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Analysis of a stochastic cooperation-competition model

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Abstract

In this paper, we explore dynamic properties of a stochastic cooperation-competition model. Some sufficient conditions are established for stochastic persistence and stochastic extinction of species. Studies suggest that the noise may have a positive effect on the persistence of species. We also analyze global asymptotic stability of positive solutions and give a stationary distribution of this stochastic model which has the ergodic property. Finally, some numerical simulations are presented to illustrate or complement our mathematical findings.

Keywords: stochastic cooperation-competition model; stochastic persistence and stochastic extinction; global asymptotic stability; stationary distribution

1 Introduction

There is already growing evidence that the ecological relationship in the population is a significant global driver of biodiversity change and decline. Competition and mutualism are the two basic elements of interspecific relationship, and they are an important driving force in structuring ecological communities. The cooperation-competition model describes that two different populations exist in a relationship in which each individual benefits from the activity of the other and are both in competition with a third population. There exist some excellent results to explore the qualitative and stability properties of the cooperation-competition model [1–6]. On the other hand, it is recognized that there is a large number of random factors in the real natural environment. These random factors of the environment are not only an integral part of any realistic ecosystem, but they also may lead to complete extinction of populations. The existing research works have investigated dynamic properties of stochastic predator-prey models [7–12], stochastic competitive models [13–18], stochastic mutualism models [19–24] and stochastic three-species models [25–33]. To the best of our knowledge, there are few studies to analyze the dynamics of a stochastic cooperation-competition model.

In the present paper, our model is based on the following deterministic cooperation-competition model:

$$
\begin{aligned}
dx(t) &= x(t)\bigl(r_1 - a_{11}x(t) + a_{12}y(t) - a_{13}z(t)\bigr),\\
dy(t) &= y(t)\bigl(r_2 + a_{21}x(t) - a_{22}y(t) - a_{23}z(t)\bigr),\\
dz(t) &= z(t)\bigl(r_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)\bigr),
\end{aligned}
\tag{1.1}
$$

where $x(t)$, $y(t)$ and $z(t)$ are the densities of species, r_i ($i = 1, 2, 3$) is the per capita growth rate of species, a_{ii} ($i = 1, 2, 3$) is the intraspecific competition rate of species, a_{12} and a_{21} are interspecific mutual rates of species, a_{i3} and a_{3i} ($i = 1, 2$) are interspecific competitive rates of species. Here we assume that all the parameters involved in the model are positive, and the environmental fluctuation modelled by means of independent Gaussian white noises mainly influences the per capita growth rate r_i ($i = 1, 2, 3$) of species since May [34] have pointed out that the per capita growth rate exhibits random fluctuation to a greater or lesser extent (also see [35, 36]). Let $L(t) = (x(t), y(t), z(t))^T$ be a Markov process with the following specifications:

$$
\begin{aligned}
&E\bigl[x(t+\Delta t) - x(t)|L(t) = l\bigr] \approx x(r_1 - a_{11}x + a_{12}y - a_{13}z)\Delta t,\\
&E\bigl[y(t+\Delta t) - y(t)|L(t) = l\bigr] \approx y(r_2 + a_{21}x - a_{22}y - a_{23}z)\Delta t,\\
&E\bigl[z(t+\Delta t) - z(t)|L(t) = l\bigr] \approx z(r_3 - a_{31}x - a_{32}y - a_{33}z)\Delta t,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Var}\bigl[x(t+\Delta t) - x(t)|L(t) = l\bigr] \approx a_1^2 x^2 \Delta t,\\
&\mathrm{Var}\bigl[y(t+\Delta t) - y(t)|L(t) = l\bigr] \approx a_2^2 y^2 \Delta t,\\
&\mathrm{Var}\bigl[z(t+\Delta t) - z(t)|L(t) = l\bigr] \approx a_3^2 z^2 \Delta t,
\end{aligned}
$$

for sufficiently small Δt (see [37]). Then we consider the following stochastic cooperation-competition model of Lotka-Volterra type:

$$
\begin{aligned}
&dx(t) = x(t)\bigl(r_1 - a_{11}x(t) + a_{12}y(t) - a_{13}z(t)\bigr) + \alpha_1 x(t)\,dB_1(t),\\
&dy(t) = y(t)\bigl(r_2 + a_{21}x(t) - a_{22}y(t) - a_{23}z(t)\bigr) + \alpha_2 y(t)\,dB_2(t),\\
&dz(t) = z(t)\bigl(r_3 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)\bigr) + \alpha_3 z(t)\,dB_3(t),
\end{aligned}
\tag{1.2}
$$

where $(B_1(t), B_2(t), B_3(t))^T$ is a three-dimensional Brownian motion defined on a complete probability space (Ω, F, P), α_i^2 ($i = 1, 2, 3$) is the intensity of the white noise.

The paper focuses on two main aspects of the stochastic cooperation-competition model (1.2). One is to establish stochastic persistence and extinction of species and sufficient conditions for global asymptotic stability of the positive solutions of model (1.2). Especially, we investigate the important effect of the noise on species. Another aspect is to prove the existence of stationary distribution of model (1.2) and the fact that there exists the ergodic property in the stationary distribution. Here, if a stationary distribution has the ergodic property, then this means that the mean of population density in time with the development of time is equal to the mean of population density in space.

This work is organized as follows. In the next section, we state our main results: stochastic persistence and stochastic extinction; global asymptotic stability of positive solutions; and the existence of stationary distribution. In Section 3, we give the rigorous proofs of the main results. In Section 4, we do some numerical simulations to illustrate or complement our mathematical findings. The final section summarizes our findings.

2 Main results

Three main results are obtained, including (1) stochastic persistence and stochastic extinction; (2) global asymptotic stability of positive solutions; and (3) the existence of stationary distribution. We will achieve the three results through several steps.

At the beginning, we introduce some notations. Let

$$[f] = \frac{1}{t}\int_0^t f(s)\,ds,$$

$$[f]^* = \limsup_{t\to+\infty} \frac{1}{t}\int_0^t f(s)\,ds,$$

$$[f]_* = \liminf_{t\to+\infty} \frac{1}{t}\int_0^t f(s)\,ds,$$

$$\rho_1 = a_{22}r_1 + a_{12}r_2, \qquad \rho_2 = a_{21}r_1 + a_{11}r_2, \qquad \rho_3 = a_{33}r_1 + a_{13}r_3,$$

$$\rho_4 = a_{11}r_3 - a_{31}r_1, \qquad \rho_5 = a_{33}r_2 + a_{23}r_3, \qquad \rho_6 = a_{22}r_3 + a_{32}r_2,$$

$$\tilde{\rho}_1 = a_{22}\alpha_1^2/2 + a_{12}\alpha_2^2/2, \qquad \tilde{\rho}_2 = a_{21}\alpha_1^2/2 + a_{11}\alpha_2^2/2,$$

$$\tilde{\rho}_3 = a_{33}\alpha_1^2/2 + a_{13}\alpha_3^2/2, \qquad \tilde{\rho}_4 = a_{11}\alpha_3^2/2 - a_{31}\alpha_1^2/2,$$

$$\tilde{\rho}_5 = a_{33}\alpha_2^2/2 - a_{23}\alpha_3^2/2, \qquad \tilde{\rho}_6 = a_{22}\alpha_3^2/2 - a_{32}\alpha_2^2/2,$$

$$G = \begin{vmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad G_1 = \begin{vmatrix} r_1 & -a_{12} & a_{13} \\ r_2 & a_{22} & a_{23} \\ r_3 & a_{32} & a_{33} \end{vmatrix}, \qquad G_2 = \begin{vmatrix} a_{11} & r_1 & a_{13} \\ -a_{21} & r_2 & a_{23} \\ a_{31} & r_3 & a_{33} \end{vmatrix},$$

$$G_3 = \begin{vmatrix} a_{11} & -a_{12} & r_1 \\ -a_{21} & a_{22} & r_2 \\ a_{31} & a_{32} & r_3 \end{vmatrix}, \qquad \tilde{G}_1 = \begin{vmatrix} \alpha_1^2/2 & -a_{12} & a_{13} \\ \alpha_2^2/2 & a_{22} & a_{23} \\ \alpha_3^2/2 & a_{32} & a_{33} \end{vmatrix}, \qquad \tilde{G}_2 = \begin{vmatrix} a_{11} & \alpha_1^2/2 & a_{13} \\ -a_{21} & \alpha_2^2/2 & a_{23} \\ a_{31} & \alpha_3^2/2 & a_{33} \end{vmatrix},$$

$$\tilde{G}_3 = \begin{vmatrix} a_{11} & -a_{12} & \alpha_1^2/2 \\ -a_{21} & a_{22} & \alpha_2^2/2 \\ a_{31} & a_{32} & \alpha_3^3/2 \end{vmatrix}.$$

Theorem 2.1 *For any given initial value $(x(0), y(0), z(0))^T \in \mathbb{R}_+^3$, there exists a unique solution $(x(t), y(t), z(t))^T$ of* (1.2) *on $t \geq 0$, and this solution will remain in $\mathbb{R}_+^3$ with probability* 1.

Proof It is clear that, for any given initial value $(x(0), y(0), z(0))^T \in \mathbb{R}_+^3$, there exists a unique local solution $(x(t), y(t), z(t))^T$ of (1.2) on $t \in [0, t_e)$, where t_e is the explosion time. Let $k_0 > 0$ be sufficiently large such that any given initial value $(x(t), y(t), z(t))^T$ lies within $[1/k_0, k_0] \times [1/k_0, k_0] \times [1/k_0, k_0]$. For each integer $k > k_0$, define the stopping time.

$$\tau_k = \inf\bigl\{t \in [0, t_e) : x(t) \notin (1/k, k) \text{ or } y(t) \notin (1/k, k) \text{ or } z(t) \notin (1/k, k)\bigr\}.$$

Carrying out similar arguments of Theorem 2.1 in [38], we conclude that $\tau_\infty > \infty$ a.s. where $\tau_\infty = \lim_{k\to\infty} \tau_k$. Here we omit some details. □

The first part of our results solves stochastic persistence of the species in model (1.2).

