24,12,34)\bigr) = p(21+34,23,14) = g_5,\\
(1,3,2)(g_1) &= (1,3,2)\bigl(p(13+24,12,34)\bigr) = p(32+14,31,24) = g_4.
\end{aligned} \tag{3.2}$$

Consider the normal subgroup $V_4 = \{(1), \gamma = (1,2)(3,4), \delta = (1,3)(2,4), \gamma\delta = (1,4)(2,3)\}$ of S_4.

$$\begin{aligned}
\gamma(g_1) &= (1,2)(3,4)\bigl(p(13+24,12,34)\bigr) = p(24+13,21,34) = g_1,\\
\delta(g_1) &= (1,3)(2,4)\bigl(p(13+24,12,34)\bigr) = p(31+42,34,12) = g_1.
\end{aligned} \tag{3.3}$$

Hence V_4 fixes g_1. Since $|S_3| \times |V_4| = |S_4|$, we obtain the following theorem.

Theorem 3.1 *Let X be the set of all generating functions of the form*

$$p(ab+cd,jk,lm) = \sum_{i=0}^{w_j w_k - 1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a w_b + w_c w_d)x + w_j w_k y + w_l w_m i]_q t} \, d\mu_{-1}(y)$$

and S_4 act on X by permuting w_1, w_2, w_3, w_4. Take $g_1 = p(13+24,12,34) \in X$. Then the orbit of g_1 is $X_2 = \{g_1,\dots,g_6\}$ and the isotropy subgroup of g_1 is $V_4 = \{(1),(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$. Moreover, S_3 acts transitively on X_2.

Remark The same computation in (3.3) shows that the isotropy group of each g_i is the normal subgroup $V_4 = \{(1),(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$ of S_4.

Next we adapt the computation in (2.4) for the change of the q-analogue in $g_1 = p(13+24,12,34)$ and obtain the corresponding modified q-Euler polynomials.

$$\bigl[(w_1 w_3 + w_2 w_4)x + w_1 w_2 y + w_3 w_4 i\bigr]_q = [w_1 w_2]_q \left[\left(\frac{w_3}{w_2} + \frac{w_4}{w_1}\right)x + y + \frac{w_3 w_4}{w_1 w_2} i\right]_{q^{w_1 w_2}}, \tag{3.4}$$

$$
\begin{aligned}
g_1 &= p(13+24,12,34)\\
&= \sum_{i=0}^{w_1w_2-1}(-1)^i\int_{\mathbb{Z}_p} e^{[(w_1w_3+w_2w_4)x+w_1w_2y+w_3w_4i]_q t}\,d\mu_{-1}(y)\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{w_1w_2-1}(-1)^i[w_1w_2]_q^n E_{n,q^{w_1w_2}}\left(\left(\frac{w_3}{w_2}+\frac{w_4}{w_1}\right)x+\frac{w_3w_4}{w_1w_2}i\right)\frac{t^n}{n!}.
\end{aligned}
\tag{3.5}
$$

Theorem 3.2 *For $q\in\mathbb{C}_p$ with $|q-1|_p<p^{-\frac{1}{p-1}}$ and $n\geq 0$, the modified q-Euler polynomials*

$$
\sum_{i=0}^{w_1w_2-1}(-1)^i[w_1w_2]_q^n E_{n,q^{w_1w_2}}\left(\left(\frac{w_3}{w_2}+\frac{w_4}{w_1}\right)x+\frac{w_3w_4}{w_1w_2}i\right)
\tag{3.6}
$$

are invariant under any permutation in the normal subgroup $V_4=\{(1),(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$ of S_4.

Now we consider the set X_3 of six polynomials $h_1=p(12+34,12,12)$, $h_2=p(13+24,13,13)$, $h_3=p(14+23,14,14)$, $h_4=p(12+34,34,34)$, $h_5=p(13+24,24,24)$ and $h_6=p(14+23,23,23)$. That is, each h_i is a polynomial of the form

$$
p(ab+cd,ab,ab)=\sum_{i=0}^{w_aw_b-1}(-1)^i\int_{\mathbb{Z}_p} e^{[(w_aw_b+w_cw_d)x+w_aw_by+w_aw_bi]_q t}\,d\mu_{-1}(y),
$$

where $1\leq a,b,c,d\leq 4$ and a, b, c, d are distinct.

