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Solvability of nonlocal boundary value problem for a class of nonlinear fractional differential coupled system with impulses

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Abstract

This paper is considered with a class of nonlinear fractional differential coupled system with fractional differential boundary value conditions and impulses. By means of the Banach contraction principle and the Schauder fixed point theorem, some sufficient criteria are established to guarantee the existence of solutions. As applications, some interesting examples are given to illustrate the effectiveness of our main results.

MSC: 34B10; 34B15; 34B37

Keywords: nonlinear fractional differential coupled system; fractional differential boundary value conditions; impulses; existence of solutions; fixed point theorem

1 Introduction

The fractional differential system as a mathematical model has been used to describe many phenomena and processes in a lot of fields such as financial mathematics, control theory, physics, chemistry, bioscience, optical and thermal systems, rheology materials and mechanical systems, signal processing and system identification, control and robotics, and so on. Some experimental results show that the fractional order differential model is more accurate than the integer order differential model. Therefore, fractional differential systems have received extensive attention and research in recent few decades. In particular, boundary value problems for fractional order differential systems have been studied and developed more extensively. There has been much research on boundary value problems of fractional order differential systems (see [1–20]).

It is well known that the effects of a pulse are inevitable in many phenomena and processes. For example, in the population dynamics systems, there are abrupt changes of population size due to the effects such as harvesting, diseases, and so on. So some authors have used an impulsive differential system to describe these kinds of phenomena since the last century. Therefore, some scholars have begun to study the boundary value problems for impulsive fractional differential equations and obtained good results (see [10–14]). In addition, it is impossible to describe complex systems and processes with a single equation because of the influence of many factors. Therefore, the boundary value problem of fractional order coupled systems has also attracted much attention and research (see [15–20]).

However, the research on boundary value problems of fractional order coupled systems is relatively scarce. Inspired by the above-mentioned issues, the main aim of this paper is to study the existence and uniqueness of solutions to four-point boundary value problem for a class of nonlinear fractional differential coupling system with impulses as follows:

$$\begin{cases} {}^cD^p_{0^+}u(t) + f(t, {}^cD^\alpha_{0^+}u(t), {}^cD^\beta_{0^+}v(t)) = 0, \quad t \neq t_k, k = 1, 2, \dots, n, \\ {}^cD^q_{0^+}v(t) + g(t, {}^cD^\alpha_{0^+}u(t), {}^cD^\beta_{0^+}v(t)) = 0, \quad t \neq t_k, k = 1, 2, \dots, n, \\ \Delta u|_{t=t_k} = I_{1k}(u(t_k)), \qquad \Delta u'|_{t=t_k} = J_{1k}(u(t_k)), \quad k = 1, 2, \dots, n, \\ \Delta v|_{t=t_k} = I_{2k}(v(t_k)), \qquad \Delta v'|_{t=t_k} = J_{2k}(v(t_k)), \quad k = 1, 2, \dots, n, \\ {}^cD^\gamma_{0^+}u(\eta) = u(1), \qquad u(0) = 0, \qquad {}^cD^\delta_{0^+}v(\xi) = v(1), \qquad v(0) = 0, \end{cases} \tag{1.1}$$

where $1 < p, q < 2$, $0 < \alpha, \beta, \gamma, \delta, \eta, \xi < 1$, ${}^cD^p_{0^+}$, ${}^cD^q_{0^+}$, ${}^cD^\alpha_{0^+}$, ${}^cD^\beta_{0^+}$, ${}^cD^\gamma_{0^+}$ and ${}^cD^\delta_{0^+}$ are the Caputo fractional derivatives. $f, g \in C(J \times R^2, R)$, $I_{1k}, I_{2k}, J_{1k}, J_{2k} \in C(R, R)$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $\Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, $\Delta v|_{t=t_k} = v(t_k^+) - v(t_k^-)$, $\Delta v'|_{t=t_k} = v'(t_k^+) - v'(t_k^-)$, $u(t_k^+)$, $v(t_k^+)$ and $u(t_k^-)$, $v(t_k^-)$ represent the right and the left limits of $u(t)$, $v(t)$ at $t = t_k$.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and lemmas of the Caputo and Riemann-Liouville fractional calculus. In Section 3, we shall prove the existence and uniqueness of solutions for system (1.1). In Section 4, some examples are also provided to illustrate the effectiveness of our main results. Finally, the conclusion is given to simply recall our studies and results obtained in Section 5.

2 Preliminaries

Definition 2.1 ([21, 22]) The Riemann-Liouville fractional integral of order α of a continuous function $f \in L^1(a, \infty)$ is defined as

$$I^\alpha_{a^+}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right side is pointwise defined on (a, ∞).

Definition 2.2 ([21, 22]) If $f \in C^n((a, \infty), R) \cap L^1(a, \infty)$ and $\alpha > 0$, then the Caputo fractional derivative of order α is defined as

$${}^cD^\alpha_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n-1 < \alpha < n, n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number $\alpha > 0$, provided that the right side is pointwise defined on (a, ∞).

Lemma 2.1 ([21]) *Assume that $u \in C(a, b) \cap L(a, b)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $C^n(a, b)$ then*

$$I^\alpha_{a^+}{}^cD^\alpha_{a^+}u(t) = u(t) + c_1 + c_2(t-a) + \cdots + c_n(t-a)^{n-1},$$

for some $c_i \in R$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 2.2 ([21, 23]) *If $\alpha,\beta > 0$, $t \in [a,b]$ and $u(t) \in L[a,b]$, then*

$$ {}^{c}D^{\alpha}_{a^+} I^{\alpha}_{a^+} u(t) = u(t), \qquad I^{\alpha}_{a^+} I^{\beta}_{a^+} u(t) = I^{\alpha+\beta}_{a^+} u(t). $$

Lemma 2.3 ([24]) *If E is a real Banach space and $F : E \to E$ is a contraction mapping, then F has a unique fixed point in E.*

Lemma 2.4 (Schauder fixed point theorem [24]) *If U is a closed bounded convex subset of a Banach space X and $F : \overline{U} \to \overline{U}$ is complete continuous, then F has at least one fixed point in $\overline{U}$.*

For the sake of convenience, we introduce the Banach spaces as follows.

Let $J = [0,1]$, $J' = J \setminus \{t_1, t_2, \ldots, t_n\}$. Define the set by

$$ PC(J) = \bigl\{ w(t) : w(t), w'(t), {}^{c}D^{\alpha}_{0^+} w(t), {}^{c}D^{\beta}_{0^+} w(t) \in C(J), w\bigl(t_k^+\bigr) \text{ and } w\bigl(t_k^-\bigr) \text{ exists} \\ \text{satisfying } w\bigl(t_k^-\bigr) = w(t_k), 1 \le k \le n \bigr\}. $$

It is easy to verify that $PC(J)$ is a Banach space equipped with the norm

$$ \|w\|_0 = \max\Bigl\{ \sup_{t\in J} \bigl| w(t) \bigr|, \sup_{t\in J} \bigl| w'(t) \bigr|, \sup_{t\in J} \bigl| {}^{c}D^{\alpha}_{0^+} w(t) \bigr|, \sup_{t\in J} \bigl| {}^{c}D^{\beta}_{0^+} w(t) \bigr| \Bigr\}, \quad \forall w(t) \in PC(J). $$

Define the Banach Space $X = PC(J) \times PC(J)$ with the norm $\|(u,v)\| = \max\{\|u\|_0, \|v\|_0\}$.

