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Pinning and adaptive synchronization of fractional-order complex dynamical networks with and without time-varying delay

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Abstract

The synchronization problem for a class of fractional-order complex dynamical networks with and without time-varying delay is investigated in this paper. By utilizing generalized Barbalat's lemma, Razumikhin-type stability theory and matrix inequality technique, some sufficient criteria ensuring synchronization under pinning control and pinning adaptive feedback control are derived. Finally, three numerical simulations are presented to demonstrate the effectiveness of the obtained results.

Keywords: complex dynamical networks; fractional-order systems; synchronization; pinning control; adaptive control

1 Introduction

In the past decade, many researchers have drawn increasing attention to dynamical analysis of complex dynamical networks due to a variety of their application fields, such as biology, physics, mathematics, sociology and so on [1–6]. On the basis of complex network models, the complex dynamical networks have been extensively investigated, especially in the interaction between the overall structure and complexity, and the local dynamical properties of the coupled nodes. A complex network is usually composed of a set of coupled interconnected nodes, and each of node states is a dynamical system.

Note that fractional calculus, governed by fractional derivative and integral, has become a focal research topic in many fields such as dielectric polarization, engineering optimization, electromagnetic wave and so on [7, 8]. These research efforts have shown that, compared with integer calculus, fractional calculus has a greater advantage in describing the memory and hereditary properties of manifold material and processes, and fractional calculus has plenty of freedom when we simulate real-world problems. In recent years, there has been a great deal of work to study fractional-order systems in dynamics and control [9–11]. In the real world, the complex networks are composed of a large number of interconnected fractional-order dynamical units; therefore, it is necessary to investigate fractional-order complex dynamical networks.

Synchronization, as one of the most important collective behaviors in complex dynamic networks, has been extensively studied [12–16]. Synchronization in complex networks plays a significant role in the fields of signal generator, image processing, engineering, etc. It is well known that synchronization of fractional-order complex dynamical networks has become an important research field [17–19]. Yang et al. [17] investigated the out synchronization with two complex dynamical networks of fractional-order chaotic nodes. Wong et al. [18] addressed the robust synchronization of coupled fractional-order complex dynamical networks with parametric uncertainties. The hybrid synchronization problem of coupled fractional-order complex networks was investigated in [19].

As is well known, various control techniques, such as pinning control [20], impulsive control [21], adaptive control [22], intermittent control [23] and observer-based control [24], have been adopted to realize synchronization. But in the real world, it is too costly and impractical if all of the nodes in the network are controlled. However, many existing works show that we can synchronize the whole network by using pinning control [25–30]. Li et al. [25] provided several low-dimensional criteria for the synchronization of fractional-order complex dynamical networks with periodically intermittent pinning control. Wang et al. [26] showed that pinning synchronization problem of fractional-order complex networks can also deal with Lipschitz-type nonlinear nodes and directed communication topology. The advantage of adaptive control is that the control parameter can be adaptively adjusted according to the appropriate update law, and adaptive pinning control method has been widely used to synchronize coupled fractional-order dynamical networks. For instance, Chai et al. [27] investigated the synchronization of fractional-order complex networks via adaptive pinning control. The problem about cluster synchronization of fractional-order complex dynamical networks was studied via adaptive pinning control in [28]. However, few people studied a synchronization problem of fractional-order complex dynamical networks with and without time-varying delay via pinning and adaptive control. Thus, it is very significant to further study the synchronization of fractional-order complex dynamical networks by utilizing pinning adaptive control strategy.

Motivated by the above discussions, this paper will investigate the synchronization of fractional-order complex dynamical networks with and without time-varying delay via pinning and adaptive control. We establish some sufficient conditions to guarantee the synchronization of fractional-order complex dynamical networks with and without time-varying delay by using the pinning state feedback controller. In addition, we design adaptive control to adjust coupling strength designed for fractional-order complex dynamical networks with directed topologies. There are two adaptive plans for updating the feedback gains such that delayed fractional-order complex dynamical networks with directed topologies under the designed pinning controllers are synchronized. Moreover, the obtained results can be used to achieve anti-synchronization and complete synchronization. Compared with [29], the results in the paper are less conservative and more general.

This paper is composed as follows. Section 2 describes some preliminaries. Main results are presented in Sections 3 and 4. Three numerical examples are given in Section 5. Finally, conclusions are drawn in Section 6.

2 Preliminaries and model description

2.1 Preliminaries about fractional-order calculus

In the following, we will introduce some notations and definitions.

The superscript T represents the transpose. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times n}$ is the set of $n \times n$ real matrices. The matrix $0 < P \in \mathbb{R}^{n\times n}$ or $0 > P \in \mathbb{R}^{n\times n}$ means P is symmetric and positive definite or negative definite. $A \otimes B$ represents the Kronecker product of matrices A and B. For any matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the largest eigenvalue and the smallest one of the matrix, respectively.

2.2 Definitions and lemmas

In this subsection, some useful definitions and lemmas are given.

Definition 2.1 ([7]) $\Gamma(\cdot)$ denotes the gamma function. The Caputo fractional derivative of order $\alpha > 0$ for a function $\mathcal{F}(t)$ is defined as

$$D^{\alpha}_{t_0,t}\mathcal{F}(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{(n-\alpha-1)}\mathcal{F}^{(n)}(s)\,\mathrm{d}s, \quad t \geq t_0,$$

where $n-1 < \alpha < n$, $n \in \mathbb{Z}^{+}$.

Definition 2.2 ([7]) The fractional integral of order α for a function $\mathcal{F}(t)$ is defined as

$$I^{\alpha}_{t_0,t} = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\mathcal{F}(s)\,\mathrm{d}s, \quad \alpha > 0.$$

Definition 2.3 ([31]) The matrix $\mathcal{A}$ of order n is called reducible if there is a permutation matrix $\mathcal{B} \in \mathbb{R}^{n\times n}$ satisfying

$$\mathcal{B}\mathcal{A}\mathcal{B}^{T} = \begin{pmatrix} \mathcal{A}_1 & 0 \\ \mathcal{A}_{21} & \mathcal{A}_2 \end{pmatrix},$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are square matrices of order at least one. If $\mathcal{A}$ is not reducible, $\mathcal{A}$ is called irreducible.

Lemma 2.1 ([31]) *Suppose that* $\mathcal{L} = (\mathcal{L}_{ij})_{N\times N}(N > 2)$ *is an irreducible matrix, where*

$$\mathcal{L}_{ij} \geq 0 \quad (i \neq j), \qquad \mathcal{L}_{ii} \leq -\sum_{i=1,j\neq i}^{N}\mathcal{L}_{ij},$$

then there exists a diagonal matrix $0 < \mathcal{K} = \operatorname{diag}(K_1, K_2, \ldots, K_N) \in P^{N\times N}$ *such that*

$$\mathcal{K}\mathcal{L} + \mathcal{L}^{T}\mathcal{K} \leq 0.$$

Lemma 2.2 ([32]) *Suppose that* $\mathcal{A}$ *is an n order matrix, then there exist* $\mathcal{B} \in \mathbb{R}^{n\times n}$ *and an integer* $r \geq 1$ *satisfying*

$$\mathcal{B}\mathcal{A}\mathcal{B}^{T} = \begin{pmatrix} \mathcal{A}_1 & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1r} \\ 0 & \mathcal{A}_2 & \cdots & \mathcal{A}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_r \end{pmatrix},$$

where $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}$ *are square irreducible matrices. The matrices* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r$ *that occur as diagonal blocks are uniquely determined within simultaneous permutation of their lines, but their ordering is not necessarily unique.*

This form is called the Frobenius normal form of the square matrix $\mathcal{A}$.

Lemma 2.3 ([33]) *Suppose that the function $g(t)$ is nondecreasing and differential on $t \in [t_0, +\infty)$, and then, for any constant h and $t \in [t_0, +\infty)$,*

$$D^{\alpha}_{t_0,t}\big(g(t) - h\big)^2 \leq 2\big(g(t) - h\big)D^{\alpha}_{t_0,t}g(t),$$

where $0 < \alpha < 1$.