Definition 2.1 (see [25]) Species $N(t)$ is said to be stochastically persistent in the mean if $[N]_* > 0$.

It is noteworthy that stochastic persistence for the stochastic population model has many forms of definition; for example, stochastic persistence defined in the form of distribution, or stochastic persistence defined in the form of expectation. Here stochastic persistence is defined as the mean of the sample paths of the solution in time.

Theorem 2.2 *If $a_{11}a_{22} > a_{12}a_{21}$ and $r_i < \alpha_i^2/2$ for $i = 1,2,3$, then all the species of model* (1.2) *go to extinction a.s.*

Theorem 2.3 *Let $G > 0$, $a_{11} > a_{12}$ and $a_{22} > a_{21}$. Then*

(i) *if $r_1 > \alpha_1^2/2$, $\rho_2 < \tilde{\rho}_2$, $G_3 < \tilde{G}_3$, then the species $y(t)$ and $z(t)$ go to extinction a.s. and $x(t)$ is stochastically persistent with*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{r_1 - \alpha_1^2/2}{a_{11}} \quad a.s.$$

(ii) *if $r_2 > \alpha_2^2/2$, $\rho_1 < \tilde{\rho}_1$, $G_3 < \tilde{G}_3$, then the species $x(t)$ and $z(t)$ go to extinction a.s. and $y(t)$ is stochastically persistent with*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{r_2 - \alpha_2^2/2}{a_{22}} \quad a.s.$$

(iii) *if $r_3 > \alpha_3^2/2$ and one of the following conditions holds*:

(A$_1$) $\rho_1 < \tilde{\rho}_1$, $\rho_5 < \tilde{\rho}_5$ *and* $a_{22}a_{33} > a_{23}a_{32}$;

(A$_2$) $\rho_1 < \tilde{\rho}_1$, $\rho_3 < \tilde{\rho}_3$ *and* $a_{11}a_{33} > a_{13}a_{31}$;

(A$_3$) $\rho_2 < \tilde{\rho}_2$, $\rho_5 < \tilde{\rho}_5$ *and* $a_{22}a_{33} > a_{23}a_{32}$;

(A$_4$) $\rho_2 < \tilde{\rho}_2$, $\rho_3 < \tilde{\rho}_3$ *and* $a_{11}a_{33} > a_{13}a_{31}$,

then the species $x(t)$ and $y(t)$ go to extinction a.s. and $z(t)$ is stochastically persistent with

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{r_3 - \alpha_3^2/2}{a_{33}} \quad a.s.$$

Theorem 2.4 *Let $G > 0$, $a_{11} > a_{12}$ and $a_{22} > a_{21}$. Then*

(iv) *if $\rho_1 > \tilde{\rho}_1$, $\rho_2 > \tilde{\rho}_2$ and $G_3 < \tilde{G}_3$, then the species $z(t)$ goes to extinction a.s. and the species $x(t)$ and $y(t)$ are stochastically persistent with*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{\rho_1 - \tilde{\rho}_1}{a_{11}a_{22} - a_{12}a_{21}} \quad a.s.,$$
$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{\rho_2 - \tilde{\rho}_2}{a_{11}a_{22} - a_{12}a_{21}} \quad a.s.$$

(v) *if $a_{11}a_{33} > a_{13}a_{31}$, $\rho_3 > \tilde{\rho}_3$, $\rho_4 > \tilde{\rho}_4$ and $G_2 < \tilde{G}_2$, then the species $y(t)$ goes to extinction a.s. and the species $x(t)$ and $z(t)$ are stochastically persistent with*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{\rho_3 - \tilde{\rho}_3}{a_{11}a_{33} - a_{13}a_{31}} \quad a.s.,$$
$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{\rho_4 - \tilde{\rho}_4}{a_{11}a_{33} - a_{13}a_{31}} \quad a.s.$$

(vi) *if* $a_{22}a_{33} > a_{23}a_{32}$, $\rho_5 > r_5$, $\rho_6 > r_6$ *and* $G_1 < \tilde{G}_1$, *then the species* $x(t)$ *goes to extinction a.s. and the species* $y(t)$ *and* $z(t)$ *are stochastically persistent with*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{\rho_5 - \tilde{\rho}_5}{a_{22}a_{33} - a_{23}a_{32}} \quad a.s.,$$
$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{\rho_6 - \tilde{\rho}_6}{a_{22}a_{33} - a_{23}a_{32}} \quad a.s.$$

Theorem 2.5 *Let* $G > 0$, $a_{11} > a_{12}$, $a_{22} > a_{21}$, $a_{11}a_{33} > a_{13}a_{31}$ *and* $a_{22}a_{33} > a_{23}a_{32}$. *If* $G_i > \tilde{G}_i$, $i = 1, 2, 3$, *then all the species are stochastically persistent with*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{G_1 - \tilde{G}_1}{G} \quad a.s.,$$
$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{G_2 - \tilde{G}_2}{G} \quad a.s.,$$
$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{G_3 - \tilde{G}_3}{G} \quad a.s.$$

The second part of our results is to establish global asymptotic stability of the positive solutions of model (1.2).

Definition 2.2 (see [25]) (1.2) is said to be globally asymptotically stable (or globally attractive) if $\lim_{t\to\infty} \max\{|x_1(t) - x_2(t)|, |y_1(t) - y_2(t)|, |z_1(t) - z_2(t)|\} = 0$ a.s., where $(x_i(t), y_i(t), z_i(t))$, $i = 1, 2$, are two arbitrary solutions of (1.2) with the initial values $(x_i(0), y_i(0), z_i(0)) \in \mathbb{R}^3_+$, $i = 1, 2$.

Theorem 2.6 *If there exists a positive constant* λ_i, $i = 1, 2, 3$, *such that*

$$\lambda_1 a_{11} \ge \lambda_2 a_{21} + \lambda_3 a_{31}, \qquad \lambda_2 a_{22} \ge \lambda_1 a_{12} + \lambda_3 a_{32}, \qquad \lambda_3 a_{33} \ge \lambda_1 a_{13} + \lambda_2 a_{23}, \tag{2.1}$$

then model (1.2) *is globally asymptotically stable*.

The final part of our results is to investigate the existence of the stationary distribution of model (1.2) and to prove that this stationary distribution has the ergodic property.

Definition 2.3 (see [39]) The distribution $\mu(s)$ is called stationary if $\mu(s)$ satisfies

$$\mu(\omega) = \int P(t, s, \omega)\,d\mu(\omega)$$

for all $t > 0$, where $P(t, s, \omega)$ is a transition probability function.

Let

$$\delta_1 = a_{11} - (a_{12} - a_{13} - a_{21} - a_{31})/2,$$
$$\delta_2 = a_{22} - (a_{12} - a_{21} - a_{23} - a_{32})/2,$$
$$\delta_3 = a_{33} - (a_{13} - a_{23} - a_{31} - a_{32})/2$$

and $(x^*, y^*, z^*)^T$ be the positive equilibrium point of the corresponding deterministic model of model (1.2), that is, $x^* = G_1/G > 0$, $y^* = G_2/G > 0$, $z^* = G_3/G > 0$.

Theorem 2.7 *If* $\delta_i > 0$, $i = 1, 2, 3$ *and*

$$\left(\alpha_1^2 x^* + \alpha_2^2 y^* + \alpha_3^2 z^*\right)/2 < \min\left\{\delta_1\left(x^*\right)^2, \delta_2\left(y^*\right)^2, \delta_3\left(z^*\right)^2\right\}, \tag{2.2}$$

then there is a stationary distribution $\mu(\cdot)$ *for model* (1.2) *and it has the ergodic property*

$$\begin{aligned} &P\left\{\lim_{t\to+\infty}\frac{1}{t}\int_0^t x(s)\,ds = \int_{\mathbb{R}_+^3}\omega_1\mu(d\omega_1, d\omega_2, d\omega_3)\right\} = 1, \\ &P\left\{\lim_{t\to+\infty}\frac{1}{t}\int_0^t y(s)\,ds = \int_{\mathbb{R}_+^3}\omega_2\mu(d\omega_1, d\omega_2, d\omega_3)\right\} = 1, \\ &P\left\{\lim_{t\to+\infty}\frac{1}{t}\int_0^t z(s)\,ds = \int_{\mathbb{R}_+^3}\omega_3\mu(d\omega_1, d\omega_2, d\omega_3)\right\} = 1. \end{aligned} \tag{2.3}$$

The above theorem shows that if a stationary distribution has the ergodic property, then the mean of population density in time with the development of time is equal to the mean of population density in space with probability one.

3 Proofs of the main results

This section presents the proofs of Theorems 2.2-2.7.

3.1 The proofs of Theorems 2.2-2.5

In order to complete the proofs of Theorems 2.2-2.5, we first introduce two important lemmas.

Lemma 3.1 (see [25]) *Let* $z \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$.