Definition 2.3 A pair of functions $(u(t), v(t)) \in X = PC(J) \times PC(J)$ is called a solution of (1.1) if $(u(t), v(t))$ satisfy all the equations and boundary value conditions of system (1.1).

Lemma 2.5 *Assume that $h \in C(J \times R^2, R)$. A function $u \in PC(J)$ is a solution of the boundary value problem*

$$ \begin{cases} {}^{c}D^{p}_{0^+} u(t) + h(t) = 0, \quad 1 < p < 2, \\ \Delta u|_{t=t_k} = I_{1k}(u(t_k)), \qquad \Delta u'|_{t=t_k} = J_{1k}(u(t_k)), \quad k = 1,2,\ldots,n, \\ {}^{c}D^{\gamma}_{0^+} u(\eta) = u(1), \qquad u(0) = 0, \quad 0 < \gamma < 1, 0 < \eta < 1, \end{cases} \tag{2.1} $$

if and only if $u \in PC(J)$ is a solution of the integral equation

$$ u(t) = \begin{cases} -\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s)\,ds + c^* t, \quad t \in [0,t_1], \\ -\frac{1}{\Gamma(p)} \int_{t_k}^t (t-s)^{p-1} h(s)\,ds - \frac{1}{\Gamma(p)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i - s)^{p-1} h(s)\,ds \\ \quad - \frac{1}{\Gamma(p-1)} \sum_{i=1}^k (t - t_i) \int_{t_{i-1}}^{t_i} (t_i - s)^{p-2} h(s)\,ds + \sum_{i=1}^k I_{1i}(u(t_i)) \\ \quad + \sum_{i=1}^k (t - t_i) J_{1i}(u(t_i)) + c^* t, \quad t \in (t_k, t_{k+1}], 1 \le k \le n, \end{cases} \tag{2.2} $$

where $j \in \{0,1,\ldots,n\}$ such that $\eta \in (t_j, t_{j+1}]$ and

$$ c^* = -\frac{1}{\Gamma(p-\gamma)} \int_{t_j}^{\eta} (\eta - s)^{p-\gamma-1} h(s)\,ds - \frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} (t_i - s)^{p-2} h(s)\,ds \\ + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)} \sum_{i=1}^{j} J_{1i}\bigl(u(t_i)\bigr) + \frac{1}{\Gamma(p)} \int_{t_n}^{1} (1-s)^{p-1} h(s)\,ds $$

$$+\frac{1}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}h(s)\,ds+\frac{1}{\Gamma(p-1)}\sum_{i=1}^{n}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}h(s)\,ds$$
$$-\sum_{i=1}^{n}I_{1i}\bigl(u(t_i)\bigr)-\sum_{i=1}^{n}(1-t_i)J_{1i}\bigl(u(t_i)\bigr).$$

Proof Assume that h satisfies (2.1). Applying Lemma 2.1, for some constants $c_0, c_1 \in R$, we have

$$u(t)=-I_{0^+}^{p}h(t)+c_0+c_1t=-\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}h(s)\,ds+c_0+c_1t,\quad t\in[0,t_1]. \tag{2.3}$$

Then we obtain

$$u'(t)=-\frac{1}{\Gamma(p-1)}\int_0^t(t-s)^{p-2}h(s)\,ds+c_1,\quad t\in[0,t_1]. \tag{2.4}$$

If $t\in(t_1,t_2)$, then we have

$$u(t)=-I_{t_1^+}^{p}h(t)+d_0+d_1(t-t_1)=-\frac{1}{\Gamma(p)}\int_{t_1}^t(t-s)^{p-1}h(s)\,ds+d_0+d_1(t-t_1) \tag{2.5}$$

and

$$u'(t)=-\frac{1}{\Gamma(p-1)}\int_{t_1}^t(t-s)^{p-2}h(s)\,ds+d_1, \tag{2.6}$$

where d_0, d_1 are arbitrary constants. According to (2.3)-(2.6), we find that

$$u\bigl(t_1^-\bigr)=-\frac{1}{\Gamma(p)}\int_0^{t_1}(t_1-s)^{p-1}h(s)\,ds+c_0+c_1t_1,\qquad u\bigl(t_1^+\bigr)=d_0, \tag{2.7}$$

and

$$u'\bigl(t_1^-\bigr)=-\frac{1}{\Gamma(p-1)}\int_0^{t_1}(t_1-s)^{p-2}h(s)\,ds+c_1,\qquad u'\bigl(t_1^+\bigr)=d_1. \tag{2.8}$$

In view of $\Delta u|_{t=t_1}=I_{11}(u(t_1))$ and $\Delta u'|_{t=t_1}=J_{11}(u(t_1))$, (2.7) and (2.8) give

$$d_0=-\frac{1}{\Gamma(p)}\int_0^{t_1}(t_1-s)^{p-1}h(s)\,ds+c_0+c_1t_1+I_{11}\bigl(u(t_1)\bigr)$$

and

$$d_1=-\frac{1}{\Gamma(p-1)}\int_0^{t_1}(t_1-s)^{p-2}h(s)\,ds+c_1+J_{11}\bigl(u(t_1)\bigr).$$

Substituting d_0 and d_1 into (2.5), for $t\in(t_1,t_2]$, we obtain

$$u(t)=-\frac{1}{\Gamma(p)}\int_{t_1}^t(t-s)^{p-1}h(s)\,ds-\frac{1}{\Gamma(p)}\int_0^{t_1}(t_1-s)^{p-1}h(s)\,ds+c_0+c_1t_1+I_{11}\bigl(u(t_1)\bigr)$$
$$-\frac{t-t_1}{\Gamma(p-1)}\int_0^{t_1}(t_1-s)^{p-2}h(s)\,ds+c_1(t-t_1)+(t-t_1)J_{11}\bigl(u(t_1)\bigr)$$

$$\begin{aligned}
&= -\frac{1}{\Gamma(p)}\int_{t_1}^{t}(t-s)^{p-1}h(s)\,ds - \frac{1}{\Gamma(p)}\int_0^{t_1}(t_1-s)^{p-1}h(s)\,ds \\
&\quad - \frac{t-t_1}{\Gamma(p-1)}\int_0^{t_1}(t_1-s)^{p-2}h(s)\,ds + I_{11}\big(u(t_1)\big) \\
&\quad + (t-t_1)J_{11}\big(u(t_1)\big) + c_0 + c_1 t.
\end{aligned} \tag{2.9}$$

Repeating the process in this way, the solution $u(t)$ for $t \in (t_k, t_{k+1}]$ $(k = 1, 2, \dots, n)$ can be formulated as

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(p)}\int_{t_k}^{t}(t-s)^{p-1}h(s)\,ds - \frac{1}{\Gamma(p)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}h(s)\,ds \\
&\quad - \frac{1}{\Gamma(p-1)}\sum_{i=1}^{k}(t-t_i)\int_{i-1}^{t_i}(t_i-s)^{p-2}h(s)\,ds + \sum_{i=1}^{k}I_{1i}\big(u(t_i)\big) \\
&\quad + \sum_{i=1}^{k}(t-t_i)J_{1i}\big(u(t_i)\big) + c_0 + c_1 t.
\end{aligned} \tag{2.10}$$