Lemma 2.4 ([34]) *Let $x(t) \in \mathbb{R}^n$ be a continuous and derivable vector-valued function. Then, for any $t \geq t_0$,*

$$\frac{1}{2}D^{\alpha}_{t_0,t}x^T(t)Px(t) \leq \big(x^T(t)P\big)D^{\alpha}_{t_0,t}x(t),$$

where $P \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $\alpha \in (0, 1)$.

Lemma 2.5 ([35]) *For any vector $x, y \in \mathbb{R}^n$, scalar $\epsilon > 0$ and positive definite matrix $Q \in \mathbb{R}^{n\times n}$, the following inequality holds*:

$$2x^Ty \leq \epsilon x^TQx + \epsilon^{-1}y^TQ^{-1}y.$$

3 Pinning synchronization of fractional-order complex dynamical networks

In this section, we consider coupled complex dynamical networks consisting of N identical nodes, which is described as follows:

$$D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf\big(x_i(t)\big) + c\sum_{j=1}^{N}G_{ij}\Gamma x_j(t) + J, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $N \geq 2$ is the number of subnetworks. $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ denotes the state vector of the ith nodes. $A = \operatorname{diag}\{a_1, a_2, \ldots, a_n\} > 0$. $B = (b_{pq})_{n\times n}$ $(p, q = 1, 2, \ldots, n)$ is the connection weight matrix, respectively. $f_i(x_i(t)) = (f_{i1}(x_{i1}(t)), f_{i2}(x_{i1}(t)), \ldots, f_{in}(x_{in}(t)))^T$. $c > 0$ represents the overall coupling strength. $G = (G_{ij})_{N\times N}$ is the coupling matrix, which is defined as follows: if there is a link from node j to node i, then $G_{ij} > 0$; otherwise $G_{ij} = 0$, the diagonal elements are defined as $G_{ii} = -\sum_{j=1,j\neq i}^{N}G_{ij}$. $0 < \Gamma \in \mathbb{R}^{n\times n}$ stands for an inner coupling matrix. $J = (J_1, J_2, \ldots, J_N)$ is a constant external input vector.

Assumption 3.1 In this paper, the function $f_j(\cdot)$ $(j = 1, 2, \ldots, n)$ is continuous, and there exist $\Pi_i > 0$. For any vectors $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, we have

$$\big|f_j(\varepsilon_1) - f_j(\varepsilon_2)\big| \leq \Pi_i|\varepsilon_1 - \varepsilon_2|.$$

The desired trajectory of $s(t)$ satisfies

$$D^{\alpha}_{t_0,t}s(t) = -As(t) + Bf\big(s(t)\big) + J. \tag{2}$$

We design the proper controller $u_i(t)$ to make system (1) synchronized to $s(t)$, that is,

$$\lim_{t\to+\infty}\big\|x_i(t) - s(t)\big\|_2 = 0, \quad i = 1, 2, \ldots, N.$$

As we all know, coupled complex dynamical network (1) cannot achieve synchronization by itself. In this section, some controllers should be used to control partial nodes for realizing synchronization. Hence, the pinning control can be presented as follows:

$$u_i(t) = \begin{cases} -ck_i\Gamma(x_i(t) - s(t)), & i \in \mathcal{C}, \\ 0, & i \notin \mathcal{C}, \end{cases} \tag{3}$$

where $\mathcal{C} = \{l_1, l_2, \dots, l_m\}$ and l_i ($i = 1, 2, \dots, m$, $1 \le m < N$) are controlled nodes. $k_i > 0$ represents feedback gains.

Under controller (3), system (1) can be rewritten as

$$D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf\bigl(x_i(t)\bigr) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j(t) + J - c\hat{k}_i\Gamma\bigl(x_i(t) - s(t)\bigr), \tag{4}$$

where $i = 1, 2, \dots, N$, $K = \operatorname{diag}(\hat{k}_1, \hat{k}_2, \dots, \hat{k}_N) = \operatorname{diag}(0, \dots, 0, \underbrace{k_1}_{l_1}, 0, \dots, 0, \underbrace{k_2}_{l_2}, 0, \dots, 0, \underbrace{k_r}_{l_m}, \dots)$.

Define the error signal $e_i(t) = x_i(t) - s(t)$, $i = 1, 2, \dots, N$. Then we can obtain

$$D^{\alpha}_{t_0,t}e_i(t) = -Ae_i(t) + Bf\bigl(x_i(t)\bigr) - Bf\bigl(s(t)\bigr) + c\sum_{j=1}^{N} \hat{G}_{ij}\Gamma e_j(t), \tag{5}$$

where $m = 1, 2, \dots, N$, $\hat{G} = (\hat{G}_{ij})_{N\times N} = G - K$.

Suppose that $\hat{G}$ is in the Frobenius normal form, that is,

$$\hat{G} = \begin{pmatrix} \bar{G}_1 & \bar{G}_{12} & \cdots & \bar{G}_{1r} \\ 0 & \bar{G}_2 & \cdots & \bar{G}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{G}_r \end{pmatrix},$$

where $\bar{G}_1 \in \mathbb{R}^{p_1\times p_1}$, $\bar{G}_2 \in \mathbb{R}^{p_2\times p_2}$, ..., $\bar{G}_r \in \mathbb{R}^{p_r\times p_r}$ are square irreducible matrices.

Denote

$$\begin{aligned}
\hat{e}_1(t) &= \bigl(e_1^T(t), e_2^T(t), \dots, e_{p_1}^T(t)\bigr)^T, \\
\hat{e}_2(t) &= \bigl(e_{p_1+1}^T(t), e_{p_1+2}^T(t), \dots, e_{p_1+p_2}^T(t)\bigr)^T, \\
&\vdots \\
\hat{e}_r(t) &= \bigl(e_{N-p_r+1}^T(t), e_{N-p_r+2}^T(t), \dots, e_N^T(t)\bigr)^T, \\
\hat{f}_i\bigl(x(t)\bigr) &= \bigl(f^T\bigl(x_{p_1+\cdots+p_{i-1}+1}(t)\bigr), f^T\bigl(x_{p_1+\cdots+p_{i-1}+2}(t)\bigr), \\
&\qquad \dots, f^T\bigl(x_{p_1+\cdots+p_{i-1}+1}(t)\bigr)\bigr)^T, \\
S(t) &= \bigl(s^T(t), s^T(t), \dots, s^T(t)\bigr)^T, \\
e(t) &= \bigl(e_1^T(t), e_2^T(t), \dots, e_N^T(t)\bigr)^T, \\
\hat{f}_i\bigl(S(t)\bigr) &= \bigl(f^T\bigl(s(t)\bigr), f^T\bigl(s(t)\bigr), \dots, f^T\bigl(s(t)\bigr)\bigr).
\end{aligned}$$

Then system (1) can be transformed as

$$\begin{aligned}D^{\alpha}_{t_0,t}\hat{e}_i(t) &= -(I_{p_i}\otimes A)\hat{e}_i(t) + (I_{p_i}\otimes B)\big(\hat{f}_i\big(x(t)\big) - \hat{f}_i\big(s(t)\big)\big)\\ &\quad + c\sum_{j=i+1}^{r}(\bar{G}_{ij}\otimes\Gamma)\hat{e}_j(t) + c(\bar{G}_i\otimes\Gamma)\hat{e}_i(t),\end{aligned}\tag{6}$$

where $i = 1, 2, \ldots, r$.