Now we have the action of S_4 to collect all of h_i in an orbit.

$$
\begin{aligned}
&(2,3)(h_1)=(2,3)\bigl(p(12+34,12,12)\bigr)=p(13+24,13,13)=h_2,\\
&(2,4)(h_1)=(2,3)\bigl(p(12+34,12,12)\bigr)=p(14+32,14,14)=h_3,\\
&(1,3)(2,4)(h_1)=(1,3)(2,4)\bigl(p(12+34,12,12)\bigr)=\bigl(p(34+12,12)\bigr)=h_4,\\
&(1,2,4)(h_1)=(1,2,4)\bigl(p(12+34,12,12)\bigr)=\bigl(p(24+31,24,24)\bigr)=h_5,\\
&(1,2,3)(h_1)=(1,2,3)\bigl(p(12+34,12,12)\bigr)=\bigl(p(23+14,23,23)\bigr)=h_6.
\end{aligned}
\tag{3.7}
$$

It is obvious that each permutation in the subgroup $\widehat{V_4}=\langle(1,2),(3,4)\rangle$ of S_4 fixes h_1. As $|\hat{V}_4|\times 6=24$, we conclude the following.

Theorem 3.3 *Let X be the set of all generating functions of the form*

$$
p(ab+cd,jk,lm)=\sum_{i=0}^{w_jw_k-1}(-1)^i\int_{\mathbb{Z}_p} e^{[(w_aw_b+w_cw_d)x+w_jw_ky+w_lw_mi]_q t}\,d\mu_{-1}(y)
$$

and S_4 act on X by permuting w_1, w_2, w_3, w_4. Take $h_1=p(12+34,12,12)\in X$. Then the orbit of h_1 is $X_3=\{h_1,\dots,h_6\}$ and the isotropy subgroup of h_1 is $\widehat{V_4}=\langle(1,2),(3,4)\rangle$ in S_4.

Remark The subgroup $\widehat{V_4}=\langle(1,2),(3,4)\rangle$ is not normal in S_4. Further, none of four subgroups of order 6 in S_4 act transitively on X_3.

A slight change of the formula in (2.4) makes $h_1 = p(13 + 24, 12, 34)$ to be modified q-Euler polynomials.

$$\bigl[(w_1w_2 + w_3w_4)x + w_1w_2y + w_1w_2i\bigr]_q = [w_1w_2]_q\biggl[\biggl(1 + \frac{w_3w_4}{w_1w_2}\biggr)x + y + i\biggr]_{q^{w_1w_2}}, \tag{3.8}$$

$$\begin{aligned} h_1 &= p(12 + 34, 12, 12) \\ &= \sum_{i=0}^{w_1w_2-1}(-1)^i \int_{\mathbb{Z}_p} e^{[(w_1w_2+w_3w_4)x+w_1w_2y+w_1w_2i]_q t}\, d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{w_1w_2-1}(-1)^i[w_1w_2]_q^n E_{n,q^{w_1w_2}}\biggl(\biggl(1 + \frac{w_3w_4}{w_1w_2}\biggr)x + i\biggr)\frac{t^n}{n!}. \end{aligned} \tag{3.9}$$

Theorem 3.4 *For $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $n \geq 0$, the modified q-Euler polynomials*

$$\sum_{i=0}^{w_1w_2-1}(-1)^i[w_1w_2]_q^n E_{n,q^{w_1w_2}}\biggl(\biggl(1 + \frac{w_3w_4}{w_1w_2}\biggr)x + i\biggr) \tag{3.10}$$

are invariant under any permutation in the subgroup $\widehat{V_4} = \langle(1,2),(3,4)\rangle$ of S_4.

4 Less symmetric modified q-Euler polynomials

When S_4 acts on the set X of 315 generating functions listed in (1.8), there are seven orbits having 12 generating functions and nine orbits having 24 generating functions. That is, these are the generating functions whose isotropy subgroup is a cyclic group of order 2 or a trivial group. In this section we just illustrate a typical one for each case.

Let X_4 be the set of all generating functions of the form

$$p(ac + ad, ab, ab) = \sum_{i=0}^{w_aw_b-1}(-1)^i \int_{\mathbb{Z}_p} e^{[(w_aw_c+w_aw_d)x+w_aw_by+w_aw_bi]_q t}\, d\mu_{-1}(y), \tag{4.1}$$

where $1 \leq a, b, c, d \leq 4$ and a, b, c, d are distinct.