In the light of (2.3) and $u(0) = 0$, we have $c_0 = 0$. In addition, it follows from (2.10) that

$$\begin{aligned}
u(1) &= -\frac{1}{\Gamma(p)}\int_{t_n}^{1}(1-s)^{p-1}h(s)\,ds - \frac{1}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}h(s)\,ds \\
&\quad - \frac{1}{\Gamma(p-1)}\sum_{i=1}^{n}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}h(s)\,ds + \sum_{i=1}^{n}I_{1i}\big(u(t_i)\big) \\
&\quad + \sum_{i=1}^{n}(1-t_i)J_{1i}\big(u(t_i)\big) + c_1.
\end{aligned} \tag{2.11}$$

In view of $\eta \in (0, 1)$, $t_0 = 0$ and $t_{n+1} = 1$, there exists $j \in \{0, 1, \dots, n\}$ such that $\eta \in (t_j, t_{j+1}]$. So we have

$$\begin{aligned}
u(\eta) &= -\frac{1}{\Gamma(p)}\int_{t_j}^{\eta}(\eta-s)^{p-1}h(s)\,ds - \frac{1}{\Gamma(p)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}h(s)\,ds \\
&\quad - \frac{1}{\Gamma(p-1)}\sum_{i=1}^{j}(\eta-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}h(s)\,ds + \sum_{i=1}^{j}I_{1i}\big(u(t_i)\big) \\
&\quad + \sum_{i=1}^{j}(\eta-t_i)J_{1i}\big(u(t_i)\big) + c_1.
\end{aligned} \tag{2.12}$$

Applying Lemma 2.2, we obtain

$$\begin{aligned}
{}^{c}D_{0^+}^{\gamma}u(\eta) &= -\frac{1}{\Gamma(p-\gamma)}\int_{t_j}^{\eta}(\eta-s)^{p-\gamma-1}h(s)\,ds - \frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)} \\
&\quad \times \sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}h(s)\,ds + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{j}J_{1i}\big(u(t_i)\big).
\end{aligned} \tag{2.13}$$

According to ${}^{c}D_{0^+}^{\gamma}u(\eta) = u(1)$ and (2.12)-(2.13), we derive

$$\begin{aligned}
c_1 = {}& -\frac{1}{\Gamma(p-\gamma)}\int_{t_j}^{\eta}(\eta-s)^{p-\gamma-1}h(s)\,ds - \frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}h(s)\,ds \\
& + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{j}J_{1i}\bigl(u(t_i)\bigr) + \frac{1}{\Gamma(p)}\int_{t_n}^{1}(1-s)^{p-1}h(s)\,ds \\
& + \frac{1}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}h(s)\,ds + \frac{1}{\Gamma(p-1)}\sum_{i=1}^{n}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}h(s)\,ds \\
& - \sum_{i=1}^{n}I_{1,i}\bigl(u(t_i)\bigr) - \sum_{i=1}^{n}(1-t_i)J_{1i}\bigl(u(t_i)\bigr).
\end{aligned}$$

Substituting the values of c_0 and c_1 into (2.3) and (2.10), one can easily reach (2.2).

Vice versa, we assume that $u(t)$ is a solution of the integral equation (2.2). By a direct computation, it follows that the solution defined by (2.2) satisfies (2.1). The proof is complete. □

Similarly, we conclude the following lemma.

Lemma 2.6 *Assume that $y \in C(J \times R^2, R)$. A function $v \in PC(J)$ is a solution of the boundary value problem*

$$\begin{cases}
{}^{c}D_{0^+}^{q}v(t) + y(t) = 0, \quad 1 < q < 2, \\
\Delta v|_{t=t_k} = I_{2k}(v(t_k)), \qquad \Delta v'|_{t=t_k} = J_{2k}(v(t_k)), \quad k = 1,2,\ldots,n, \\
{}^{c}D_{0^+}^{\delta}v(\xi) = v(1), \qquad v(0) = 0, \quad 0 < \delta < 1, 0 < \xi < 1,
\end{cases} \tag{2.14}$$

if and only if $v \in PC(J)$ is a solution of the integral equation

$$v(t) = \begin{cases}
-\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\,ds + d^*t, \quad t \in [0,t_1], \\
-\frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1}y(s)\,ds - \frac{1}{\Gamma(q)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}y(s)\,ds \\
\quad - \frac{1}{\Gamma(q-1)}\sum_{i=1}^k (t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\,ds + \sum_{i=1}^k I_{2i}(v(t_i)) \\
\quad + \sum_{i=1}^k (t-t_i)J_{2i}(v(t_i)) + d^*t, \quad t \in (t_k,t_{k+1}], 1 \le k \le n,
\end{cases} \tag{2.15}$$

where $l \in \{0,1,\ldots,n\}$ such that $\xi \in (t_l,t_{l+1}]$ and

$$\begin{aligned}
d^* = {}& -\frac{1}{\Gamma(q-\delta)}\int_{t_l}^{\xi}(\xi-s)^{q-\delta-1}y(s)\,ds - \frac{\xi^{1-\delta}}{\Gamma(q-1)\Gamma(2-\delta)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\,ds \\
& + \frac{\xi^{1-\delta}}{\Gamma(2-\delta)}\sum_{i=1}^{l}J_{2i}\bigl(v(t_i)\bigr) + \frac{1}{\Gamma(q)}\int_{t_n}^{1}(1-s)^{q-1}y(s)\,ds \\
& + \frac{1}{\Gamma(q)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)\,ds + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{n}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\,ds \\
& - \sum_{i=1}^{n}I_{2i}\bigl(v(t_i)\bigr) - \sum_{i=1}^{n}(1-t_i)J_{2i}\bigl(v(t_i)\bigr).
\end{aligned}$$

3 Main results

In this section, we shall employ the fixed point theorems of the operator to prove the existence of solutions to system (1.1). According to Lemmas 2.5-2.6, we define the operator $F : X = PC(J) \times PC(J) \to X$ by

$$F(u,v)(t) = \big(F_1(u,v)(t), F_2(u,v)(t)\big)^T, \quad \forall (u,v) \in X, t \in [0,1], \tag{3.1}$$

where

$$F_1(u,v)(t) = \begin{cases} -\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds + C^* t, \quad t \in [0,t_1], \\ -\frac{1}{\Gamma(p)} \int_{t_k}^t (t-s)^{p-1} f(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds \\ \quad - \frac{1}{\Gamma(p)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{p-1} f(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds \\ \quad - \frac{1}{\Gamma(p-1)} \sum_{i=1}^k (t-t_i) \int_{t_{i-1}}^{t_i} (t_i-s)^{p-2} f(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds \\ \quad + \sum_{i=1}^k I_{1i}(u(t_i)) + \sum_{i=1}^k (t-t_i) J_{1i}(u(t_i)) + C^* t, \\ \quad t \in (t_k, t_{k+1}], 1 \le k \le n, \end{cases}$$

$$F_2(u,v)(t) = \begin{cases} -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds + D^* t, \quad t \in [0,t_1], \\ -\frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} g(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds \\ \quad - \frac{1}{\Gamma(q)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{p-1} g(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds \\ \quad - \frac{1}{\Gamma(q-1)} \sum_{i=1}^k (t-t_i) \int_{t_{i-1}}^{t_i} (t_i-s)^{q-2} g(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s))\,ds \\ \quad + \sum_{i=1}^k I_{2i}(v(t_i)) + \sum_{i=1}^k (t-t_i) J_{2i}(v(t_i)) + D^* t, \\ \quad t \in (t_k, t_{k+1}], 1 \le k \le n, \end{cases}$$