Lemma 3.1 ([36]) *Let f be a nonnegative uniformly continuous function. If for all $t \geq 0$, $I^{\alpha}_0 f(t) < C$ with C a positive constant, then f converges to zero.*

Lemma 3.2 ([10]) *If the Caputo fractional derivative $D^{\alpha}_{t_0,t}\mathcal{F}(t)$ is integrable, then*

$$I^{\alpha}_{t_0,t}D^{\alpha}_{t_0,t}\mathcal{F}(t) = \mathcal{F}(t) - \sum_{k=0}^{n-1}\frac{f^{(}k)(t_0)}{k!}(t-t_0)^k.$$

When $0 < \alpha < 1$, one can obtain

$$I^{\alpha}_{t_0,t}D^{\alpha}_{t_0,t}\mathcal{F}(t) = \mathcal{F}(t) - \mathcal{F}(0).$$

3.1 Fixed coupled strength

Theorem 3.1 *If there exist matrices $0 < P_i = \operatorname{diag}(P_{i1}, P_{i2}, \ldots, P_{ip_i}) \in \mathbb{R}^{p_i\times p_i}$ and a positive scalar $\lambda_1 > 0$ such that*

$$\Lambda = P\otimes\big(-2A + BB^T + \Xi\big) + c\big(P\hat{G} + \hat{G}^T P\big)\otimes\Gamma - \lambda_1 P < 0,\tag{7}$$

where $P = \operatorname{diag}(P_1, P_2, \ldots, P_r)$, $\Xi = \operatorname{diag}(\Pi_1^2, \Pi_2^2, \ldots, \Pi_n^2)$, then system (1) can achieve synchronization.

Proof Let us consider the function $V_1(t) = \sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\otimes I_n)\hat{e}_i(t)$.

From Lemmas 2.3 and 2.4, we can obtain

$$\begin{aligned}D^{\alpha}_{t_0,t}V_1(t) &\leq 2\sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\otimes I_n)D^{\alpha}_{0,t}\hat{e}_i(t)\\ &= 2\sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\otimes I_n)\Bigg[-(I_{p_i}\otimes A)\hat{e}_i(t) + c\sum_{j=i+1}^{r}(\bar{G}_{ij}\otimes\Gamma)\hat{e}_j(t)\\ &\quad + c(\bar{G}_i\otimes\Gamma)\hat{e}_i(t) + (I_{p_i}\otimes B)\big(\hat{f}_i\big(x(t)\big) - \hat{f}_i\big(S(t)\big)\big)\Bigg]\\ &= 2\sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\otimes I_n)(-I_{p_i}\otimes A)\hat{e}_i(t)\\ &\quad + 2c\sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\otimes I_n)\sum_{j=i+1}^{r}(\bar{G}_{ij}\otimes\Gamma)\hat{e}_j(t)\\ &\quad + 2c\sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\otimes I_n)(\bar{G}_i\otimes\Gamma)\hat{e}_i(t)\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i=1}^{r}\hat{e}_i^T(t)(\Phi_i\otimes I_n)(I_{p_i}\otimes B)\big(\hat{f}_i\big(x(t)\big)-\hat{f}_i\big(S(t)\big)\big)\\
&=-2\sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\otimes A)+2c\sum_{i=1}^{r}\sum_{j=i+1}^{r}\hat{e}_i^T(t)\big[(P_i\bar{G}_{ij})\otimes\Gamma\big]\hat{e}_j(t)\\
&+2c\sum_{i=1}^{r}\hat{e}_i^T(t)(P_i\bar{G}_i)\otimes\Gamma)\hat{e}_i^T(t)\\
&+2\sum_{i=1}^{r}\sum_{j=1}^{p_i}P_{ij}e_{p_1+\cdots+p_{i-1}+j}^T(t)\big(BB^T+\Xi\big)e_{p_1+\cdots+p_{i-1}+j}(t)\\
&=e^T(t)\big[P\otimes\big(-2A+BB^T+\Xi\big)+c\big(P\hat{G}+\hat{G}^TP\big)\otimes\Gamma\big]e(t)\\
&\le e^T(t)\Lambda e(t)-\lambda_1V_1(t)\\
&\le -\lambda_1V_1(t).
\end{aligned}
\tag{8}
$$

That is,

$$
D_{t_0,t}^{\alpha}V_1(t)\le-\lambda_1V_1(t)<0. \tag{9}
$$

Let $Q(t)=\lambda_1V_1(t)$, we divide $[t_0,t)$ into ι intervals $[t_0,t)=[t_0,t_1)\cup[t_1,t_2)\cup\cdots\cup[t_\iota,t)$. From Definition 2.1, Definition 2.2 and Lemma 3.2, we have

$$
\begin{aligned}
I_{t_0,t}^{\alpha}Q(t)&=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{(\alpha-1)}Q(s)\,\mathrm{d}s\\
&=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}(t-s)^{(\alpha-1)}Q(s)\,\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t-s)^{(\alpha-1)}Q(s)\,\mathrm{d}s\\
&\quad+\cdots+\frac{1}{\Gamma(\alpha)}\int_{t_\iota}^{t}(t-s)^{(\alpha-1)}Q(s)\,\mathrm{d}s\\
&=I_{t_0,t_1}^{\alpha}Q(t_1)+I_{t_1,t_2}^{\alpha}Q(t_2)+\cdots+I_{t_\iota,t}^{\alpha}Q(t)\\
&\le-\big(I_{t_0,t_1}^{\alpha}D_{t_0,t_1}^{\alpha}V_1(t_1)+I_{t_1,t_2}^{\alpha}D_{t_1,t_2}^{\alpha}V_1(t_2)\big)\\
&\quad+\cdots+I_{t_\iota,t}^{\alpha}D_{t_\iota,t}^{\alpha}V_1(t))\\
&=V_1(t_0)-V_1(t_1)+V_1(t_1)-V_1(t_2)+V_1(t_2)\\
&\quad-V_1(t_3)+\cdots+V_1(t_\iota)-V_1(t)\\
&=V_1(t_0)-V_1(t)\\
&\le V_1(t_0).
\end{aligned}
\tag{10}
$$

From Lemma 3.1 and Lemma 7 in [37], we have $\lim_{t\to\infty}I_{t_0,t}^{\alpha}Q(t)$ is bounded, that is, $\lim_{t\to\infty}\|e(t)\|=0$, then system (4) can achieve synchronization. The proof is completed. □

3.2 Adaptive coupled strength

In this part, the main work is to solve the synchronization problem of system (4) by updating the coupled strength. Now, we design a suitable adaptive controller:

$$D^{\alpha}_{t_0,t}c(t) = \beta_1 \sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes \Gamma)\hat{e}_i(t), \tag{11}$$

where $0 < \beta_1 \in \mathbb{R}$.

Combining system (6) with adaptive law (11), we have

$$\begin{aligned} D^{\alpha}_{t_0,t}\hat{e}_i(t) &= -(I_{p_i} \otimes A)\hat{e}_i(t) + c(t)(\bar{G}_i \otimes \Gamma)\hat{e}_i(t) \\ &\quad + c(t)\sum_{j=i+1}^{r}(\bar{G}_{ij} \otimes \Gamma)\hat{e}_j(t) + (I_{p_i} \otimes B)\big(\hat{f}_i\big(x(t)\big) - \hat{f}_i\big(S(t)\big)\big). \end{aligned} \tag{12}$$

Theorem 3.2 *If there exist matrices* $0 < P_i = \operatorname{diag}(P_{i1}, P_{i2}, \ldots, P_{ip_i}) \in \mathbb{R}^{p_i \otimes p_i}$ *and positive scalars* $\delta_1 > 0$ *such that*

$$P\hat{G} + \hat{G}^T P + \delta_1 P < 0, \tag{13}$$

where $P = \operatorname{diag}(P_1, P_2, \ldots, P_r)$, *then* (12) *can achieve synchronization under adaptive law* (11).

Proof Define the following function: $V_2(t) = \sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n)\hat{e}_i(t) + \frac{\delta_1}{2\beta_1}(c(t) - c^*)^2$.