That is, X_4 consists of the following 12 polynomials:

$$\begin{aligned} &f_1 = p(13 + 14, 12, 12), && f_2 = p(23 + 24, 12, 12), \\ &f_3 = p(12 + 14, 13, 13), && f_4 = p(32 + 34, 13, 13), \\ &f_5 = p(12 + 13, 14, 14), && f_6 = p(42 + 43, 14, 14), \\ &f_7 = p(21 + 24, 23, 23), && f_8 = p(31 + 34, 23, 23), \\ &f_9 = p(21 + 23, 24, 24), && f_{10} = p(41 + 43, 24, 24), \\ &f_{11} = p(31 + 32, 34, 34), && f_{12} = p(41 + 42, 34, 34). \end{aligned} \tag{4.2}$$

We can check that the alternating group A_4 acts on X_4 transitively.

$$\begin{aligned}
&(1,2,3)f_1 = f_7, \qquad (1,3,2)f_1 = f_4,\\
&(1,2,4)f_1 = f_9, \qquad (1,4,2)f_1 = f_6,\\
&(1,3,4)f_1 = f_8, \qquad (1,4,3)f_1 = f_{10},\\
&(2,3,4)f_1 = f_3, \qquad (2,4,3)f_1 = f_5,\\
&(1,2)(3,4)f_1 = f_2, \qquad (1,3)(2,4))f_1 = f_{11},\\
&(1,4)(2,3))f_1 = f_{12}.
\end{aligned} \tag{4.3}$$

As $(3,4)f_1 = (3,4)(p(13+14,12,12)) = p(14+13,12,12) = f_1$, the isotropy subgroup of f_1 is $\langle(3,4)\rangle$ and X_4 is the orbit of f. Note that $\langle(3,4)\rangle$ consists of one even permutation (1) and one odd permutation $(3,4)$. So, for any $p(ac+ad,ab,ab) \in X_4$, there is an even permutation τ such that $\tau(f_1) = p(ac+ad,ab,ab)$. Hence the alternating subgroup A_4 of S_4 acts transitively on X_4.

Theorem 4.1 *Let X be the set of all generating functions of the form*

$$p(ab+cd,jk,lm) = \sum_{i=0}^{w_j w_k - 1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a w_b + w_c w_d)x + w_j w_k y + w_l w_m i]_q t}\, d\mu_{-1}(y)$$

and S_4 act on X by permuting w_1, w_2, w_3, w_4. Take $f_1 = p(13+14,12,12) \in X$. Then the orbit of f_1 is X_4 consisting of 12 polynomials as above and the isotropy subgroup of f_1 is $\langle(3,4)\rangle$. Moreover, A_4 acts transitively on X_4.

Finally, we illustrate one of the nine orbits each consisting of 24 generating functions.

Let X_5 be the set of all generating functions of the form

$$p(ab+ac,ab,ab) = \sum_{i=0}^{w_a w_b - 1} (-1)^i \int_{\mathbb{Z}_p} e^{[(w_a w_b + w_a w_c)x + w_a w_b y + w_a w_b i]_q t}\, d\mu_{-1}(y), \tag{4.4}$$

where $1 \le a,b,c \le 4$ and a, b, c are distinct. Note that there are six choices for the coefficient $w_a w_b$ of y and four choices for the coefficient $w_a w_b + w_a w_c$ of x. Hence X_5 consists of 24 generating functions. We use $g = p(12+13,12,12) \in X_5$, then we can see that the action of S_4 on X_5 is transitive.

5 Conclusion

In this paper we investigate the symmetric property of the Euler polynomials. If we use n weights $w_1,\dots,w_n$ in the definition of the generating functions of Euler polynomials, then the symmetric group S_n naturally acts on a prescribed set of generating functions of the Euler polynomials. This paper uses four weights $w_1,\dots,w_4$ in a quadratic form. We find that the dihedral group D_4 (or the Klein 4 group V_4) is an isotropy group of some generating function. As a result, the corresponding Euler polynomial is fixed by D_4 (or V_4). The results in the paper extend to other special polynomials such as degenerate Euler polynomials and Catalan polynomials. It is a further problem to find special polynomials that are invariant under the alternating group A_n.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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