(i) *If there exist two positive constants* T *and* λ_0 *such that*

$$\ln z(t) \le \lambda t - \lambda_0 \int_0^t z(s)\,ds + \sum_{i=1}^n \sigma_i B_i(t)$$

for all $t \ge T$, *where* $B_i(t)$, $1 \le i \le n$, *are independent standard Brownian motions and* σ_i, $1 \le i \le n$, *are constants, then* $[z]^* \le \lambda/\lambda_0$ *a.s. if* $\lambda \ge 0$ *or* $\lim_{t\to+\infty} z(t) = 0$ *a.s. if* $\lambda < 0$.

(ii) *If there exist three positive constants* T, λ *and* λ_0 *such that*

$$\ln z(t) \ge \lambda t - \lambda_0 \int_0^t z(s)\,ds + \sum_{i=1}^n \sigma_i B_i(t)$$

for all $t \ge T$, *where* $B_i(t)$, $1 \le i \le n$, *are independent standard Brownian motions and* σ_i, $1 \le i \le n$, *are constants, then* $[z]_* \ge \lambda/\lambda_0$ *a.s.*

Similar to Lemmas 3.1, 3.4 in [14], Lemma 3.9 in [19] and Theorem 2.5 in [26], we have the following lemma.

Lemma 3.2 *If $a_{11} > a_{12}$ and $a_{22} > a_{21}$, then for any $p \geq 1$ and solution $(x(t), y(t), z(t))^T \in \mathbb{R}^3_+$ of (1.2), there is a constant $K = K(p)$ such that*

$$\limsup_{t\to\infty} E\bigl(x(t)^p\bigr) \leq K, \qquad \limsup_{t\to\infty} E\bigl(y(t)^p\bigr) \leq K, \qquad \limsup_{t\to\infty} E\bigl(z(t)^p\bigr) \leq K$$

and

$$\begin{aligned} &\limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \leq 1 \quad a.s., \\ &\limsup_{t\to\infty} \frac{\ln y(t)}{\ln t} \leq 1 \quad a.s., \\ &\limsup_{t\to\infty} \frac{\ln z(t)}{\ln t} \leq 1 \quad a.s. \end{aligned} \tag{3.1}$$

Proof of Theorem 2.2 By applying Itô's formula to model (1.2), we get

$$\begin{aligned} d\ln x(t) &= \bigl[r_1 - \alpha_1^2/2 - a_{11}x(t) + a_{12}y(t) - a_{13}z(t)\bigr]\,dt + \alpha_1\,dB_1(t), \\ d\ln y(t) &= \bigl[r_2 - \alpha_2^2/2 + a_{21}x(t) - a_{22}y(t) - a_{23}z(t)\bigr]\,dt + \alpha_2\,dB_2(t), \\ d\ln z(t) &= \bigl[r_3 - \alpha_3^2/2 - a_{31}x(t) - a_{32}y(t) - a_{33}z(t)\bigr]\,dt + \alpha_3\,dB_3(t). \end{aligned}$$

Integrating from 0 to t on both sides of the above equation and dividing by t, we obtain

$$\frac{1}{t}\ln\frac{x(t)}{x(0)} = r_1 - \frac{\alpha_1^2}{2} - a_{11}\bigl[x(t)\bigr] + a_{12}\bigl[y(t)\bigr] - a_{13}\bigl[z(t)\bigr] + \frac{\alpha_1 B_1(t)}{t}. \tag{3.2}$$

Similarly, we have

$$\frac{1}{t}\ln\frac{y(t)}{y(0)} = r_2 - \frac{\alpha_2^2}{2} + a_{21}\bigl[x(t)\bigr] - a_{22}\bigl[y(t)\bigr] - a_{23}\bigl[z(t)\bigr] + \frac{\alpha_2 B_2(t)}{t}, \tag{3.3}$$

$$\frac{1}{t}\ln\frac{z(t)}{z(0)} = r_3 - \frac{\alpha_3^2}{2} - a_{31}\bigl[x(t)\bigr] - a_{32}\bigl[y(t)\bigr] - a_{33}\bigl[z(t)\bigr] + \frac{\alpha_3 B_3(t)}{t}. \tag{3.4}$$

A direct calculation shows that

$$\frac{1}{t}\ln\frac{z(t)}{z(0)} \leq r_3 - \frac{\alpha_3^2}{2} - a_{33}\bigl[z(t)\bigr] + \frac{\alpha_3 B_3(t)}{t}. \tag{3.5}$$

By Lemma 3.1, we have $\lim_{t\to+\infty} z(t) = 0$ a.s. if $r_3 < \alpha_3^2/2$ holds. It follows from $a_{11}a_{22} > a_{12}a_{21}$ that there are positive constants c_1 and d_1 such that $a_{22}/a_{12} \geq c_1/d_1 \geq a_{21}/a_{11}$. Equation (3.2) multiplied by c_1 plus equation (3.3) multiplied by d_1 gives

$$\begin{aligned} c_1\frac{1}{t}\ln\frac{x(t)}{x(0)} + d_1\frac{1}{t}\ln\frac{y(t)}{y(0)} &= \frac{1}{t}\ln\frac{x(t)^{c_1}y(t)^{d_1}}{x(0)^{c_1}y(0)^{d_1}} \\ &= c_1\bigl(r_1 - \alpha_1^2/2\bigr) + d_1\bigl(r_2 - \alpha_2^2/2\bigr) - (c_1a_{11} - d_1a_{21})\bigl[x(t)\bigr] \\ &\quad - (d_1a_{22} - c_1a_{12})\bigl[y(t)\bigr] - (c_1a_{13} + d_1a_{23})\bigl[z(t)\bigr] \\ &\quad + c_1\alpha_1B_1(t)/t + d_1\alpha_2B_2(t)/t. \end{aligned} \tag{3.6}$$

In view of $r_i < \alpha_i^2/2$, $i = 1,2$, $c_1 a_{11} - d_1 a_{21} \geq 0$ and $d_1 a_{22} - c_1 a_{12} \geq 0$, we obtain $\lim_{t\to+\infty} x(t)^{c_1} y(t)^{d_1} = 0$, a.s. Then there are two cases:

$$\lim_{t\to+\infty} x(t) = 0 \quad \text{a.s.} \tag{3.7}$$

or

$$\lim_{t\to+\infty} y(t) = 0 \quad \text{a.s.} \tag{3.8}$$

If (3.7) holds, substituting (3.7) into (3.3), we have

$$\frac{1}{t}\ln\frac{y(t)}{y(0)} \leq r_2 - \frac{\alpha_2^2}{2} + \varepsilon - a_{22}\big[y(t)\big] + \frac{\alpha_2 B_2(t)}{t}.$$

It follows from Lemma 3.1 that $\lim_{t\to+\infty} y(t) = 0$ a.s. if ε is sufficiently small such that $r_2 - \alpha_2^2/2 + \varepsilon < 0$. Similarly, we conclude that $\lim_{t\to+\infty} x(t) = 0$ a.s. if (3.8) holds. The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3 (i) Let

$$c_2 = (a_{21}a_{32} + a_{22}a_{31})/(a_{11}a_{22} - a_{12}a_{21}), \qquad d_2 = (a_{12}a_{31} + a_{11}a_{32})/(a_{11}a_{22} - a_{12}a_{21}).$$

By multiplying both sides of (3.2), (3.3) and (3.4) by c_2, d_2 and -1, respectively, and then adding these three equalities, we get

$$\begin{aligned}\frac{1}{t}\ln\frac{z(t)}{z(0)} &= \frac{c_2}{t}\ln\frac{x(t)}{x(0)} + \frac{d_2}{t}\ln\frac{y(t)}{y(0)} + \frac{G_3 - \tilde{G}_3}{a_{11}a_{22} - a_{12}a_{21}} - \frac{G}{a_{11}a_{22} - a_{12}a_{21}}\big[z(t)\big] \\ &\quad + \frac{\alpha_3 B_3(t) - \alpha_1 c_2 B_1(t) - \alpha_2 d_2 B_2(t)}{t} \\ &\leq \frac{G_3 - \tilde{G}_3}{a_{11}a_{22} - a_{12}a_{21}} + \varepsilon - \frac{G}{a_{11}a_{22} - a_{12}a_{21}}\big[z(t)\big] \\ &\quad + \frac{\alpha_3 B_3(t) - \alpha_1 c_2 B_1(t) - \alpha_2 d_2 B_2(t)}{t}\end{aligned} \tag{3.9}$$

for sufficiently large t since (3.1) holds and $c_2, d_2 > 0$. By Lemma 3.1, we have

$$\lim_{t\to+\infty} z(t) = 0 \quad \text{a.s.} \tag{3.10}$$

if $G_3 < \tilde{G}_3$ and ε is sufficiently small. Combining (3.10) with equation (3.2) gives

$$\frac{1}{t}\ln\frac{x(t)}{x(0)} \geq r_1 - \frac{\alpha_1^2}{2} - a_{11}\big[x(t)\big] - \varepsilon + \frac{\alpha_1 B_1(t)}{t}$$

for sufficiently large t. It follows from Lemma 3.1 that

$$[x]_* \geq \big(r_1 - \alpha_1^2/2\big)/a_{11} > 0 \quad \text{a.s.,} \tag{3.11}$$

which implies that

$$\limsup_{t\to\infty} \ln x(t)/t = 0 \quad \text{a.s.} \tag{3.12}$$

if (3.1) holds. Let $c_1 = a_{21}$ and $d_1 = a_{11}$ in (3.6). It follows that

$$\begin{aligned} a_{11}\frac{1}{t}\ln\frac{y(t)}{y(0)} &= -a_{21}\frac{1}{t}\ln\frac{x(t)}{x(0)} + \rho_2 - \tilde{\rho}_2 - (a_{11}a_{22} - a_{12}a_{21})\bigl[y(t)\bigr] \\ &\quad - (a_{13}a_{21} + a_{11}a_{23})\bigl[z(t)\bigr] + \frac{a_{21}\alpha_1 B_1(t)}{t} + \frac{a_{11}\alpha_2 B_2(t)}{t} \\ &\le \rho_2 - \tilde{\rho}_2 - (a_{11}a_{22} - a_{12}a_{21})\bigl[y(t)\bigr] + \varepsilon + \frac{a_{21}\alpha_1 B_1(t)}{t} + \frac{a_{11}\alpha_2 B_2(t)}{t} \end{aligned} \tag{3.13}$$