$$\begin{aligned} C^* = {} & -\frac{1}{\Gamma(p-\gamma)} \int_{t_j}^{\eta} (\eta-s)^{p-\gamma-1} f\big(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s)\big)\,ds \\ & - \frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} (t_i-s)^{p-2} f\big(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s)\big)\,ds \\ & + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)} \sum_{i=1}^{j} J_{1i}\big(u(t_i)\big) + \frac{1}{\Gamma(p)} \int_{t_n}^{1} (1-s)^{p-1} f\big(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s)\big)\,ds \\ & + \frac{1}{\Gamma(p)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i-s)^{p-1} f\big(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s)\big)\,ds \\ & + \frac{1}{\Gamma(p-1)} \sum_{i=1}^{n} (1-t_i) \int_{t_{i-1}}^{t_i} (t_i-s)^{p-2} f\big(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s)\big)\,ds \\ & - \sum_{i=1}^{n} I_{1i}\big(u(t_i)\big) - \sum_{i=1}^{n} (1-t_i) J_{1i}\big(u(t_i)\big), \end{aligned}$$

and

$$\begin{aligned} D^* = {} & -\frac{1}{\Gamma(q-\delta)} \int_{t_l}^{\xi} (\xi-s)^{q-\delta-1} g\big(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s)\big)\,ds \\ & - \frac{\xi^{1-\delta}}{\Gamma(q-1)\Gamma(2-\delta)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i-s)^{q-2} g\big(s, {}^cD_{0^+}^{\alpha} u(s), {}^cD_{0^+}^{\beta} v(s)\big)\,ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{\xi^{1-\delta}}{\Gamma(2-\delta)}\sum_{i=1}^{l}J_{2i}\big(v(t_i)\big)+\frac{1}{\Gamma(q)}\int_{t_n}^{1}(1-s)^{q-1}g\big(s,{}^cD_{0^+}^{\alpha}u(s),{}^cD_{0^+}^{\beta}v(s)\big)\,ds\\
&+\frac{1}{\Gamma(q)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}g\big(s,{}^cD_{0^+}^{\alpha}u(s),{}^cD_{0^+}^{\beta}v(s)\big)\,ds\\
&+\frac{1}{\Gamma(q-1)}\sum_{i=1}^{n}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}g\big(s,{}^cD_{0^+}^{\alpha}u(s),{}^cD_{0^+}^{\beta}v(s)\big)\,ds\\
&-\sum_{i=1}^{n}I_{2i}\big(v(t_i)\big)-\sum_{i=1}^{n}(1-t_i)J_{2i}\big(v(t_i)\big).
\end{aligned}
$$

Thus, solving the boundary value problem (1.1) is equivalent to finding the fixed point of the operator F defined by (3.1). Next we shall give our main results.

Theorem 3.1 *If the following conditions* (A_1)-(A_3) *hold, then the boundary value problem* (1.1) *has a unique solution pair* $(u^*(t),v^*(t))$.

(A_1) *For all* $t\in[0,1]$ *and* $\mathcal{U}_i,\mathcal{V}_i\in R$ $(i=1,2)$, *there exist some positive constants* $\mathcal{L}_i$, $\hat{\mathcal{L}}_i$ $(i=1,2)$ *such that*

$$
\begin{aligned}
&\big|f(t,\mathcal{U}_1,\mathcal{V}_1)-f(t,\mathcal{U}_2,\mathcal{V}_2)\big|\le\mathcal{L}_1|\mathcal{U}_1-\mathcal{U}_2|+\mathcal{L}_2|\mathcal{V}_1-\mathcal{V}_2|,\\
&\big|g(t,\mathcal{U}_1,\mathcal{V}_1)-g(t,\mathcal{U}_2,\mathcal{V}_2)\big|\le\hat{\mathcal{L}}_1|\mathcal{U}_1-\mathcal{U}_2|+\hat{\mathcal{L}}_2|\mathcal{V}_1-\mathcal{V}_2|;
\end{aligned}
$$

(A_2) *For all* $\mathcal{U},\mathcal{V}\in R$, *there exist some positive constants* L_{ik}, $\hat{L}_{ik}$ $(i=1,2;k=1,2,\ldots,n)$ *such that*

$$
\big|I_{ik}(\mathcal{U})-I_{ik}(\mathcal{V})\big|\le L_{ik}|\mathcal{U}-\mathcal{V}|,\qquad\big|J_{ik}(\mathcal{U})-J_{ik}(\mathcal{V})\big|\le\hat{L}_{ik}|\mathcal{U}-\mathcal{V}|;
$$

(A_3)

$$
\begin{aligned}
&\left[\frac{2}{\Gamma(p)}+\frac{1}{\Gamma(p-\gamma)}+\frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}+\frac{2}{\Gamma(p-1)}\right](\mathcal{L}_1+\mathcal{L}_2)\\
&\quad+2\sum_{i=1}^{n}(L_{1i}+\hat{L}_{1i})+\frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}<1,\\
&\left[\frac{2}{\Gamma(q)}+\frac{1}{\Gamma(q-\delta)}+\frac{\xi^{1-\delta}}{\Gamma(q-1)\Gamma(2-\delta)}+\frac{2}{\Gamma(q-1)}\right](\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2)\\
&\quad+2\sum_{i=1}^{n}(L_{2i}+\hat{L}_{2i})+\frac{\xi^{1-\delta}}{\Gamma(2-\delta)}\sum_{i=1}^{n}\hat{L}_{2i}<1.
\end{aligned}
$$

Proof Now, we will use the Banach contraction principle to prove that $F:X\to X$ defined by (3.1) has a fixed point. We first show that F is a contraction. In fact, when $t\in J=[0,1]$, from (3.1) and conditions (A_1)-(A_2), for all $(u_1,v_1),(u_2,v_2)\in X$, we have