From Lemmas 2.3 and 2.4, we can get

$$\begin{aligned} D^{\alpha}_{t_0,t}V_2(t) &\le 2\sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n)D^{\alpha}_{t_0,t}\hat{e}_i(t) + D^{\alpha}_{t_0,t}\frac{\delta_1}{2\beta_1}\big(c(t) - c^*\big)^2 \\ &= \sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n)\Bigg(-(I_{p_i} \otimes A)\hat{e}_i(t) \\ &\quad + c(t)(\bar{G}_i \otimes \Gamma)\hat{e}_i(t) + c(t)\sum_{j=i+1}^{r}(\bar{G}_{ij} \otimes \Gamma)\hat{e}_j(t) \\ &\quad + (I_{p_i} \otimes B)\big(\hat{f}_i\big(x(t)\big) - \hat{f}_i\big(S(t)\big)\big)\Bigg) + \frac{\delta_1}{\beta_1}\big(c(t) - c^*\big)D^{\alpha}_{t_0,t}c(t) \\ &= e^T(t)\big[P \otimes \big(-2A + BB^T + \Xi\big) + c(t)\big(P\hat{G} + \hat{G}^T P \\ &\quad + \delta_1 P\big) \otimes \Gamma - \delta_1 c^*(P \otimes \Gamma)\big]e(t). \end{aligned} \tag{14}$$

We know that $c(t_0) > 0$ and $c(t)$ is monotonically increasing, then we can obtain $c(t) > 0$. According to (13), we have

$$c(t)\big(P\hat{G} + \hat{G}^T P + \delta_1 P\big) < 0.$$

By setting c^* large enough such that

$$P \otimes \big(-2A + BB^T + \Xi - \delta_1 c^*\Gamma\big) < -I_{Nn},$$

we have

$$D^{\alpha}_{t_0,t}V_2(t) \le -e^T(t)e(t), \quad t \ge t_0. \tag{15}$$

From Definition 2.2 and (15), we can get

$$V_2(t) - V_2(t_0) \le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\bigl(-e^T(t)e(t)\bigr)\,\mathrm{d}s \le 0. \tag{16}$$

Hence $V_2(t) \le V_2(t_0)$, we can conclude that $\|\hat{e}_i(t)\|$ $(i = 1,2,\dots,r)$ and $c(t)$ are bounded on $t \ge t_0$. Let $U(t) = x^T(t)x(t)$. From the definition of $V_2(t)$, we can conclude that $\|x(t)\|$ and $c(t)$ are bounded on $t \ge t_0$. Therefore, there exists a positive constant M satisfying

$$\bigl|D^{\alpha}_{t_0,t}U(t)\bigr| \le M, \quad t \ge t_0.$$

Next, we will prove $U(t)$ is uniformly continuous. For $0 \le \mathcal{T}_1 < \mathcal{T}_2$, we have

$$\begin{aligned}
\bigl|U(\mathcal{T}_1) - U(\mathcal{T}_2)\bigr| &= \bigl|I^{\alpha}_{t_0,t}D^{\alpha}_{t_0,t}U(\mathcal{T}_1) - I^{\alpha}_{t_0,t}D^{\alpha}_{t_0,t}U(\mathcal{T}_2)\bigr| \\
&= \frac{1}{\Gamma(\alpha)}\biggl|\int_{t_0}^{\mathcal{T}_1}(\mathcal{T}_1 - s)^{\alpha-1}D^{\alpha}_{t_0,t}U(s)\,\mathrm{d}s \\
&\quad - \int_{t_0}^{\mathcal{T}_2}(\mathcal{T}_2 - s)^{\alpha-1}D^{\alpha}_{t_0,t}U(s)\,\mathrm{d}s\biggr| \\
&= \frac{1}{\Gamma(\alpha)}\biggl|\int_{t_0}^{\mathcal{T}_1}\bigl((\mathcal{T}_1 - s)^{\alpha-1} - (\mathcal{T}_2 - s)^{\alpha-1}\bigr)D^{\alpha}_{t_0,t}U(s)\,\mathrm{d}s \\
&\quad - \int_{\mathcal{T}_1}^{\mathcal{T}_2}(\mathcal{T}_2 - s)^{\alpha-1}D^{\alpha}_{t_0,t}U(s)\,\mathrm{d}s\biggr| \\
&\le \frac{1}{\Gamma(\alpha)}\biggl(\biggl|\int_{t_0}^{\mathcal{T}_1}\bigl((\mathcal{T}_1 - s)^{\alpha-1} - (\mathcal{T}_2 - s)^{\alpha-1}\bigr)D^{\alpha}_{t_0,t}U(s)\,\mathrm{d}s\biggr| \\
&\quad + \biggl|\int_{\mathcal{T}_1}^{\mathcal{T}_2}(\mathcal{T}_2 - s)^{\alpha-1}D^{\alpha}_{t_0,t}U(s)\,\mathrm{d}s\biggr|\biggr) \\
&\le \frac{M}{\Gamma(\alpha)}\biggl(\int_{t_0}^{\mathcal{T}_1}\bigl[(\mathcal{T}_1 - s)^{\alpha-1} - (\mathcal{T}_2 - s)^{\alpha-1}\bigr]\,\mathrm{d}s \\
&\quad + \int_{\mathcal{T}_1}^{\mathcal{T}_2}(\mathcal{T}_2 - s)^{\alpha-1}\,\mathrm{d}s\biggr) \\
&\le \frac{M}{\Gamma(\alpha+1)}\bigl[\mathcal{T}_1^{\alpha} - \mathcal{T}_2^{\alpha} + 2(\mathcal{T}_2 - \mathcal{T}_1)^{\alpha}\bigr] \\
&\le 2\frac{M}{\Gamma(\alpha+1)}(\mathcal{T}_2 - \mathcal{T}_1)^{\alpha} < \hat{\varepsilon},
\end{aligned}$$

where $|\mathcal{T}_2 - \mathcal{T}_1| < \Delta(\hat{\varepsilon}) = (\frac{\hat{\varepsilon}\Gamma(\alpha+1)}{2M})^{\frac{1}{\alpha}}$. Then $U(t)$ is uniformly continuous by the definition of uniform continuity. From Lemma 3.1, we obtain $\lim_{t\to\infty} U(t) = 0$; obviously, $\lim_{t\to\infty}\|e(t)\| = 0$, that is, system (12) can achieve synchronization under adaptive law (11). The proof is completed. □

4 Pinning synchronization of complex dynamical networks with time-varying delay

In this section, we will consider the following complex dynamical networks with time-varying delay:

$$D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf\big(x_i(t)\big) + c\sum_{j=1}^{N} G_{ij}\Gamma x_j\big(t-\tau(t)\big) + J, \tag{17}$$

where $i = 1,2,\dots,N, N \geq 2, x_i(t) = (x_{i1}(t), x_{i2}(t),\dots,x_{in}(t))^T \in \mathbb{R}^n$ denotes the state vector of the ith nodes. $A = \operatorname{diag}\{a_1,a_2,\dots,a_n\} > 0$. $B = (b_{pq})_{n\times n}$ $(p,q = 1,2,\dots,n)$ is the connection weight matrix, respectively. $f_i(x_i(t)) = (f_{i1}(x_{i1}(t)), f_{i2}(x_{i1}(t)),\dots,f_{in}(x_{in}(t)))^T$. $c > 0$ represents the overall coupling strength. $G = (G_{ij})_{N\times N}$ is the coupling matrix, which is defined as follows: if there is a link from node j to node i, then $G_{ij} > 0$; otherwise $G_{ij} = 0$, the diagonal elements are defined as $G_{ii} = -\sum_{j=1,j\neq i}^{N} G_{ij}$. $0 < \Gamma \in \mathbb{R}^{n\times n}$ stands for an inner coupling matrix. $J = (J_1,J_2,\dots,J_N)$ is a constant external input vector. $\tau(t)$ stands for the transmission delay with $0 \leq \tau(t) \leq \tau$.