for sufficiently large t if (3.10), (3.11) and (3.1) hold. It follows from Lemma 3.1 that

$$\lim_{t\to+\infty} y(t) = 0 \quad \text{a.s.} \tag{3.14}$$

if $\rho_2 < \tilde{\rho}_2$ and ε is sufficiently small. Combining equation (3.2) with (3.10) and (3.14) gives

$$\frac{1}{t}\ln\frac{y(t)}{y(0)} \le r_1 - \frac{\alpha_1^2}{2} + \varepsilon - a_{11}\bigl[x(t)\bigr] + \frac{\alpha_1 B_1(t)}{t}$$

for sufficiently large t. By applying Lemma 3.1, we get

$$[x]^* \le \bigl(r_1 - \alpha_1^2/2\bigr)/a_{11} \quad \text{a.s.} \tag{3.15}$$

Then we can combine (3.11) and (3.15) to obtain

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t x(s)\,ds = \frac{r_1 - \alpha_1^2/2}{a_{11}} \quad \text{a.s.} \tag{3.16}$$

In view of (3.10), (3.14) and (3.16), (i) of Theorem 2.3 holds.

(ii) It follows from (3.10), (3.3), (3.1) and Lemma 3.1 that

$$\frac{1}{t}\ln\frac{y(t)}{y(0)} \ge r_2 - \frac{\alpha_2^2}{2} - a_{22}\bigl[y(t)\bigr] - \varepsilon + \frac{\alpha_1 B_2(t)}{t}, \qquad [y]_* \ge \bigl(r_2 - \alpha_2^2/2\bigr)/a_{22} > 0 \quad \text{a.s.} \tag{3.17}$$

for sufficiently large t and

$$\limsup_{t\to\infty} \ln y(t)/t = 0 \quad \text{a.s.} \tag{3.18}$$

Let $c_1 = a_{22}$ and $d_1 = a_{12}$ in (3.6). It follows that

$$\begin{aligned} a_{22}\frac{1}{t}\ln\frac{x(t)}{x(0)} &= -a_{12}\frac{1}{t}\ln\frac{y(t)}{y(0)} + \rho_1 - \tilde{\rho}_1 - (a_{11}a_{22} - a_{12}a_{21})\bigl[x(t)\bigr] \\ &\quad - (a_{13}a_{22} + a_{12}a_{23})\bigl[z(t)\bigr] + \frac{a_{22}\alpha_1 B_1(t)}{t} + \frac{a_{12}\alpha_2 B_2(t)}{t} \end{aligned}$$

$$\le \rho_1 - \tilde{\rho}_1 + \varepsilon - (a_{11}a_{22} - a_{12}a_{21})\bigl[x(t)\bigr] + \frac{a_{22}\alpha_1 B_1(t)}{t} + \frac{a_{12}\alpha_2 B_2(t)}{t} \quad (3.19)$$

for sufficiently large t if (3.1) and (3.18) hold. Then

$$\lim_{t\to+\infty} x(t) = 0 \quad \text{a.s.} \quad (3.20)$$

if $\rho_1 < \tilde{\rho}_1$ and ε is sufficiently small. From (3.10), (3.20) and Lemma 3.1, we have

$$\frac{1}{t}\ln\frac{y(t)}{y(0)} \le r_2 - \frac{\alpha_2^2}{2} + \varepsilon - a_{22}\bigl[y(t)\bigr] + \frac{\alpha_2 B_2(t)}{t}$$

for sufficiently large t and

$$[y]^* \le \bigl(r_2 - \alpha_2^2/2\bigr)/a_{22} \quad \text{a.s.} \quad (3.21)$$

if $r_2 > \alpha_2^2/2$ and ε is sufficiently small. It follows from (3.10), (3.20), (3.17) and (3.21) that (ii) of Theorem 2.3 holds.

(iii) We first prove that

$$\lim_{t\to+\infty} x(t) = 0 \quad \text{a.s.} \quad \text{and} \quad \lim_{t\to+\infty} y(t) = 0 \quad \text{a.s.} \quad (3.22)$$

Let $c_1 = a_{22}$ and $d_1 = a_{12}$ in (3.6). From (3.19) and $\rho_1 < \tilde{\rho}_1$, we conclude that (3.7) or (3.8) holds. If (3.7) holds, then equation (3.3) multiplied by a_{33} minus equation (3.4) multiplied by a_{23} gives

$$\begin{aligned} a_{33}\frac{1}{t}\ln\frac{y(t)}{y(0)} &= a_{23}\frac{1}{t}\ln\frac{z(t)}{z(0)} + \rho_5 - \tilde{\rho}_5 + (a_{21}a_{33} + a_{23}a_{31})\bigl[x(t)\bigr] \\ &\quad - (a_{22}a_{33} - a_{23}a_{32})\bigl[y(t)\bigr] + \frac{a_{33}\alpha_2 B_2(t)}{t} - \frac{a_{23}\alpha_3 B_3(t)}{t} \\ &\le \rho_5 - \tilde{\rho}_5 + \varepsilon - (a_{22}a_{33} - a_{23}a_{32})\bigl[y(t)\bigr] + \frac{a_{33}\alpha_2 B_2(t)}{t} - \frac{a_{23}\alpha_3 B_3(t)}{t}. \end{aligned} \quad (3.23)$$

If follows from (A_1) and Lemma 3.1 that $\lim_{t\to+\infty} y(t) = 0$ a.s. If (3.8) holds, then equation (3.2) multiplied by a_{33} minus equation (3.4) multiplied by a_{13} gives

$$\begin{aligned} a_{33}\frac{1}{t}\ln\frac{x(t)}{x(0)} &= a_{13}\frac{1}{t}\ln\frac{z(t)}{z(0)} + \rho_3 - \tilde{\rho}_3 - (a_{11}a_{33} - a_{13}a_{31})\bigl[x(t)\bigr] \\ &\quad + (a_{12}a_{33} + a_{13}a_{32})\bigl[y(t)\bigr] + \frac{a_{33}\alpha_1 B_1(t)}{t} - \frac{a_{13}\alpha_3 B_3(t)}{t} \\ &\le \rho_3 - \tilde{\rho}_3 + \varepsilon - (a_{11}a_{33} - a_{13}a_{31})\bigl[x(t)\bigr] + \frac{a_{33}\alpha_1 B_1(t)}{t} - \frac{a_{13}\alpha_3 B_3(t)}{t}. \end{aligned} \quad (3.24)$$

From (A_2) and Lemma 3.1, we have $\lim_{t\to+\infty} x(t) = 0$ a.s. Similarly, if (A_3) or (A_4) holds, then (3.22) holds.

It follows from (3.5) and Lemma 3.1 that

$$[z]^* \le \bigl(r_3 - \alpha_3^2/2\bigr)/a_{33} \quad \text{a.s.} \quad (3.25)$$

if $r_3 > \alpha_3^2/2$. On the other hand, by applying (3.22) and Lemma 3.1, we have

$$\frac{1}{t}\ln\frac{z(t)}{z(0)} \geq r_3 - \frac{\alpha_3^2}{2} - \varepsilon - a_{33}\bigl[z(t)\bigr] + \frac{\alpha_3 B_3(t)}{t}$$

and

$$[z]_* \geq \bigl(r_3 - \alpha_3^2/2\bigr)/a_{33} \quad \text{a.s.} \tag{3.26}$$

It follows from (3.22), (3.25) and (3.26) that (iii) of Theorem 2.3 holds. □

Proof of Theorem 2.4 We first establish (iv) of Theorem 2.4. It follows from (3.9) and Lemma 3.1 that $\lim_{t\to+\infty} z(t) = 0$ a.s. From (3.13), we have

$$a_{11}\frac{1}{t}\ln\frac{y(t)}{y(0)} \geq \rho_2 - \tilde{\rho}_2 - \varepsilon - (a_{11}a_{22} - a_{12}a_{21})\bigl[y(t)\bigr] + \frac{a_{21}\alpha_1 B_1(t)}{t} + \frac{a_{11}\alpha_2 B_2(t)}{t}$$

for sufficiently large t. By applying (6), Lemma 3.1 and the arbitrariness of ε, we obtain

$$[y]_* \geq (\rho_2 - \tilde{\rho}_2)/(a_{11}a_{22} - a_{12}a_{21}) \quad \text{a.s.} \tag{3.27}$$

and

$$\limsup_{t\to\infty} \ln y(t)/t = 0 \quad \text{a.s.} \tag{3.28}$$

Substituting (3.27) into (3.2) leads to

$$\frac{1}{t}\ln\frac{x(t)}{x(0)} \geq r_1 - \frac{\alpha_1^2}{2} - a_{11}\bigl[x(t)\bigr] + a_{12}[y]_* - a_{13}\bigl[z(t)\bigr] + \frac{\alpha_1 B_1(t)}{t}.$$

This implies that

$$[x]_* \geq (\rho_1 - \tilde{\rho}_1)/(a_{11}a_{22} - a_{12}a_{21}) \quad \text{a.s.} \tag{3.29}$$

by Lemma 3.1 and ε is sufficiently small. On the other hand, it follows from (3.19) that

$$a_{22}\frac{1}{t}\ln\frac{x(t)}{x(0)} \leq \rho_1 - \tilde{\rho}_1 + \varepsilon - (a_{11}a_{22} - a_{12}a_{21})\bigl[x(t)\bigr] + \frac{a_{22}\alpha_1 B_1(t)}{t} + \frac{a_{12}\alpha_2 B_2(t)}{t}$$

for sufficiently large t if (3.1) and (3.28) hold. It follows from Lemma 3.1, the arbitrariness of ε and $\rho_1 > \tilde{\rho}_1$ that

$$[x]^* \leq (\rho_1 - \tilde{\rho}_1)/(a_{11}a_{22} - a_{12}a_{21}) \quad \text{a.s.} \tag{3.30}$$

Similarly, from (3.13) and $\rho_2 > \tilde{\rho}_2$, we get

$$[y]^* \leq (\rho_2 - \tilde{\rho}_2)/(a_{11}a_{22} - a_{12}a_{21}) \quad \text{a.s.} \tag{3.31}$$

It follows from (3.27), (3.29), (3.30) and (3.31) that (iv) of Theorem 2.4 holds.