$$
\begin{aligned}
&\big|F_1(u_1,v_1)(t)-F_1(u_2,v_2)(t)\big|\\
&\quad=\bigg|-\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}\big[f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{t}{\Gamma(p-\gamma)}\int_{t_j}^{\eta}(\eta-s)^{p-\gamma-1}\big[f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)\\
&-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds\\
&-\frac{t\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}\big[f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)\\
&-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds+\frac{t\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{j}\big[J_{1i}\big(u_1(t_i)\big)-J_{1i}\big(u_2(t_i)\big)\big]\\
&+\frac{t}{\Gamma(p)}\int_{t_n}^{1}(1-s)^{p-1}\big[f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds\\
&+\frac{t}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}\big[f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)\\
&-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds\\
&+\sum_{i=1}^{n}\frac{t(1-t_i)}{\Gamma(p-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}\big[f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)\\
&-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds-t\sum_{i=1}^{n}\big[I_{1i}\big(u_1(t_i)\big)-I_{1i}\big(u_2(t_i)\big)\big]\\
&-t\sum_{i=1}^{n}(1-t_i)\big[J_{1i}\big(u_1(t_i)\big)-J_{1i}\big(u_2(t_i)\big)\big]\bigg|\\
\le{}&\frac{1}{\Gamma(p)}\int_0^t(t-s)^{p-1}\big|f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big|\,ds\\
&+\frac{t}{\Gamma(p-\gamma)}\int_{t_j}^{\eta}(\eta-s)^{p-\gamma-1}\big|f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)\\
&-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big|\,ds\\
&+\frac{t\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}\big|f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)\\
&-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big|\,ds+\frac{t\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{j}\big|J_{1i}\big(u_1(t_i)\big)-J_{1i}\big(u_2(t_i)\big)\big|\\
&+\frac{t}{\Gamma(p)}\int_{t_n}^{1}(1-s)^{p-1}\big|f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big|\,ds\\
&+\frac{t}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}\big|f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big|\,ds\\
&+\sum_{i=1}^{n}\frac{t(1-t_i)}{\Gamma(p-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}\big|f\big(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\big)\\
&-f\big(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\big)\big|\,ds+t\sum_{i=1}^{n}\big|I_{1i}\big(u_1(t_i)\big)-I_{1i}\big(u_2(t_i)\big)\big|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + t\sum_{i=1}^{n}(1-t_i)\big|J_{1i}\big(u_1(t_i)\big)-J_{1i}\big(u_2(t_i)\big)\big| \\
&\le \frac{1}{\Gamma(p)}\int_0^1\big[\mathcal{L}_1\big|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\big|+\mathcal{L}_2\big|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\big|\big]\,ds \\
&\quad + \frac{1}{\Gamma(p-\gamma)}\int_0^1\big[\mathcal{L}_1\big|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\big|+\mathcal{L}_2\big|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\big|\big]\,ds \\
&\quad + \frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}\big[\mathcal{L}_1\big|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\big| \\
&\quad + \mathcal{L}_2\big|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\big|\big]\,ds + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\big|u_1(t_i)-u_2(t_i)\big| \\
&\quad + \frac{1}{\Gamma(p)}\int_{t_n}^1\big[\mathcal{L}_1\big|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\big|+\mathcal{L}_2\big|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\big|\big]\,ds \\
&\quad + \frac{1}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\big[\mathcal{L}_1\big|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\big|+\mathcal{L}_2\big|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\big|\big]\,ds \\
&\quad + \sum_{i=1}^{n}\frac{1}{\Gamma(p-1)}\int_{t_{i-1}}^{t_i}\big[\mathcal{L}_1\big|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\big|+\mathcal{L}_2\big|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\big|\big]\,ds \\
&\quad + \sum_{i=1}^{n}L_{1i}\big|u_1(t_i)-u_2(t_i)\big| + \sum_{i=1}^{n}\hat{L}_{1i}\big|u_1(t_i)-u_2(t_i)\big| \\
&\le \frac{1}{\Gamma(p)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\frac{1}{\Gamma(p-\gamma)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big] \\
&\quad + \frac{\eta^{1-\gamma}t_j}{\Gamma(p-1)\Gamma(2-\gamma)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\|u_1-u_2\|_0 \\
&\quad + \frac{1-t_n}{\Gamma(p)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\frac{t_n}{\Gamma(p)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big] \\
&\quad + \frac{t_n}{\Gamma(p-1)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\sum_{i=1}^{n}L_{1i}\|u_1-u_2\|_0 \\
&\quad + \sum_{i=1}^{n}\hat{L}_{1i}\|u_1-u_2\|_0 \\
&\le \frac{1}{\Gamma(p)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|+\frac{1}{\Gamma(p-\gamma)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\| \\
&\quad + \frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\| \\
&\quad + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\big\|(u_1-u_2,v_1-v_2)\big\|+\frac{1}{\Gamma(p)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\| \\
&\quad + \frac{1}{\Gamma(p-1)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|+\sum_{i=1}^{n}L_{1i}\big\|(u_1-u_2,v_1-v_2)\big\| \\
&\quad + \sum_{i=1}^{n}\hat{L}_{1i}\big\|(u_1-u_2,v_1-v_2)\big\|
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg[\frac{2(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(p)} + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p-\gamma)} + \frac{\eta^{1-\gamma}(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(p-1)\Gamma(2-\gamma)} + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p-1)} + \sum_{i=1}^{n} L_{1i} + \sum_{i=1}^{n} \hat{L}_{1i} \\
&\quad + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)} \sum_{i=1}^{n} \hat{L}_{1i}\Bigg] \big\|(u_1-u_2, v_1-v_2)\big\|, \quad t \in [0,t_1],
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
&\big|F_1(u_1,v_1)(t) - F_1(u_2,v_2)(t)\big| \\
&= \bigg| -\frac{1}{\Gamma(p)} \int_{t_k}^{t} (t-s)^{p-1}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds \\
&\quad - \frac{1}{\Gamma(p)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{p-1}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) \\
&\quad - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds \\
&\quad - \sum_{i=1}^{k} \frac{t-t_i}{\Gamma(p-1)} \int_{t_{i-1}}^{t_i} (t_i-s)^{p-2}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) \\
&\quad - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds + \sum_{i=1}^{k}\big[I_{1i}\big(u_1(t_i)\big) - I_{1i}\big(u_2(t_i)\big)\big] \\
&\quad + \sum_{i=1}^{k}(t-t_i)\big[J_{1i}\big(u_1(t_i)\big) - J_{1i}\big(u_2(t_i)\big)\big] \\
&\quad - \frac{t}{\Gamma(p-\gamma)} \int_{t_j}^{\eta} (\eta-s)^{p-\gamma-1}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) \\
&\quad - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds \\