In the following, some controllers should be used to control partial nodes for realizing synchronization. Hence, the pinning control can be presented as follows:

$$u_i(t) = \begin{cases} -ck_i\Gamma(x_i(t)-s(t)), & i \in \mathcal{C}, \\ 0, & i \notin \mathcal{C}, \end{cases} \tag{18}$$

where $\mathcal{C} = \{l_1,l_2,\dots,l_m\}$ and l_i $(i = 1,2,\dots,m,\ 1 \leq m < N)$ are controlled nodes. $k_i > 0$ represents feedback gains.

Under controller (18), system (17) can be rewritten as

$$D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf\big(x_i(t)\big) + c\sum_{j=1}^{N} G_{ij}\Gamma x_i\big(t-\tau(t)\big) - c\hat{k}_i\Gamma\big(x_i(t)-s(t)\big) + J, \tag{19}$$

where $i = 1,2,\dots,N$, $N \geq 2$, $K = \operatorname{diag}(\hat{k}_1,\hat{k}_2,\dots,\hat{k}_N) = \operatorname{diag}(0,\dots,0,\underbrace{k_1}_{l_1},0,\dots,0,\underbrace{k_2}_{l_2},0,\dots,0,\underbrace{k_m}_{l_m},\dots)$.

Let $e_i(t) = x_i(t) - s(t)$, we have

$$\begin{cases} D^{\alpha}_{t_0,t}e_i(t) = -Ae_i(t) + Bf(x_i(t)) - Bf(s(t)) - c\hat{k}_i\Gamma e_i(t) \\ \qquad\qquad + c\sum_{j=1}^{N} G_{ij}\Gamma e_i(t-\tau(t)), \\ e_i(s) = \Phi_i(s), \quad -\tau \leq s \leq 0, \end{cases} \tag{20}$$

where $i = 1,2,\dots,N$.

Lemma 4.1 ([38]) *The Caputo fractional-order differential system*

$$D^{\alpha}_{t_0,t}x(t) = f\big(t,x(t),x\big(t-\tau(t)\big)\big),$$

where $x \in \mathbb{R}^n$, $0 < \alpha < 1$. *Suppose that* $w_1(s)$, $w_2(s)$ *are continuous nondecreasing functions,* $w_1(s)$ *and* $w_2(s)$ *are positive for* $s > 0$ *and* $w_1(0) = w_2(0)$, w_2 *is strictly increasing. If there* *exists a continuously differentiable function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that* $w_1(\|x\|) \leq V(t,x) \leq w_2(\|x\|)$ *for* $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. *Besides this, there exist two constants* $p, q > 0$ *with* $p < q$, *so that*

$$D^{\alpha}_{t_0,t} V\bigl(t, x(t)\bigr) \leq -qV\bigl(t, x(t)\bigr) + p \sup_{-\tau \leq \theta \leq 0} V\bigl(t+\theta, x(t+\theta)\bigr)$$

for $t \geq 0$. *Then system* $D^{\alpha}_{t_0,t} x(t) = f(t, x(t), x(t-\tau(t)))$ *is globally uniformly asymptotically stable.*

Assume

$$G = \begin{pmatrix} \tilde{G}_1 & \tilde{G}_{12} & \cdots & \tilde{G}_{1r} \\ 0 & \tilde{G}_2 & \cdots & \tilde{G}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{G}_r \end{pmatrix},$$

where $\tilde{G}_1 \in \mathbb{R}^{p_1 \times p_1}$, $\tilde{G}_2 \in \mathbb{R}^{p_2 \times p_2}$, $\ldots$, $\tilde{G}_r \in \mathbb{R}^{p_r \times p_r}$ are square irreducible matrices, then system (20) can be rewritten as

$$\begin{aligned} D^{\alpha}_{t_0,t} \hat{e}_i(t) = {} & -(I_{p_i} \otimes A)\hat{e}_i(t) + (I_{p_i} \otimes B)\bigl(\hat{f}_i\bigl(x(t)\bigr) - \hat{f}_i\bigl(S(t)\bigr)\bigr) \\ & + c\sum_{j=i+1}^{r} (\tilde{G}_{ij} \otimes \Gamma)\hat{e}_j\bigl(t-\tau(t)\bigr) - c(K_i \otimes \Gamma)\hat{e}_i(t) \\ & + c(\tilde{G}_i \otimes \Gamma)\hat{e}_i\bigl(t-\tau(t)\bigr), \end{aligned} \tag{21}$$

where $i = 1, 2, \ldots, m$, the initial value of (21) is given by $\hat{e}_i(s) = \hat{\Phi}_i(s)$,

$$\begin{aligned} & \hat{\Phi}_i(s) = \bigl(\Phi_1(s), \Phi_2(s), \ldots, \Phi_{p_i}(s)\bigr), \\ & K_i = \operatorname{diag}(K_{i1}, K_{i2}, \ldots, K_{ip_i}) \in \mathbb{R}^{p_i \times p_i}, \\ & K = \operatorname{diag}(K_1, K_2, \ldots, K_r), \\ & e(t) = \bigl(\hat{e}_1(t), \hat{e}_2(t), \ldots, \hat{e}_r\bigr)^T, \\ & \hat{e}_1(t) = \bigl(e_1^T(t), e_2^T(t), \ldots, e_{p_1}^T(t)\bigr), \\ & \hat{e}_2(t) = \bigl(e_{p_1+1}^T(t), e_{p_1+2}^T(t), \ldots, e_{p_1+p_2}^T(t)\bigr), \\ & \vdots \\ & \hat{e}_r(t) = \bigl(e_{N-p_r+1}^T(t), e_{N-p_r+2}^T(t), \ldots, e_N^T(t)\bigr), \\ & e\bigl(t-\tau(t)\bigr) = \bigl(\hat{e}_1\bigl(t-\tau(t)\bigr), \hat{e}_2\bigl(t-\tau(t)\bigr), \ldots, \hat{e}_r\bigl(t-\tau(t)\bigr)\bigr)^T, \\ & \hat{e}_1\bigl(t-\tau(t)\bigr) = \bigl(e_1^T\bigl(t-\tau(t)\bigr), e_2^T\bigl(t-\tau(t)\bigr), \ldots, e_{p_1}^T\bigl(t-\tau(t)\bigr)\bigr)^T, \\ & \hat{e}_2\bigl(t-\tau(t)\bigr) = \bigl(e_{p_1+1}^T\bigl(t-\tau(t)\bigr), e_{p_1+2}^T\bigl(t-\tau(t)\bigr), \ldots, e_{p_1+p_2}^T\bigl(t-\tau(t)\bigr)\bigr)^T, \\ & \vdots \\ & \hat{e}_r\bigl(t-\tau(t)\bigr) = \bigl(e_{N-p_r+1}^T\bigl(t-\tau(t)\bigr), e_{N-p_r+2}^T\bigl(t-\tau(t)\bigr), \ldots, e_N^T\bigl(t-\tau(t)\bigr)\bigr)^T, \end{aligned}$$

$$S(t) = \big(s^T(t), s^T(t), \dots, s^T(s)\big)^T,$$
$$\hat{f}_i\big(S(t)\big) = \underbrace{\big(f^T\big(s(t)\big), f^T\big(s(t)\big), \dots, f^T\big(s(t)\big)\big)^T}_{p_i},$$
$$x(t) = \big(x_1^T(t), x_2(t), \dots, x_N^T(t)\big)^T,$$
$$\hat{f}_i\big(x(t)\big) = \big(f^T\big(x_{p_1+\cdots+p_{i-1}+1}(t)\big), f^T\big(x_{p_1+\cdots+p_{i-1}+2}(t)\big), \cdots, f^T\big(x_{p_1+\cdots+p_{i-1}+p_i}(t)\big)\big)^T.$$