We now establish (v) of Theorem 2.4. From (3.9) and Lemma 3.1, we conclude that

$$[z]^* \le (G_3 - \tilde{G}_3)/G. \tag{3.32}$$

It follows from (3.19) and (3.32) that

$$\begin{aligned} a_{22}\frac{1}{t}\ln\frac{x(t)}{x(0)} &\ge \rho_1 - \tilde{\rho}_1 - \varepsilon - (a_{11}a_{22} - a_{12}a_{21})\big[x(t)\big] \\ &\quad - (a_{13}a_{22} + a_{12}a_{23})[z]^* + \frac{a_{22}\alpha_1 B_1(t)}{t} + \frac{a_{12}\alpha_2 B_2(t)}{t} \\ &\ge (a_{11}a_{22} - a_{12}a_{21})(G_1 - \tilde{G}_1)/G - \varepsilon \\ &\quad - (a_{11}a_{22} - a_{12}a_{21})\big[x(t)\big] + \frac{a_{22}\alpha_1 B_1(t)}{t} + \frac{a_{12}\alpha_2 B_2(t)}{t} \end{aligned} \tag{3.33}$$

for sufficiently large t. By applying Lemma 3.1, one obtains

$$[x]_* \ge (G_1 - \tilde{G}_1)/G > 0 \quad \text{a.s.} \tag{3.34}$$

for $G_1 > \tilde{G}_1$, and ε is sufficiently small. Let

$$\begin{aligned} c_3 &= -(a_{21}a_{33} + a_{23}a_{31})/(a_{11}a_{33} - a_{13}a_{31}), \\ d_3 &= (a_{11}a_{23} + a_{13}a_{21})/(a_{11}a_{33} - a_{13}a_{31}). \end{aligned}$$

Multiplying both sides of (3.2), (3.3) and (3.4) by c_3, -1 and d_3, respectively, and then adding these three equalities yield

$$\begin{aligned} \frac{1}{t}\ln\frac{y(t)}{y(0)} &= \frac{c_3}{t}\ln\frac{x(t)}{x(0)} + \frac{d_3}{t}\ln\frac{z(t)}{z(0)} + \frac{G_2 - \tilde{G}_2}{a_{11}a_{33} - a_{13}a_{31}} - \frac{G}{a_{11}a_{33} - a_{13}a_{31}}\big[y(t)\big] \\ &\quad + \frac{\alpha_2 B_2(t) - \alpha_1 c_3 B_1(t) - \alpha_3 d_3 B_3(t)}{t} \\ &\le \frac{G_2 - \tilde{G}_2}{a_{11}a_{33} - a_{13}a_{31}} + \varepsilon - \frac{G}{a_{11}a_{33} - a_{13}a_{31}}\big[y(t)\big] \\ &\quad + \frac{\alpha_2 B_2(t) - \alpha_1 c_3 B_1(t) - \alpha_3 d_3 B_3(t)}{t} \end{aligned} \tag{3.35}$$

for sufficiently large t if (3.34) and (3.1) hold. It follows from Lemma 3.1 that

$$\lim_{t\to+\infty} y(t) = 0 \quad \text{a.s.} \tag{3.36}$$

if $G_2 < \tilde{G}_2$ and ε is sufficiently small. Next, using (3.24), Lemma 3.1 and $\rho_3 > \tilde{\rho}_3$, one gets

$$[x]^* \le (\rho_3 - \tilde{\rho}_3)/(a_{11}a_{33} - a_{13}a_{31}) \quad \text{a.s.} \tag{3.37}$$

Equation (3.4) multiplied by a_{11} minus equation (3.2) multiplied by a_{31} gives

$$a_{11}\frac{1}{t}\ln\frac{z(t)}{z(0)} \le \rho_4 - \tilde{\rho}_4 + \varepsilon - (a_{11}a_{33} - a_{13}a_{31})\big[z(t)\big] + \frac{a_{11}\alpha_3 B_3(t)}{t} - \frac{a_{31}\alpha_1 B_1(t)}{t}.$$

It follows from Lemma 3.1 that

$$[z]^* \le (\rho_4 - \tilde{\rho}_4)/(a_{11}a_{33} - a_{13}a_{31}) \quad \text{a.s.} \tag{3.38}$$

for $\rho_4 > \tilde{\rho}_4$. On the other hand, substituting (3.38) and (3.36) into (3.2) yields

$$\begin{aligned}\frac{1}{t}\ln\frac{x(t)}{x(0)} &\ge r_1 - \frac{\alpha_1^2}{2} - \varepsilon - a_{11}\bigl[x(t)\bigr] - a_{13}[z]^* + \frac{\alpha_1 B_1(t)}{t} \\ &\ge \frac{a_{11}(\rho_3 - \tilde{\rho}_3)}{a_{11}a_{33} - a_{13}a_{31}} - \varepsilon - a_{11}\bigl[x(t)\bigr] + \frac{\alpha_1 B_1(t)}{t}\end{aligned}$$

for sufficiently large t. Then

$$[x]_* \ge (\rho_3 - \tilde{\rho}_3)/(a_{11}a_{33} - a_{13}a_{31}) \quad \text{a.s.} \tag{3.39}$$

if ε is sufficiently small and $\rho_3 > \tilde{\rho}_3$. Applying Lemma 3.1 and substituting (3.39) and (3.36) into (3.4) yield

$$\begin{aligned}\frac{1}{t}\ln\frac{z(t)}{z(0)} &\ge r_3 - \frac{\alpha_3^2}{2} - a_{31}[x]^* - \varepsilon - a_{33}\bigl[z(t)\bigr] + \frac{\alpha_3 B_3(t)}{t} \\ &\ge \frac{a_{33}(\rho_4 - \tilde{\rho}_4)}{a_{11}a_{33} - a_{13}a_{31}} - \varepsilon - a_{33}\bigl[z(t)\bigr] + \frac{\alpha_3 B_3(t)}{t}\end{aligned}$$

and

$$[z]_* \ge (\rho_4 - \tilde{\rho}_4)/(a_{11}a_{33} - a_{13}a_{31}) \quad \text{a.s.} \tag{3.40}$$

It follows from (3.36)-(3.40) that (v) of Theorem 2.4 holds.

We finally establish (vi) of Theorem 2.4. Substituting (3.32) into (3.13) and applying Lemma 3.1 yield

$$\begin{aligned}a_{11}\frac{1}{t}\ln\frac{y(t)}{y(0)} &\ge \rho_2 - \tilde{\rho}_2 - \varepsilon - (a_{11}a_{22} - a_{12}a_{21})\bigl[y(t)\bigr] \\ &\quad - (a_{13}a_{21} + a_{11}a_{23})[z]^* + \frac{a_{21}\alpha_1 B_1(t)}{t} + \frac{a_{11}\alpha_2 B_2(t)}{t} \\ &\ge (a_{11}a_{22} - a_{12}a_{21})(G_2 - \tilde{G}_2)/G - \varepsilon \\ &\quad - (a_{11}a_{22} - a_{12}a_{21})\bigl[y(t)\bigr] + \frac{a_{21}\alpha_1 B_1(t)}{t} + \frac{a_{11}\alpha_2 B_2(t)}{t}\end{aligned} \tag{3.41}$$

for sufficiently large t and

$$[y]_* \ge (G_2 - \tilde{G}_2)/G > 0 \quad \text{a.s.} \tag{3.42}$$

for $G_2 > \tilde{G}_2$, and ε is sufficiently small. Let

$$c_4 = -(a_{12}a_{33} + a_{13}a_{32})/(a_{22}a_{33} - a_{23}a_{32}), \qquad d_4 = (a_{12}a_{23} + a_{13}a_{22})/(a_{22}a_{33} - a_{23}a_{32}).$$