&\quad - \frac{t\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)} \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} (t_i-s)^{p-2}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) \\
&\quad - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds + \frac{t\eta^{1-\gamma}}{\Gamma(2-\gamma)} \sum_{i=1}^{j}\big[J_{1i}\big(u_1(t_i)\big) - J_{1i}\big(u_2(t_i)\big)\big] \\
&\quad + \frac{t}{\Gamma(p)} \int_{t_n}^{1} (1-s)^{p-1}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds \\
&\quad + \frac{t}{\Gamma(p)} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (t_i-s)^{p-1}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) \\
&\quad - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds \\
&\quad + \sum_{i=1}^{n} \frac{t(1-t_i)}{\Gamma(p-1)} \int_{t_{i-1}}^{t_i} (t_i-s)^{p-2}\big[f\big(s, {}^cD_{0^+}^{\alpha}u_1(s), {}^cD_{0^+}^{\beta}v_1(s)\big) \\
&\quad - f\big(s, {}^cD_{0^+}^{\alpha}u_2(s), {}^cD_{0^+}^{\beta}v_2(s)\big)\big]\,ds - t\sum_{i=1}^{n}\big[I_{1i}\big(u_1(t_i)\big) - I_{1i}\big(u_2(t_i)\big)\big] \\
&\quad - t\sum_{i=1}^{n}(1-t_i)\big[J_{1i}\big(u_1(t_i)\big) - J_{1i}\big(u_2(t_i)\big)\big]\bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(p)}\int_{t_k}^{t}(t-s)^{p-1}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds\\
&\quad+\frac{1}{\Gamma(p)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds\\
&\quad+\sum_{i=1}^{k}\frac{|t-t_i|}{\Gamma(p-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)\\
&\quad-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds+\sum_{i=1}^{k}\bigl|I_{1i}\bigl(u_1(t_i)\bigr)-I_{1i}\bigl(u_2(t_i)\bigr)\bigr|\\
&\quad+\sum_{i=1}^{k}|t-t_i|\bigl|J_{1i}\bigl(u_1(t_i)\bigr)-J_{1i}\bigl(u_2(t_i)\bigr)\bigr|\\
&\quad+\frac{t}{\Gamma(p-\gamma)}\int_{t_j}^{\eta}(\eta-s)^{p-\gamma-1}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)\\
&\quad-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds\\
&\quad+\frac{t\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)\\
&\quad-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds+\frac{t\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{j}\bigl|J_{1i}\bigl(u_1(t_i)\bigr)-J_{1i}\bigl(u_2(t_i)\bigr)\bigr|\\
&\quad+\frac{t}{\Gamma(p)}\int_{t_n}^{1}(1-s)^{p-1}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds\\
&\quad+\frac{t}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-1}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds\\
&\quad+\sum_{i=1}^{n}\frac{t(1-t_i)}{\Gamma(p-1)}\int_{t_{i-1}}^{t_i}(t_i-s)^{p-2}\bigl|f\bigl(s,{}^cD_{0^+}^{\alpha}u_1(s),{}^cD_{0^+}^{\beta}v_1(s)\bigr)\\
&\quad-f\bigl(s,{}^cD_{0^+}^{\alpha}u_2(s),{}^cD_{0^+}^{\beta}v_2(s)\bigr)\bigr|\,ds+t\sum_{i=1}^{n}\bigl|I_{1i}\bigl(u_1(t_i)\bigr)-I_{1i}\bigl(u_2(t_i)\bigr)\bigr|\\
&\quad+t\sum_{i=1}^{n}(1-t_i)\bigl|J_{1i}\bigl(u_1(t_i)\bigr)-J_{1i}\bigl(u_2(t_i)\bigr)\bigr|\\
&\leq \frac{1}{\Gamma(p)}\int_{t_k}^{1}\bigl[\mathcal{L}_1\bigl|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\bigr|+\mathcal{L}_2\bigl|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\bigr|\bigr]\,ds\\
&\quad+\frac{1}{\Gamma(p)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\bigl[\mathcal{L}_1\bigl|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\bigr|+\mathcal{L}_2\bigl|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\bigr|\bigr]\,ds\\
&\quad+\sum_{i=1}^{k}\frac{1}{\Gamma(p-1)}\int_{t_{i-1}}^{t_i}\bigl[\mathcal{L}_1\bigl|{}^cD_{0^+}^{\alpha}u_1(s)-{}^cD_{0^+}^{\alpha}u_2(s)\bigr|+\mathcal{L}_2\bigl|{}^cD_{0^+}^{\beta}v_1(s)-{}^cD_{0^+}^{\beta}v_2(s)\bigr|\bigr]\,ds\\
&\quad+\sum_{i=1}^{n}L_{1i}\bigl|u_1(t_i)-u_2(t_i)\bigr|+\sum_{i=1}^{n}\hat{L}_{1i}\bigl|u_1(t_i)-u_2(t_i)\bigr|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(p-\gamma)}\int_0^1\big[\mathcal{L}_1\big|{}^cD^\alpha_{0^+}u_1(s)-{}^cD^\alpha_{0^+}u_2(s)\big|+\mathcal{L}_2\big|{}^cD^\beta_{0^+}v_1(s)-{}^cD^\beta_{0^+}v_2(s)\big|\big]\,ds\\
&+\frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i}\big[\mathcal{L}_1\big|{}^cD^\alpha_{0^+}u_1(s)-{}^cD^\alpha_{0^+}u_2(s)\big|\\
&+\mathcal{L}_2\big|{}^cD^\beta_{0^+}v_1(s)-{}^cD^\beta_{0^+}v_2(s)\big|\big]\,ds+\frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\big|u_1(t_i)-u_2(t_i)\big|\\
&+\frac{1}{\Gamma(p)}\int_{t_n}^1\big[\mathcal{L}_1\big|{}^cD^\alpha_{0^+}u_1(s)-{}^cD^\alpha_{0^+}u_2(s)\big|+\mathcal{L}_2\big|{}^cD^\beta_{0^+}v_1(s)-{}^cD^\beta_{0^+}v_2(s)\big|\big]\,ds\\
&+\frac{1}{\Gamma(p)}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\big[\mathcal{L}_1\big|{}^cD^\alpha_{0^+}u_1(s)-{}^cD^\alpha_{0^+}u_2(s)\big|+\mathcal{L}_2\big|{}^cD^\beta_{0^+}v_1(s)-{}^cD^\beta_{0^+}v_2(s)\big|\big]\,ds\\
&+\sum_{i=1}^{n}\frac{1}{\Gamma(p-1)}\int_{t_{i-1}}^{t_i}\big[\mathcal{L}_1\big|{}^cD^\alpha_{0^+}u_1(s)-{}^cD^\alpha_{0^+}u_2(s)\big|+\mathcal{L}_2\big|{}^cD^\beta_{0^+}v_1(s)-{}^cD^\beta_{0^+}v_2(s)\big|\big]\,ds\\
&+\sum_{i=1}^{n}L_{1i}\big|u_1(t_i)-u_2(t_i)\big|+\sum_{i=1}^{n}\hat{L}_{1i}\big|u_1(t_i)-u_2(t_i)\big|\\
\le{}&\frac{1-t_k}{\Gamma(p)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\frac{t_k}{\Gamma(p)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]\\
&+\frac{t_k}{\Gamma(p-1)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\sum_{i=1}^{n}L_{1i}\|u_1-u_2\|_0\\
&+\sum_{i=1}^{n}\hat{L}_{1i}\|u_1-u_2\|_0+\frac{1}{\Gamma(p-\gamma)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]\\
&+\frac{\eta^{1-\gamma}t_j}{\Gamma(p-1)\Gamma(2-\gamma)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\|u_1-u_2\|_0\\
&+\frac{1-t_n}{\Gamma(p)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\frac{t_n}{\Gamma(p)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]\\
&+\frac{t_n}{\Gamma(p-1)}\big[\mathcal{L}_1\|u_1-u_2\|_0+\mathcal{L}_2\|v_1-v_2\|_0\big]+\sum_{i=1}^{n}L_{1i}\|u_1-u_2\|_0\\
&+\sum_{i=1}^{n}\hat{L}_{1i}\|u_1-u_2\|_0\\
\le{}&\frac{1}{\Gamma(p)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|+\frac{1}{\Gamma(p-1)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|\\
&+\sum_{i=1}^{n}[L_{1i}+\hat{L}_{1i}]\big\|(u_1-u_2,v_1-v_2)\big\|+\frac{1}{\Gamma(p-\gamma)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|\\
&+\frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|\\
&+\frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\big\|(u_1-u_2,v_1-v_2)\big\|+\frac{1}{\Gamma(p)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|\\
&+\frac{1}{\Gamma(p-1)}[\mathcal{L}_1+\mathcal{L}_2]\big\|(u_1-u_2,v_1-v_2)\big\|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n} L_{1i}\left\|(u_1-u_2,v_1-v_2)\right\|+\sum_{i=1}^{n}\hat{L}_{1i}\left\|(u_1-u_2,v_1-v_2)\right\| \\
&=\left[\frac{2(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(p)}+\frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p-\gamma)}+\frac{\eta^{1-\gamma}(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(p-1)\Gamma(2-\gamma)}+\frac{2(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(p-1)}+2\sum_{i=1}^{n}L_{1i}+2\sum_{i=1}^{n}\hat{L}_{1i}\right. \\
&\quad\left.+\frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\right]\left\|(u_1-u_2,v_1-v_2)\right\|,\quad t\in(t_k,t_{k+1}],1\le k\le n. \qquad (3.3)
\end{aligned}
$$