4.1 Fixed feedback gains

Theorem 4.1 *If there exist matrices* $0 < \Sigma_1 \in \mathbb{R}^{n\times n}$, $0 < P_i \in \mathbb{R}^{n\times n}$ *satisfying*

$$\begin{aligned} H &= \big(P \otimes \big(2A - BB^T - \Xi\big) + c(PK + KP) \otimes \Gamma \\ &\quad - c^2(PG) \otimes \Gamma\big)\big(I_N \otimes \Sigma_1^{-1}\big)\big(G^T P\big) \otimes \Gamma \\ &> 0, \end{aligned} \tag{22}$$

$$T = I_N \otimes \Sigma_1 > 0, \tag{23}$$

where $P = \operatorname{diag}(P_1, P_2, \dots, P_r)$, $\sigma_1 = \frac{\lambda_{\min(H)}}{\xi_2}$, $\sigma_2 = \frac{\lambda_{\max}(T)}{\xi_1}$, $\sigma_1\sigma_2 > 0$, *then system* (19) *can achieve synchronization.*

Proof Define the following function for system (21):

$$V_3(t) = \sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n)\hat{e}_i(t).$$

From Lemmas 2.3 and 2.4, we can get

$$\begin{aligned} D_{t_0,t}^{\alpha} V_3(t) &\le 2\sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n) D_{t_0,t}^{\alpha}\hat{e}_i(t) \\ &= 2\sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n)\big(-(I_{p_i} \otimes A)\hat{e}_i(t) \\ &\quad + (I_{p_i} \otimes B)\big(\hat{f}_i\big(x(t)\big) - \hat{f}_i\big(S(t)\big)\big) \\ &\quad + c\sum_{j=i+1}^{r} (\tilde{G}_{ij} \otimes \Gamma)\hat{e}_j\big(t - \tau(t)\big) \\ &\quad - c(K_i \otimes \Gamma)\hat{e}_i(t) + c(\tilde{G}_i \otimes \Gamma)\hat{e}_i\big(t - \tau(t)\big)\big) \\ &= \sum_{i=1}^{r} \hat{e}_i^T(t)\big[-2(P_i \otimes A) + \big(P_i \otimes \big(BB^T + \Xi\big)\big) \\ &\quad - c(P_iK_i + K_iP_i) \otimes \Gamma\big]\hat{e}_i(t) \\ &\quad + 2\left(\sum_{i=1}^{r}\sum_{j=i+1}^{r} \hat{e}_i(t)\tilde{G}_{ij} \otimes \Gamma\right)\hat{e}_j\big(t - \tau(t)\big) \end{aligned}$$

$$\begin{aligned}
&+ 2c\sum_{i=1}^{r}\hat{e}_i(t)\big((P_i\tilde{G}_i)\otimes\Gamma\big)\hat{e}_j\big(t-\tau(t)\big)\\
&= e^T(t)\big(-2(P\otimes A) + \big(P\otimes\big(BB^T+\Xi\big)\big) - c(PK+KP)\otimes\Gamma\big)e(t)\\
&+ 2ce^T(t)\big((PG)\otimes\Gamma\big)e\big(t-\tau(t)\big).
\end{aligned}$$

From Lemma 2.5, we can obtain

$$\begin{aligned}
&2ce^T(t)\big[(PG)\otimes\Gamma\big]e\big(t-\tau(t)\big)\\
&\le c\big[e^T(t)\big((PG)\otimes\Gamma\big)\big]e\big(t-\tau(t)\big) + ce^T\big(t-\tau(t)\big)\big[\big(G^TP\big)\otimes\Gamma\big]e(t)\\
&\le c^2e^T(t)\big(\big((PG)\otimes\Gamma\big)\big(I_N\otimes\Sigma_1^{-1}\big)\big(G^TP\big)\otimes\Gamma\big)e(t)\\
&\quad + e^T\big(t-\tau(t)\big)(I_N\otimes\Sigma_1)e\big(t-\tau(t)\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
D_{t_0,t}^{\alpha}V_3(t) &\le e^T(t)\big[-2(P\otimes A) + \big(P\otimes\big(BB^T+\Xi\big)\big) - c(PK+KP)\otimes\Gamma\\
&\quad + c^2\big((PG)\otimes\Gamma\big)\big(I_N\otimes\Sigma_1^{-1}\big)\big(G^TP\big)\Gamma\big]e(t)\\
&\quad + e^T\big(t-\tau(t)\big)(I_N\otimes\Sigma_1)e\big(t-\tau(t)\big)\\
&\le -\lambda_{\min}(H)e^T(t)e(t) + \lambda_{\max}(T)e^T\big(t-\tau(t)\big)e\big(t-\tau(t)\big),
\end{aligned}$$

where

$$\begin{aligned}
H &= (P\otimes\big(2A - BB^T - \Xi\big) + c(PK+KP)\otimes\Gamma\\
&\quad - c^2\big((PG)\otimes\Gamma\big)\big(I_N\otimes\Sigma_1^{-1}\big)\big(G^TP\big)\Gamma\\
&> 0,\\
T &= I_N\otimes\Sigma_1 > 0.
\end{aligned}$$

From the definition of $V(t)$, we get

$$\xi_1\big\|x(t)\big\|^2 \le V_1(t) \le \xi_2\big\|x(t)\big\|^2,$$

where $\xi_1 = \min_{i=1,2,\dots,r}\{\lambda_{\min}(P_i)\}$, $\xi_2 = \min_{i=1,2,\dots,r}\{\lambda_{\max}(P_i)\}$, then

$$D_{t_0,t}^{\alpha}V_3(t) \le \frac{-\lambda_{\min}(H)}{\xi_2}V_3(t) + \frac{\lambda_{\max}(T)}{\xi_1}V_3\big(t-\tau(t)\big). \tag{24}$$

Let $\sigma_1 = \frac{\lambda_{\min}(H)}{\xi_2}$, $\sigma_2 = \frac{\lambda_{\max}(T)}{\xi_1}$, then

$$D_{t_0,t}^{\alpha}V_3(t) \le \sigma_1V_3(t) + \sigma_2V_3\big(t-\tau(t)\big).$$

Thus, we can obtain

$$D_{t_0,t}^{\alpha}V_3(t) \le \sigma_1V_3(t) + \sigma_2\sup_{-\tau\le\theta\le 0}V_3(t+\theta). \tag{25}$$

From Lemma 4.1, error system (20) will be globally uniformly asymptotically stable. Thus, the error $e(t)$ will converge to zero asymptotically, which means that (19) can achieve synchronization. □

4.2 Adaptive feedback gains

In the following, we will turn the feedback gains and propose a new synchronization criterion to realize the synchronization of system (19).

Design the following pinning controllers:

$$u_i(t) = \begin{cases} -ck_i\Gamma(x_i(t) - s(t)), & i \in \mathcal{C}, \\ 0, & i \notin \mathcal{C}, \end{cases} \tag{26}$$

where $\mathcal{C} = \{l_1, l_2, \ldots, l_m\}$, and $l_i (i = 1, 2, \ldots, m)$ $(1 \leq m < N)$ are controlled nodes. Then we can get

$$\begin{cases} D^{\alpha}_{t_0,t} x_i(t) = -Ax_i(t) + Bf(x_i(t)) + c\sum_{j=1}^{N} G_{ij}\Gamma x_i(t - \tau(t)) \\ \qquad\qquad + J - c\hat{k}_i(t)\Gamma(x_i(t) - s(t)), \\ x_i(s) = \phi_i(s), \quad -\tau \leq s \leq 0, \end{cases} \tag{27}$$

where $i = 1, 2, \ldots, N$.