Multiplying both sides of (3.2), (3.3) and (3.4) by -1, c_4 and d_4, respectively, and adding these three equalities, we have

$$\begin{aligned}\frac{1}{t}\ln\frac{x(t)}{x(0)} &= \frac{c_4}{t}\ln\frac{y(t)}{y(0)} + \frac{d_4}{t}\ln\frac{z(t)}{z(0)} \\ &\quad + \frac{G_1-\tilde{G}_1}{a_{22}a_{33}-a_{23}a_{32}} - \frac{G}{a_{22}a_{33}-a_{23}a_{32}}\bigl[x(t)\bigr] \\ &\quad + \frac{\alpha_1 B_1(t)-\alpha_2 c_4 B_2(t)-\alpha_3 d_4 B_3(t)}{t} \\ &\le \frac{G_1-\tilde{G}_1}{a_{22}a_{33}-a_{23}a_{32}} + \varepsilon - \frac{G}{a_{22}a_{33}-a_{23}a_{32}}\bigl[x(t)\bigr] \\ &\quad + \frac{\alpha_1 B_1(t)-\alpha_2 c_4 B_2(t)-\alpha_3 d_4 B_3(t)}{t}\end{aligned} \tag{3.43}$$

for sufficiently large t since (3.42) and (3.1) hold. Then

$$\lim_{t\to+\infty} x(t) = 0 \quad \text{a.s.} \tag{3.44}$$

if $G_1 < \tilde{G}_1$, $a_{22}a_{33} > a_{23}a_{32}$ and ε is sufficiently small. From (3.23), Lemma 3.1 and $\rho_5 > \tilde{\rho}_5$, one has

$$[y]^* \le (\rho_5 - \tilde{\rho}_5)/(a_{22}a_{33}-a_{23}a_{32}) \quad \text{a.s.} \tag{3.45}$$

Using (3.44), (3.3), (3.4) and similar arguments as the ones given by (v) of Theorem 2.4, we derive

$$\begin{aligned}&[y]_* \ge (\rho_5 - \tilde{\rho}_5)/(a_{22}a_{33}-a_{23}a_{32}) \quad \text{a.s.}, \\ &[z]_* \ge (\rho_6 - \tilde{\rho}_6)/(a_{22}a_{33}-a_{23}a_{32}) \quad \text{a.s.}, \\ &[z]^* \le (\rho_6 - \tilde{\rho}_6)/(a_{22}a_{33}-a_{23}a_{32}) \quad \text{a.s.}\end{aligned} \tag{3.46}$$

This yields (vi) of Theorem 2.4. □

Proof of Theorem 2.5 From the above discussion, we conclude that if $a_{11}a_{22} > a_{12}a_{21}$ and $G_i > \tilde{G}_i$, $i = 1, 2, 3$, hold, then we have (3.32), (3.34) and (3.42). Meanwhile, it follows from (3.9) and Lemma 3.1 that

$$\begin{aligned}\frac{1}{t}\ln\frac{z(t)}{z(0)} &\ge \frac{G_3-\tilde{G}_3}{a_{11}a_{22}-a_{12}a_{21}} - \varepsilon - \frac{G}{a_{11}a_{22}-a_{12}a_{21}}\bigl[z(t)\bigr] \\ &\quad + \frac{\alpha_3 B_3(t)-\alpha_1 c_2 B_1(t)-\alpha_2 d_2 B_2(t)}{t}\end{aligned}$$

for sufficiently large t, and

$$[z]_* \ge (G_3 - \tilde{G}_3)/G \quad \text{a.s.} \tag{3.47}$$

By (3.35) and (3.43), we have

$$\frac{1}{t}\ln\frac{y(t)}{y(0)} \le \frac{G_2-\tilde{G}_2}{a_{11}a_{33}-a_{13}a_{31}} + \varepsilon - \frac{G}{a_{11}a_{33}-a_{13}a_{31}}\big[y(t)\big] + \frac{\alpha_2 B_2(t) - \alpha_1 c_3 B_1(t) - \alpha_3 d_3 B_3(t)}{t}$$

and

$$\frac{1}{t}\ln\frac{x(t)}{x(0)} \le \frac{G_1-\tilde{G}_1}{a_{22}a_{33}-a_{23}a_{32}} + \varepsilon - \frac{G}{a_{22}a_{33}-a_{23}a_{32}}\big[x(t)\big] + \frac{\alpha_1 B_1(t) - \alpha_2 c_4 B_2(t) - \alpha_3 d_4 B_3(t)}{t}$$

for sufficiently large t. Then

$$[x]^* \le (G_1-\tilde{G}_1)/G \quad \text{a.s.} \quad \text{and} \quad [y]^* \le (G_2-\tilde{G}_2)/G \quad \text{a.s.} \tag{3.48}$$

From (3.32), (3.34), (3.42), (3.47) and (3.48), we conclude that Theorem 2.5 holds. This completes the proof of Theorem 2.5. □

3.2 Proof of Theorem 2.6

To prove Theorem 2.6, we need the following lemmas.

Lemma 3.3 (see [40]) *If g is a non-negative function defined on* $[0,+\infty)$ *such that g is integrable and uniformly continuous, then* $\lim_{t\to+\infty} g(t) = 0$.

Let (Ω, F, P) be a probability space, and let (E, B) be a measurable space. A family of random variables $\{X_t\}_{t\in T}$ satisfies $X_t : (\Omega, F) \to (E, B)$ for $t \in T$and for all $\omega \in \Omega$. The mapping $X_t(\cdot,\omega) : t \in T \to X_t(\omega) \in E$ is called a sample path of the process corresponding to ω.

Carrying out similar arguments as those of Lemma 15 in [25], we have the following lemma.

Lemma 3.4 *Let* $(x(t), y(t), z(t))^T$ *be a positive solution of* (1.2). *Then almost every sample path of* $x(t)$, $y(t)$ *and* $z(t)$ *is uniformly continuous.*

Proof of Theorem 2.6 Let $(x_i(t), y_i(t), z_i(t))^T$, $i = 1, 2$, be two arbitrary solutions of model (1.2) with the initial values $(x_i(0), y_i(0), z_i(0)) \in \mathbb{R}^3_+$, $i = 1, 2, 3$, $\omega_i > 0$, $i = 1, 2, 3$, and

$$V(t) = \lambda_1\big|\ln\big(x_1(t)/x_2(t)\big)\big| + \lambda_2\big|\ln\big(y_1(t)/y_2(t)\big)\big| + \lambda_3\big|\ln\big(z_1(t)/z_2(t)\big)\big|.$$

A direct calculation has

$$\begin{aligned} dV(t) = {} & \lambda_1 \operatorname{sgn}\big(x_1(t)-x_2(t)\big)\big[-a_{11}\big(x_1(t)-x_2(t)\big) \\ & + a_{12}\big(y_1(t)-y_2(t)\big) - a_{13}\big(z_1(t)-z_2(t)\big)\big]\,dt \\ & + \lambda_2 \operatorname{sgn}\big(y_1(t)-y_2(t)\big)\big[a_{21}\big(x_1(t)-x_2(t)\big) \\ & - a_{22}\big(y_1(t)-y_2(t)\big) - a_{23}\big(z_1(t)-z_2(t)\big)\big]\,dt \end{aligned}$$

$$
\begin{aligned}
&- \lambda_3 \operatorname{sgn}\bigl(z_1(t) - z_2(t)\bigr)\bigl[a_{31}\bigl(x_1(t) - x_2(t)\bigr) \\
&+ a_{32}\bigl(y_1(t) - y_2(t)\bigr) - a_{33}\bigl(z_1(t) - z_2(t)\bigr)\bigr]\,dt \\
\le{}& -(\lambda_1 a_{11} - \lambda_2 a_{21} - \lambda_3 a_{31})\bigl|x_1(t) - x_2(t)\bigr|\,dt \\
&- (\lambda_2 a_{22} - \lambda_1 a_{12} - \lambda_3 a_{32})\bigl|y_1(t) - y_2(t)\bigr|\,dt \\
&- (\lambda_3 a_{33} - \lambda_1 a_{13} - \lambda_2 a_{23})\bigl|z_1(t) - z_2(t)\bigr|\,dt \\
:={}& -R(t)\,dt.
\end{aligned}
$$

Then

$$
V(t) + \int_0^t R(s)\,ds \le V(0) < +\infty.
$$

Applying $V(t) \ge 0$, Lemma 3.3 and Lemma 3.4 shows that (1.2) is globally asymptotically stable. □

3.3 The proof of Theorem 2.7

To complete the proof of Theorem 2.7, we first introduce the theory of Has'minskii [41]. Let E^l be an l-dimensional Euclidean space and $Y(t)$ be a homogeneous Markov process in E^l. Moreover, $Y(t)$ satisfies the stochastic differential equation

$$
dY(t) = b(Y)\,dt + \sum_{m=1}^{k} g_m(Y)\,dB_m(t).
$$

Let $\Lambda(x) = (a_{ij}(x))$ be the diffusion matrix, where $a_{ij}(x) = \sum_{m=1}^{k} g_m^i(x) g_m^j(x)$.

We are now presenting a useful condition. There is a bounded domain $U \subset E^l$ with the regular boundary Γ such that

(A_1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $\Lambda(x)$ is bounded away from zero;

(A_2) If $x \in E^l \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x \tau < +\infty$ for every compact subset $K \subset E^l$.

In order to verify (A_1) and (A_2) in Assumption 3.1, we introduce two stronger conditions (see [36, 42, 43]):

(B_1) To establish (A_1), it is sufficient to show that T is uniformly elliptical in U, where $Tu = b(x)u_x + \operatorname{tr}(\Lambda(x)u_{xx})/2$, that is, there exists $c > 0$ such that $\sum_{i,j=1}^{k} a_{ij}(x)\xi_i\xi_j \ge c|\xi|^2$, $x \in U$, $\xi \in R^l$;

(B_2) To establish (A_2), it is sufficient to prove that there exist a neighborhood U and a nonnegative C^2-function $V(x)$ such that, for any $x \in E^l \setminus U$, $LV(x) < 0$.