In the same way, we obtain

$$
\begin{aligned}
&\left|F_2(u_1,v_1)(t)-F_2(u_2,v_2)(t)\right| \\
&\quad\le\left[\frac{2(\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2)}{\Gamma(q)}+\frac{\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2}{\Gamma(q-\delta)}+\frac{\xi^{1-\delta}(\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2)}{\Gamma(q-1)\Gamma(2-\delta)}+\frac{\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2}{\Gamma(q-1)}\right. \\
&\quad\left.+\sum_{i=1}^{n}(L_{2i}+\hat{L}_{2i})+\frac{\xi^{1-\delta}}{\Gamma(2-\delta)}\sum_{i=1}^{n}\hat{L}_{2i}\right]\left\|(u_1-u_2,v_1-v_2)\right\|,\quad t\in[0,t_1], \qquad (3.4)\\
&\left|F_2(u_1,v_1)(t)-F_2(u_2,v_2)(t)\right| \\
&\quad\le\left[\frac{2(\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2)}{\Gamma(q)}+\frac{\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2}{\Gamma(q-\delta)}+\frac{\xi^{1-\delta}(\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2)}{\Gamma(q-1)\Gamma(2-\delta)}+\frac{\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2}{\Gamma(q-1)}\right. \\
&\quad\left.+2\sum_{i=1}^{n}(L_{2i}+\hat{L}_{2i})+\frac{\xi^{1-\delta}}{\Gamma(2-\delta)}\sum_{i=1}^{n}\hat{L}_{2i}\right] \\
&\quad\times\left\|(u_1-u_2,v_1-v_2)\right\|,\quad t\in(t_k,t_{k+1}],1\le k\le n. \qquad (3.5)
\end{aligned}
$$

Thus, it follows from (3.2)-(3.5) and (A_3) that $F:X\to X$ defined by (3.1) is a contraction mapping. According to Lemma 2.3, for F there exists a unique fixed point $(u^*(t),v^*(t))\in X$. Therefore, the boundary value problem (1.1) has a unique solution pair $(u^*(t),v^*(t))\in X$. The proof of Theorem 3.1 is complete. □

Theorem 3.2 *Suppose that the condition* (A_1)-(A_2) *hold. Assume further that the following* (A_4) *holds. Then the boundary value problem* (1.1) *has at least a solution pair* $(u^*(t),v^*(t))$.

(A_4) *For all* $t\in[0,1]$, $f(t,0,0)=g(t,0,0)=0$, $I_{ik}(0)=J_{ik}(0)=0$ $(i=1,2;k=1,2,\dots,n)$.

Proof For the sake of convenience, we denote $\rho=[\frac{2}{\Gamma(p)}+\frac{1}{\Gamma(p-\gamma)}+\frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)}+\frac{2}{\Gamma(p-1)}](\mathcal{L}_1+\mathcal{L}_2)+2\sum_{i=1}^{n}(L_{1i}+\hat{L}_{1i})+\frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}$, $\varrho=[\frac{2}{\Gamma(q)}+\frac{1}{\Gamma(q-\delta)}+\frac{\xi^{1-\delta}}{\Gamma(q-1)\Gamma(2-\delta)}+\frac{2}{\Gamma(q-1)}](\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2)+2\sum_{i=1}^{n}(L_{2i}+\hat{L}_{2i})+\frac{\xi^{1-\delta}}{\Gamma(2-\delta)}\sum_{i=1}^{n}\hat{L}_{2i}$, $R_0=\max\{\frac{1}{\rho}+1,\frac{1}{\varrho}+1\}$. Define the operator $F:X\to X$ by (3.1) and a closed ball of Banach space X as follows:

$$
\Omega=\left\{(u,v)\in X:\left\|(u,v)\right\|\le R_0\right\}. \qquad (3.6)
$$

Similar to (3.2)-(3.5), we easily show that $F(\Omega)\subset\Omega$ by applying (A_4). $F(\Omega)\subset\Omega$ indicates that $F(\Omega)$ is uniformly bounded in X. The continuity of the operator F follows from the continuity of f, g, I_{ik} and J_{ik}. Now we need to prove that $F:\overline{\Omega}\to\overline{\Omega}$ is equicontinuous. In fact, let $(u,v)\in\Omega$ and $\tau_1,\tau_2\in[0,1]$ with $\tau_1<\tau_2$.

When $0 \leq \tau_1 < \tau_2 \leq t_1$, similar to (3.2), we have

$$
\begin{aligned}
&\big|F_1(u,v)(\tau_2) - F_1(u,v)(\tau_1)\big| \\
&\quad = \bigg|\frac{1}{\Gamma(p)}\int_0^{\tau_1}\big[(\tau_2-s)^{p-1}-(\tau_1-s)^{p-1}\big]f\big(s,{}^cD_{0^+}^{\alpha}u(s),{}^cD_{0^+}^{\beta}v(s)\big)\,ds \\
&\qquad + \frac{1}{\Gamma(p)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{p-1}f\big(s,{}^cD_{0^+}^{\alpha}u(s),{}^cD_{0^+}^{\beta}v(s)\big)\,ds - C^*(\tau_2-\tau_1)\bigg| \\
&\quad \leq \frac{1}{\Gamma(p)}\int_0^{\tau_1}\big[(\tau_2-s)^{p-1}-(\tau_1-s)^{p-1}\big]\big|f\big(s,{}^cD_{0^+}^{\alpha}u(s),{}^cD_{0^+}^{\beta}v(s)\big)\big|\,ds \\
&\qquad + \frac{1}{\Gamma(p)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{p-1}\big|f\big(s,{}^cD_{0^+}^{\alpha}u(s),{}^cD_{0^+}^{\beta}v(s)\big)\big|\,ds + \big|C^*\big|(\tau_2-\tau_1) \\
&\quad \leq \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p)}\big\|(u,v)\big\|\int_0^{\tau_1}\big[(\tau_2-s)^{p-1}-(\tau_1-s)^{p-1}\big]\,ds \\
&\qquad + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p)}\big\|(u,v)\big\|(\tau_2-\tau_1) + \big|C^*\big|(\tau_2-\tau_1) \\
&\quad = \frac{\mathcal{L}_1+\mathcal{L}_2}{p\Gamma(p)}\big\|(u,v)\big\|\big[(\tau_2-\tau_1)^p - \big(\tau_2^p-\tau_1^p\big)\big] \\
&\qquad + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p)}\big\|(u,v)\big\|(\tau_2-\tau_1) + \big|C^*\big|(\tau_2-\tau_1) \\
&\quad \leq \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p)}\big\|(u,v)\big\|(\tau_2-\tau_1)^p + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p)}\big\|(u,v)\big\|(\tau_2-\tau_1) + \big|C^*\big|(\tau_2-\tau_1) \\
&\quad \leq \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p)}\big\|(u,v)\big\|(\tau_2-\tau_1) + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p)}\big\|(u,v)\big\|(\tau_2-\tau_1) + \big|C^*\big|(\tau_2-\tau_1) \\
&\quad \leq \Bigg[\frac{2(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(p)} + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p-\gamma)} + \frac{\eta^{1-\gamma}(\mathcal{L}_1+\mathcal{L}_2)}{\Gamma(p-1)\Gamma(2-\gamma)} + \frac{\mathcal{L}_1+\mathcal{L}_2}{\Gamma(p-1)} + \sum_{i=1}^{n}L_{1i} + \sum_{i=1}^{n}\hat{L}_{1i} \\
&\qquad + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\sum_{i=1}^{n}\hat{L}_{1i}\Bigg]\big\|(u,v)\big\|(\tau_2-\tau_1) \\
&\quad \leq \rho R_0(\tau_2-\tau_1).
\end{aligned} \tag{3.7}
$$