From (26) and (27), we have

$$\begin{aligned} D^{\alpha}_{t_0,t} \hat{e}_i(t) = {} & -(I_{p_i} \otimes A)\hat{e}_i(t) + (I_{p_i} \otimes B)\big(\hat{f}_i\big(x(t)\big) - \hat{f}_i\big(S(t)\big)\big) \\ & + c\sum_{j=i+1}^{r} (\tilde{G}_{ij} \otimes \Gamma)\hat{e}_j\big(t - \tau(t)\big) \\ & - c\big(K_i(t) \otimes \Gamma\big)\hat{e}_i(t) + c(\tilde{G}_i \otimes \Gamma)\hat{e}_i\big(t - \tau(t)\big), \end{aligned} \tag{28}$$

where $i = 1, 2, \ldots, m$, the initial value of (28) is given by $\hat{e}_i(s) = \hat{\phi}_i(s)$,

$$\begin{aligned} \hat{\phi}_i(s) &= \big(\phi_1^T(s), \phi_2^T(s), \ldots, \phi_{p_i}(s)\big), \\ K(t) &= \operatorname{diag}\big(K_1(t), K_2(t), \ldots, K_r(t)\big) = \operatorname{diag}\big(\hat{k}_1(t), \hat{k}_2(t), \ldots, \hat{k}_N(t)\big) \\ &= \operatorname{diag}\big(0, \ldots, 0, \underbrace{k_1(t)}_{l_1}, 0, \ldots, 0, \underbrace{k_2(t)}_{l_2}, 0, \ldots, 0, \underbrace{k_m(t)}_{l_m}, \ldots\big), \\ K_i(t) &= \operatorname{diag}\big(K_{i1}(t), K_{i2}(t), \ldots, K_{ip_i}(t)\big) \in \mathbb{R}^{p_i \times p_i}. \end{aligned}$$

Theorem 4.2 *If there exist matrices* $0 < P_i = \operatorname{diag}(P_{i1}, P_{i2}, \ldots, P_{ip_i}) \in \mathbb{R}^{p_i \times p_i}$, $0 < \Sigma_2 \in \mathbb{R}^{n \times n}$, $0 < \Sigma_3 \in \mathbb{R}^{n \times n}$ $k_i^* = \operatorname{diag}(k_{i1}^*, k_{i2}^*, \ldots, k_{ip_i}^*) \geq 0$, $i = 1, 2, \ldots, r$, *satisfying*

$$\begin{aligned} &\mathcal{L} = P \otimes \big(2A - BB^T - \Xi\big)\big) + 2k^* \otimes \Gamma, \\ &\mathcal{M} = -c^2(PG \otimes \Gamma)\big(I_N \otimes \Sigma_2^{-1}\big)\big(G^T P\big) \otimes \Gamma, \\ &\mathcal{I} = (I_N \otimes \Gamma)\Sigma_3^{-1}(I_N \otimes \Gamma), \\ &\mathcal{H} = \mathcal{L} + \mathcal{M} + \mathcal{I} > 0, \end{aligned} \tag{29}$$

$$\lambda_{\max}(\Sigma_2) < \lambda_{\max}(\Sigma_3), \tag{30}$$

where $P = \operatorname{diag}(P_1, P_2, \ldots, P_r)$, $k^* = \operatorname{diag}(k_1^*, k_2^*, \ldots, k_r^*)$, $k_{ij}^* = 0$ $(j = 1, 2, \ldots, p_i)$ *if and only if* $K_{ij}(t) = 0$. *Then the pinning controlled network* (27) *can achieve synchronization under the following adaptive law*:

$$D_{t_0,t}^{\alpha} K(t) = \frac{-\beta_2 k^* e^T(t)e(t) - \beta_2 k^* e^T(t)e(t-\tau(t))}{K(t)} + ce^T(t)e(t),$$

where $\beta_2 > 0$.

Proof Define the following function for system (27):

$$V_4(t) = \sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n)\hat{e}_i(t) + \sum_{i=1}^{r} \frac{\hat{K}_i(t)(I_{p_i} \otimes \Gamma)\hat{K}_i(t)}{\beta_2}. \tag{31}$$

From Lemmas 2.3 and 2.4, we can get

$$\begin{aligned}
D_{t_0,t}^{\alpha} V_4(t) &\leq 2\sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n) D_{t_0,t}^{\alpha} \hat{e}_i(t) + D_{t_0,t}^{\alpha} \sum_{i=1}^{r} \frac{\hat{K}_i(t)(I_{p_i} \otimes \Gamma)\hat{K}_i(t)}{\beta_2} \\
&= 2\sum_{i=1}^{r} \hat{e}_i^T(t)(P_i \otimes I_n)\Bigg(-(I_{p_i} \otimes A)\hat{e}_i(t) + (I_{p_i} \otimes B)\big(\hat{f}_i\big(x(t)\big) \\
&\quad - \hat{f}_i\big(S(t)\big)\big) + c\sum_{i=1}^{r} (\tilde{G}_{ij} \otimes \Gamma)\hat{e}_j\big(t-\tau(t)\big) - c\big(K_i(t) \otimes \Gamma\big)\hat{e}_i(t) \\
&\quad + c(\tilde{G}_i \otimes \Gamma)\hat{e}_i\big(t-\tau(t)\big)\Bigg) - 2\sum_{i=1}^{r} k^* \hat{e}_i^T(t)(P_i \otimes I_n)\hat{e}_i(t) \\
&\quad - 2\sum_{i=1}^{r} k^* \hat{e}_i^T(t)(P_i \otimes I_n)\hat{e}_i\big(t-\tau(t)\big) + 2c\hat{e}_i^T(t)\big(K_i(t) \otimes \Gamma\big)\hat{e}_i(t) \\
&\leq e^T(t)\big(P \otimes \big(-2A + BB^T + \Xi - 2k^*\Gamma\big)\big)e(t) \\
&\quad + 2ce^T(t)(PG \otimes \Gamma)e\big(t-\tau(t)\big) - 2e^T(t)\big(I_N \otimes k^*\Gamma\big)e\big(t-\tau(t)\big) \\
&\leq e^T(t)\big(P \otimes \big(-2A + BB^T + \Xi\big) - 2k^* \otimes \Gamma\big)e(t) \\
&\quad + c^2 e^T(t)(PG \otimes \Gamma)\big(I_N \otimes \Sigma_2^{-1}\big)\big(G^T P\big) \otimes \Gamma\big)e(t) \\
&\quad + e^T\big(t-\tau(t)\big)(I_N \otimes \Sigma_2)e\big(t-\tau(t)\big) - e^T(t)(I_N \otimes \Gamma)\Sigma_3^{-1}(I_N \otimes \Gamma)e(t) \\
&\quad - e^T\big(t-\tau(t)\big)(I_N \otimes \Sigma_3)e\big(t-\tau(t)\big).
\end{aligned}$$

From (29) and (30) we know

$$\begin{aligned}
&\lambda_{\max}(I_N \otimes \Sigma_2) < \lambda_{\max}(I_N \otimes \Sigma_3), \\
&\lambda_{\min}(\mathcal{H}) > 0.
\end{aligned}$$

That is,

$$D_{t_0,t}^{\alpha} V_4(t) \leq -\lambda_{\min}(\mathcal{H}) e^T(t)e(t).$$

Then, similar to the proof of Theorem 3.2, we can conclude that system (27) can achieve synchronization. The proof is completed. □

Remark 4.1 In recent years, the synchronization of coupled fractional-order complex dynamical networks has been regarded as a popular topic in the scientific research because of its wide application in different fields. But very few authors have discussed adjusting the feedback gains and coupling strength. In [30], by the comparison principle, the synchronization of fractional-order complex dynamical networks with delay is realized via adaptive control. In this paper, we mainly use Razumikhin-type stability theory and the matrix inequality technique to realize synchronization.

Remark 4.2 In this paper, we mainly discuss $0 < \alpha < 1$; evidently, it is still true for $\alpha = 1$. However, when $\alpha > 1$, it is not suitable for this paper since Lemmas 2.2 and 2.3 are not solved for $\alpha > 1$. This is worth our deep study.

Remark 4.3 The proposed methods in this paper can be used to study the synchronization of chaotic and hyperchaotic systems or multi-synchronization systems with fractional-order derivative.

5 Numerical examples

Three examples are provided to substantiate the theoretical results.