Lemma 3.5 ([41]) *If Assumption* 3.1 *holds, then the Markov process* $Y(t)$ *has a stationary distribution* $\mu(\cdot)$. *Furthermore, if* $f(\cdot)$ *is a function integrable with respect to the measure* μ, *then*

$$
P\left\{\lim_{t\to+\infty} \frac{1}{t}\int_0^t f\bigl(Y(s)\bigr)\,ds = \int_{E^l} f(x)\mu(dx)\right\} = 1.
$$

Proof of Theorem 2.7 In order to establish our result, we only need to verify that conditions (B_1) and (B_2) hold. Note that the diffusion matrix of (1.2) is $\Lambda(x) = \mathrm{diag}(\alpha_1^2 x^2, \alpha_2^2 y^2, \alpha_3^2 z^2)$.

By $\delta_i > 0$, $i = 1, 2, 3$, and formula (2.2), we obtain that the ellipsoid

$$\delta_1 (x - x^*)^2 + \delta_2 (y - y^*)^2 + \delta_3 (z - z^*)^2 = \alpha_1^2 x^*/2 + \alpha_2^2 y^*/2 + \alpha_3^2 z^*/2$$

lies entirely in $\mathbb{R}_+^3$. A direct computation shows that there exists a positive constant γ such that

$$\sum_{i,j=1}^{3} a_{ij}(x)\omega_i\omega_j = \alpha_1^2 x^2 \omega_1^2 + \alpha_2^2 y^2 \omega_2^2 + \alpha_3^2 z^2 \omega_3^2 > \gamma |\omega|^2$$

for $x \in \bar{U}$ and $\omega \in \mathbb{R}^3$, where U is a neighborhood of the ellipsoid with $\bar{U} \subseteq \mathbb{R}_+^3$. This shows that condition (B_1) holds. On the other hand, we define

$$V(x, y, z) = x - x^* - x^* \ln(x/x^*) + y - y^* - y^* \ln(y/y^*) + z - z^* - z^* \ln(z/z^*).$$

A simple calculation has

$$dV(x, y, z) = LV(x, y, z)\, dt + (x - x^*)\alpha_1\, dB_1(t) + (y - y^*)\alpha_2\, dB_2(t) + (z - z^*)\alpha_3\, dB_3(t),$$

where

$$\begin{aligned} LV(x, y, z) &= (x - x^*)[r_1 - a_{11}x + a_{12}y - a_{13}z] + \alpha_1^2 x^*/2 \\ &\quad + (y - y^*)[r_2 + a_{21}x - a_{22}y - a_{23}z] + \alpha_2^2 y^*/2 \\ &\quad + (z - z^*)[r_3 - a_{31}x - a_{32}y - a_{33}z] + \alpha_3^2 z^*/2 \\ &\le -\delta_1 (x - x^*)^2 - \delta_2 (y - y^*)^2 - \delta_3 (z - z^*)^2 + (\alpha_1^2 x^* + \alpha_2^2 y^* + \alpha_3^2 z^*)/2 \end{aligned} \tag{3.49}$$

since $(x^*, y^*, z^*)^T$ is the positive equilibrium point of the corresponding deterministic model of (1.2). This implies that $LV(x) < 0$ for any $x \in \mathbb{R}_+^3 \setminus U$ and condition (B_2) holds. In view of the above arguments and Lemma 3.5, we conclude that there is a stationary distribution $\mu(\cdot)$ for model (1.2) and it has the ergodic property.

Finally, we claim that (5) holds. Applying the dominated convergence theorem and Lemma 3.2 gives

$$\begin{aligned} E\Big[\lim_{t\to+\infty} \frac{1}{t}\int_0^t [x(s) \wedge Q]\, ds\Big] &= \lim_{t\to+\infty} \frac{1}{t}\int_0^t E[x(s) \wedge Q]\, ds \le K, \\ E\Big[\lim_{t\to+\infty} \frac{1}{t}\int_0^t [y(s) \wedge Q]\, ds\Big] &= \lim_{t\to+\infty} \frac{1}{t}\int_0^t E[y(s) \wedge Q]\, ds \le K, \\ E\Big[\lim_{t\to+\infty} \frac{1}{t}\int_0^t [z(s) \wedge Q]\, ds\Big] &= \lim_{t\to+\infty} \frac{1}{t}\int_0^t E[z(s) \wedge Q]\, ds \le K \end{aligned}$$

for any $Q > 0$, where $\wedge$ is minimal. It follows from the ergodic property that

$$\int_{\mathbb{R}_+^3} [\omega_1 \wedge Q]\mu(d\omega_1, d\omega_2, d\omega_3) = E \lim_{t\to+\infty} \frac{1}{t}\int_0^t [x(s) \wedge Q]\, ds < K,$$

$$\int_{\mathbb{R}^3_+} [\omega_2 \wedge Q]\mu(d\omega_1, d\omega_2, d\omega_3) = E \lim_{t\to+\infty} \frac{1}{t}\int_0^t \bigl[y(s) \wedge Q\bigr]\,ds < K,$$

$$\int_{\mathbb{R}^3_+} [\omega_3 \wedge Q]\mu(d\omega_1, d\omega_2, d\omega_3) = E \lim_{t\to+\infty} \frac{1}{t}\int_0^t \bigl[z(s) \wedge Q\bigr]\,ds < K.$$

Letting $Q \to +\infty$ yields

$$\int_{\mathbb{R}^3_+} \omega_1 \mu(d\omega_1, d\omega_2, d\omega_3) \le K,$$

$$\int_{\mathbb{R}^3_+} \omega_2 \mu(d\omega_1, d\omega_2, d\omega_3) \le K,$$

$$\int_{\mathbb{R}^3_+} \omega_3 \mu(d\omega_1, d\omega_2, d\omega_3) \le K.$$

As a consequence of Lemma 3.5, (2.3) holds. This completes the proof of the theorem. □

4 Numerical simulations

In this section, we do some numerical simulations to illustrate or complement our mathematical findings by using Milstein's method [44]. By taking advantage of revising the stochastic increment, Milstein's method achieves the high order convergence and can be more accurate to approximate the exact solution.

Let

$$\begin{aligned} &r_1 = 0.5, \quad r_2 = 0.25, \quad r_3 = 0.4, \\ &a_{11} = 0.8, \quad a_{12} = 0.2, \quad a_{13} = 0.2, \\ &a_{21} = 0.3, \quad a_{22} = 0.9, \quad a_{23} = 0.1, \\ &a_{31} = 0.3, \quad a_{32} = 0.1, \quad a_{33} = 0.5. \end{aligned} \tag{4.1}$$

A simple calculation gives

$$\begin{aligned} &G = 0.256, \quad G_1 = 0.17, \quad G_2 = 0.119, \quad G_3 = 0.079, \\ &a_{11} > a_{12}, \quad a_{22} > a_{21}, \quad a_{11}a_{33} > a_{13}a_{31}, \quad a_{22}a_{33} > a_{23}a_{32}, \\ &\rho_1 = 0.5, \quad \rho_2 = 0.35, \quad \rho_3 = 0.17, \\ &\rho_4 = 0.17, \quad \rho_5 = 0.085, \quad \rho_6 = 0.335. \end{aligned}$$

We choose the appropriate parameters α_i, $i = 1, 2, 3$, satisfying the conditions of our main theorems.

Figure 1(a) means that if the white noise is sufficiently large, then it is harmful to all the species and leads to the extinction of the species. In Figure 1(b), (c) and (d), we conclude that in the appropriate conditions, one species is stochastically persistent, and the other two species are extinct a.s.

Figure 2 shows that in the complex ecological relationship, the noise may have a positive effect on the coexistence of species, such as α_1, α_2 on z and α_3 on x, y. Therefore, in a certain context, noise may be beneficial to the maintenance of biodiversity.

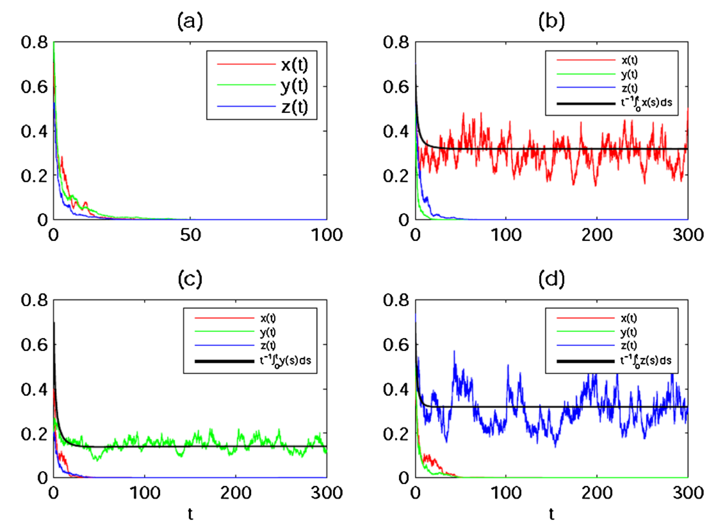

Figure 1 **(a)** Let $\alpha_1 = 1.1$, $\alpha_2 = 0.8$ and $\alpha_3 = 1$ such that $r_i < \alpha_i^2/2$, $i = 1, 2, 3$. It follows from Theorem 2.2 that all the species of model (1.2) go to extinction a.s. **(b)** Let $\alpha_1 = 0.7$, $\alpha_2 = 1$ and $\alpha_3 = 0.85$ such that $r_1 > \alpha_1^2/2$, $\rho_2 < \tilde{\rho}_2$, and $G_3 < \tilde{G}_3$ ((i) of Theorem 2.3). Then the species $y(t)$ and $z(t)$ go to extinction a.s. and $x(t)$ is stochastically persistent with $\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{r_1-\alpha_1^2/2}{a_{11}} = 0.31875$. **(c)** Let $\alpha_1 = 1.1$, $\alpha_2 = 0.5$ and $\alpha_3 = 1$ such that $r_2 > \alpha_2^2/2$, $\rho_1 < \tilde{\rho}_1$ and $G_3 < \tilde{G}_3$ ((ii) of Theorem 2.3). Then the species $x(t)$ and $z(t)$ go to extinction a.s. and $y(t)$ is stochastically persistent with $\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{r_2-\alpha_2^2/2}{a_{22}} = 0.1389$. **(d)** Let $\alpha_1 = 1$, $\alpha_2 = 0.8$ and $\alpha_3 = 0.7$ such that $r_3 > \alpha_3^2/2$, $\rho_1 < \tilde{\rho}_1$ and $\rho_5 < \tilde{\rho}_5$ ((iii) of Theorem 2.3). Then the species $x(t)$ and $y(t)$ go to extinction a.s. and $z(t)$ is stochastically persistent with $\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{r_3-\alpha_3^2/2}{a_{33}} = 0.31$.