In like manner, we get

$$\big|F_2(u,v)(\tau_2) - F_2(u,v)(\tau_1)\big| \leq \varrho R_0(\tau_2-\tau_1). \tag{3.8}$$

When $t_k < \tau_1 < \tau_2 \leq t_{k+1}$, $1 \leq k \leq n$, analogous to (3.3), we obtain

$$\big|F_1(u,v)(\tau_2) - F_1(u,v)(\tau_1)\big| \leq \rho R_0(\tau_2-\tau_1) \tag{3.9}$$

and

$$\big|F_2(u,v)(\tau_2) - F_2(u,v)(\tau_1)\big| \leq \varrho R_0(\tau_2-\tau_1). \tag{3.10}$$

Thus, it follows from (3.7)-(3.10) that, for any $\epsilon > 0$, there exists a positive constant $\sigma = \frac{\epsilon}{R_0}\min\{\frac{1}{\rho},\frac{1}{\varrho}\}$ independent of τ_1, τ_2 and (u,v) such that $\|F(u,v)(\tau_2) - F(u,v)(\tau_1)\| < \epsilon$, whenever $|\tau_2 - \tau_1| \leq \sigma$. Thereby, $F : X \to X$ is equicontinuous. By the Ascoli-Arzela theorem we know that $F : X \to X$ is completely continuous. In view of Lemma 2.4, F has at least one fixed point $(u^*(t), v^*(t)) \in \overline{\Omega}$, which is a solution pair of system (1.1). The proof is complete. □

4 Examples

Consider a class of nonlinear fractional differential coupling system with impulses as follows:

$$\begin{cases} {}^cD_{0^+}^p u(t) + f(t, {}^cD_{0^+}^\alpha u(t), {}^cD_{0^+}^\beta v(t)) = 0, & t \neq t_k, \\ {}^cD_{0^+}^q v(t) + g(t, {}^cD_{0^+}^\alpha u(t), {}^cD_{0^+}^\beta v(t)) = 0, & t \neq t_k, \\ \Delta u|_{t=t_k} = I_{1k}(u(t_k)), \qquad \Delta u'|_{t=t_k} = J_{1k}(u(t_k)), \\ \Delta v|_{t=t_k} = I_{2k}(v(t_k)), \qquad \Delta v'|_{t=t_k} = J_{2k}(v(t_k)), \\ {}^cD_{0^+}^\gamma u(\eta) = u(1), \qquad u(0) = 0, \qquad {}^cD_{0^+}^\delta v(\xi) = v(1), \qquad v(0) = 0. \end{cases} \tag{4.1}$$

Case 1: Take $J = [0,1]$, $p = \frac{3}{2}$, $q = \sqrt{2}$, $\gamma = \frac{1}{2}$, $\delta = \frac{1}{3}$, $\alpha = \frac{1}{\sqrt{2}}$, $\beta = \frac{1}{4}$, $t_1 = \frac{1}{2}$, $\eta = \frac{1}{5}$, $\xi = \frac{4}{5}$, $f(t,x,y) = e^t + \frac{x+y}{40(1+x+y)}$, $g(t,x,y) = \frac{\sin t + x + y}{40(1+x+y)}$, $I_{11}(x) = I_{21}(x) = \frac{x}{5+x}$, $J_{11}(x) = J_{21}(x) = \frac{x}{10+x}$.

By a simple calculation, we have $\mathcal{L}_1 = \mathcal{L}_2 = \hat{\mathcal{L}}_1 = \hat{\mathcal{L}}_2 = \frac{1}{40}$, $L_{11} = L_{21} = \frac{1}{5}$, $\hat{L}_{11} = \hat{L}_{21} = \frac{1}{10}$,

$$\begin{aligned} &\left[\frac{2}{\Gamma(p)} + \frac{1}{\Gamma(p-\gamma)} + \frac{\eta^{1-\gamma}}{\Gamma(p-1)\Gamma(2-\gamma)} + \frac{2}{\Gamma(p-1)}\right](\mathcal{L}_1 + \mathcal{L}_2) \\ &\quad + 2(L_{11} + \hat{L}_{11}) + \frac{\eta^{1-\gamma}}{\Gamma(2-\gamma)}\hat{L}_{11} \approx 0.8840 < 1, \\ &\left[\frac{2}{\Gamma(q)} + \frac{1}{\Gamma(q-\delta)} + \frac{\xi^{1-\delta}}{\Gamma(q-1)\Gamma(2-\delta)} + \frac{2}{\Gamma(q-1)}\right](\hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_2) \\ &\quad + 2(L_{21} + \hat{L}_{21}) + \frac{\xi^{1-\delta}}{\Gamma(2-\delta)}\hat{L}_{21} \approx 0.9391 < 1. \end{aligned}$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, system (4.1) has a unique solution pair on $[0,1]$.

Case 2: The values of p, q, γ, δ, α, β, t_1, η and ξ are the same as case 1. Take $f(t,x,y) = e^{-t}(10x+20y)$, $g(t,x,y) = (9+\cos 2t)x + (19 + |\sin\sqrt{5}t|)y$, $I_{11}(x) = I_{21}(x) = 5x$, $J_{11}(x) = J_{21}(x) = \sin x$. By a direct computation, we obtain $\mathcal{L}_1 = \hat{\mathcal{L}}_1 = 10$, $\mathcal{L}_2 = \hat{\mathcal{L}}_2 = 20$, $L_{11} = L_{21} = 5$, $\hat{L}_{11} = \hat{L}_{21} = 1$, $f(t,0,0) = g(t,0,0) = I_{11}(0) = I_{21}(0) = J_{11}(0) = J_{21}(0) = 0$. Thus, all the assumptions of Theorem 3.2 hold. Therefore, system (4.1) has at least a unique solution pair on $[0,1]$.

5 Conclusions

In this paper, we study the nonlocal boundary value problem for a class of nonlinear fractional order differential coupled system with impulses. By means of the Banach contraction principle and the Schauder fixed point theorem, some sufficient criteria are established to guarantee the existence of solutions. Compared with the single fractional differential equation, the study of a fractional order coupled system is more complicated and challenging since it is difficult to define the appropriate nonlinear operator. Our results are new and interesting. Our methods can be used to study the existence of solutions for the high order or multiple-point boundary value problems of a nonlinear fractional differential coupled system.

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