Figure 4 shows that if a stationary distribution has the ergodic property, then the mean of population density in time with the development of time is equal to the mean of population density in space.

5 Conclusions

Our results provide insight into dynamic properties of a stochastic cooperation-competition model including stochastic persistence and stochastic extinction of species, global asymptotic stability of positive solutions and the existence of stationary distribution. The theoretical analysis shows that there are eight different cases for stochastic persistence and stochastic extinction of species. Especially, we also establish a sufficient criterion for global asymptotic stability of the positive solutions of model (1.2). A stationary distribution of model (1.2) with the ergodic property is investigated. This study shows that the time average of population size with the development of time is equal to the stationary distribution in space.

It is important and interesting to explore the effect of the noise on the persistence or extinction of species. To do this, we need to investigate the conclusions of Theorems 2.2-2.5 and evaluate the impact of the white noise intensity α_i ($i = 1, 2, 3$) on these conclusions. The summary of the effect of the noise on species is listed in Table 1. Based on the analysis above, we may draw two conclusions:

- the noise is a harmful factor for single species in general, such as α_1 for x, α_2 for y and α_3 for z;
- in the complex ecological relationship, the noise may also have a positive effect on the coexistence of species, such as α_1, α_2 on z and α_3 on x, y.

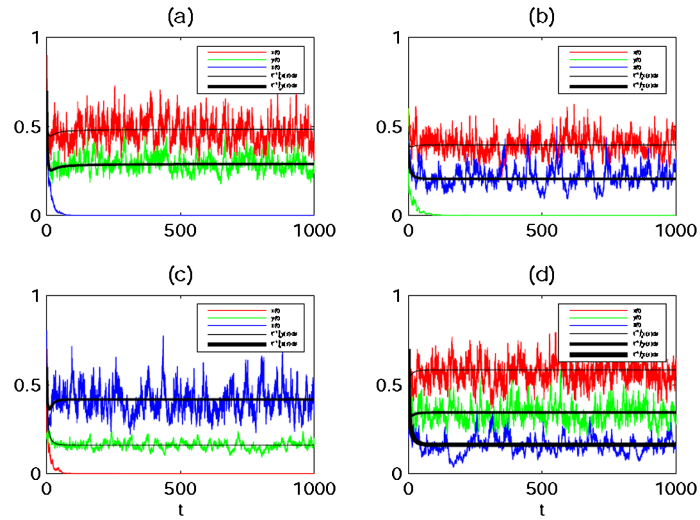

Figure 2 **(a)** $\alpha_1 = 0.6$, $\alpha_2 = 0.5$ and $\alpha_3 = 0.74$ such that $\rho_1 > \tilde{\rho}_1$, $\rho_2 > \tilde{\rho}_2$ and $G_3 < \tilde{G}_3$ ((iv) of Theorem 2.4). Then the species $z(t)$ goes to extinction a.s. and the species $x(t)$ and $y(t)$ are stochastically persistent with $\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{\rho_1-\tilde{\rho}_1}{a_{11}a_{22}-a_{12}a_{21}} = 0.4742$, $\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{\rho_2-\tilde{\rho}_2}{a_{11}a_{22}-a_{12}a_{21}} = 0.2970$. **(b)** Let $\alpha_1 = 0.53$, $\alpha_2 = 0.84$ and $\alpha_3 = 0.6$ such that $\rho_3 > \tilde{\rho}_3$, $\rho_4 > \tilde{\rho}_4$ and $G_2 < \tilde{G}_2$ ((v) of Theorem 2.4). Then the species $y(t)$ goes to extinction a.s. and the species $x(t)$ and $z(t)$ are stochastically persistent with $\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{\rho_3-\tilde{\rho}_3}{a_{11}a_{33}-a_{13}a_{31}} = 0.3993$, $\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{\rho_4-\tilde{\rho}_4}{a_{11}a_{33}-a_{13}a_{31}} = 0.2003$. **(c)** Let $\alpha_1 = 0.98$, $\alpha_2 = 0.36$ and $\alpha_3 = 0.6$ such that $\rho_5 > \tilde{\rho}_5$, $\rho_6 > \tilde{\rho}_6$ and $G_1 > \tilde{G}_1$ ((vi) of Theorem 2.4). Then the species $x(t)$ goes to extinction a.s. and the species $y(t)$ and $z(t)$ are stochastically persistent with $\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{\rho_5-\tilde{\rho}_5}{a_{22}a_{33}-a_{23}a_{32}} = 0.1605$, $\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{\rho_6-\tilde{\rho}_6}{a_{22}a_{33}-a_{23}a_{32}} = 0.4079$. **(d)** Let $\alpha_1 = 0.4$, $\alpha_2 = 0.45$ and $\alpha_3 = 0.48$ such that $G_1 > \tilde{G}_1$, $G_2 > \tilde{G}_2$ and $G_3 > \tilde{G}_3$ (Theorem 2.5). Then all the species are stochastically persistent with $\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\,ds = \frac{G_1-\tilde{G}_1}{G} = 0.5691$, $\lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)\,ds = \frac{G_2-\tilde{G}_2}{G} = 0.3371$, $\lim_{t\to+\infty} \frac{1}{t}\int_0^t z(s)\,ds = \frac{G_3-\tilde{G}_3}{G} = 0.1605$.

Figure 3 Let (4.1) hold and $\alpha_1 = 0.2$, $\alpha_2 = 0.2$, $\alpha_3 = 0.3$ such that $\lambda_1 a_{11} \geq \lambda_2 a_{21} + \lambda_3 a_{31}$, $\lambda_2 a_{22} \geq \lambda_1 a_{12} + \lambda_3 a_{32}$ and $\lambda_3 a_{33} \geq \lambda_1 a_{13} + \lambda_2 a_{23}$ (Theorem 2.6). Then (1.2) is globally asymptotically stable.

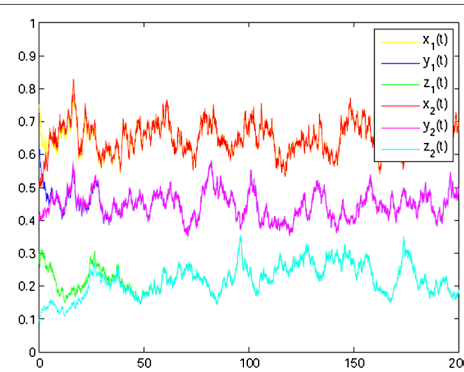

Table 1 The summary of the effect of the noise for species

Noise	Species		
	x	***y***	***z***
α_1	negative	negative	positive
α_2	negative	negative	positive
α_3	positive	positive	negative

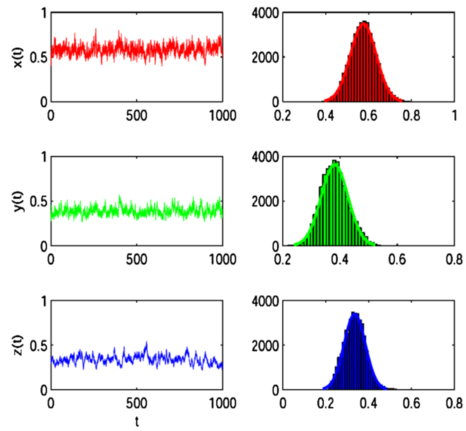

Figure 4 Let (4.1) hold and $\alpha_1 = 0.3$, $\alpha_2 = 0.3$, $\alpha_3 = 0.2$. Note that $\delta_1 = 1.1$, $\delta_2 = 1.05$, $\delta_3 = 0.65$ and $x^* = G_1/G = 0.6641$, $y^* = G_2/G = 0.4648$, $z^* = G_3/G = 0.3086$, we conclude that (2.2) holds. It follows from Theorems 2.5 and 2.7 that $\lim_{t\to+\infty} \frac{1}{t} \times \int_0^t x(s)\,ds = \int_{\mathbb{R}_+^3} \omega_1 \mu(d\omega_1, d\omega_2, d\omega_3) = \frac{G_1 - \tilde{G}_1}{G}$ a.s., $\lim_{t\to+\infty} \frac{1}{t} \int_0^t y(s)\,ds = \int_{\mathbb{R}_+^3} \omega_2 \mu(d\omega_1, d\omega_2, d\omega_3) = \frac{G_2 - \tilde{G}_2}{G}$ a.s., $\lim_{t\to+\infty} \frac{1}{t} \int_0^t z(s)\,ds = \int_{\mathbb{R}_+^3} \omega_3 \mu(d\omega_1, d\omega_2, d\omega_3) = \frac{G_3 - \tilde{G}_3}{G}$ a.s.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, and they read and approved the final version of the manuscript.

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