Example 5.1 Consider the following complex dynamical networks:

$$D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf\bigl(x_i(t)\bigr) + c\sum_{j=1}^{6} G_{ij}\Gamma x_j(t) + J, \tag{32}$$

where $\alpha = 0.98$, $i = 1,2,\ldots,6$, $f_j(\varrho) = \tanh(\varrho)$, $J = (0,0,0)^T$, $A = \operatorname{diag}(0.2,0.2,0.3)$, $\Gamma = \operatorname{diag}(0.5,0.6,0.4)$, $c = 2$, $k_1 = k_2 = 0.2$, $k_3 = k_4 = k_5 = k_6 = 0$.

$$B = \begin{pmatrix} 0.02 & -0.3 & -0.1 \\ -0.2 & 0.1 & -0.1 \\ -0.2 & -0.1 & 0.1 \end{pmatrix},$$

$$G = \begin{pmatrix} -0.1 & 0.3 & 0.1 & 0.3 & 0 & 0 \\ 0.4 & -0.5 & 0.5 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & -0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & 0.4 & 0.2 \\ 0 & 0 & 0 & 0.1 & -0.4 & 0.3 \\ 0 & 0 & 0 & 0.5 & -0.2 & -0.7 \end{pmatrix}.$$

Obviously, $f_j(\cdot)$ $(j = 1,2,3)$ satisfies the Lipschitz condition with $\Pi_j = 0.6$.

We select nodes 1 and 2 as pinned nodes. Take $S(t) = (0,0,0) \in \mathbb{R}^3$.

Case 1 By exploiting the MATLAB LMI Toolbox, we can get the matrices P_1 and P_2 satisfying (7), $P_1 = \operatorname{diag}(1.2872, 0.4371, 1.4351)$, $P_2 = \operatorname{diag}(1.8363, 1.0765, 0.7430)$.

According to Theorem 3.1, system (32) is synchronized. The simulation results are given in Figures 1 and 2.

Case 2 Let $\beta_1 = 0.05$, we can easily find matrices P_1 and P_2 satisfying (13), $P_1 = \operatorname{diag}(1.3145, 0.9764, 1.1610)$, $P_2 = \operatorname{diag}(0.6771, 1.0711, 0.5121)$.

According to Theorem 3.2, system (32) under pinning adaptive law (11) is synchronized. The simulation results are given in Figures 3 and 4.

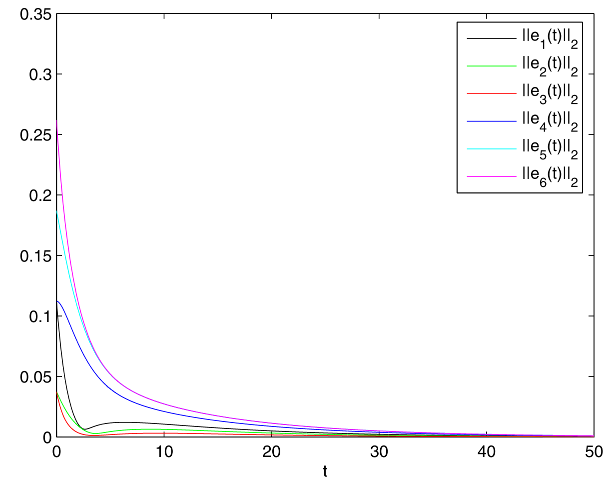

Figure 1 **The synchronized error $\|e_i(t)\|_2$, $i = 1, 2, 3, 4, 5, 6$, of system (32).**

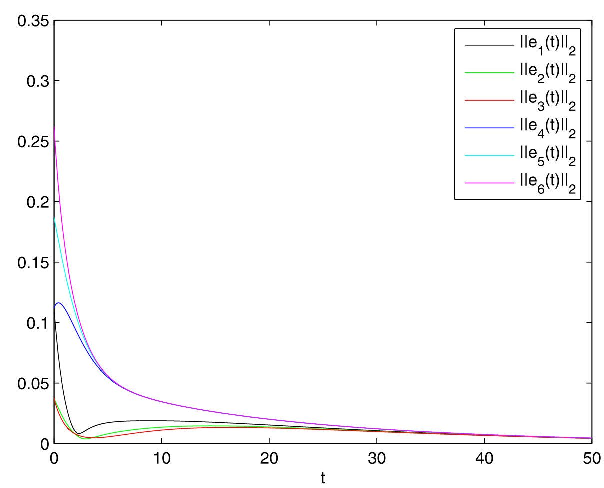

Figure 2 **The synchronized error $\|e_i(t)\|_2$, $i = 1, 2, 3, 4, 5, 6$, of system (32) under pinning control (3).**

Example 5.2 Consider the following complex dynamical network with time-varying delay:

$$D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf\bigl(x_i(t)\bigr) + c\sum_{j=1}^{6} G_{ij}\Gamma x_j\bigl(t - \tau(t)\bigr) + J, \tag{33}$$

where $\alpha = 0.98$, $i = 1, 2, 3, 4, 5, 6$ $\tau(t) = 1$, $f_j(\varrho) = \tanh(\varrho)$, $J = (0, 0, 0)^T$, $A = \operatorname{diag}(0.5, 0.4, 0.4)$, $\Gamma = \operatorname{diag}(0.5, 0.6, 0.5)$, $c = 0.2$,

$$B = \begin{pmatrix} 0.02 & -0.1 & 0.2 \\ -0.1 & 0.1 & -0.2 \\ 0.3 & -0.1 & 0.1 \end{pmatrix},$$

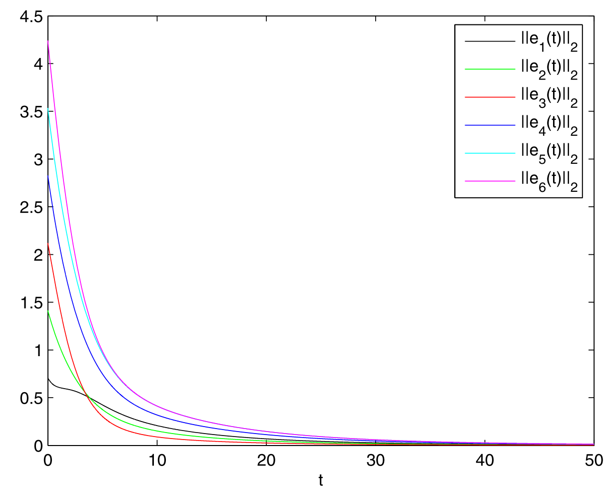

Figure 3 The synchronized error $\|e_i(t)\|_2$, $i = 1, 2, 3, 4, 5, 6$, of system (32) under adaptive control (11).

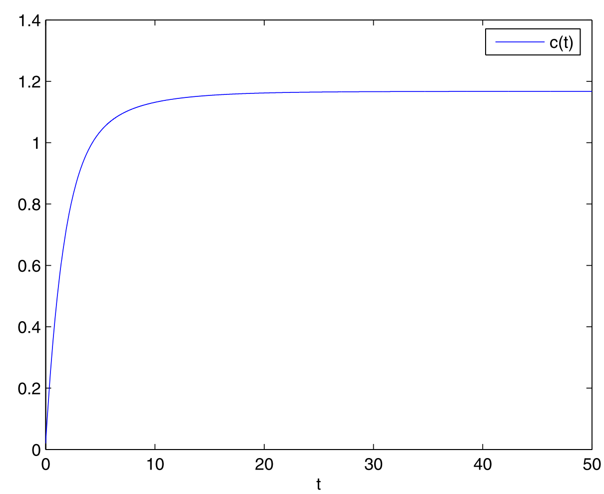

Figure 4 Adaptive coupling strength $c(0) = 0.05$.

$$G = \begin{pmatrix} -0.7 & 0.1 & 0.2 & 0 & 0 & 0 \\ 0.4 & -0.1 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.6 & -0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & -0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & -0.6 & 0.4 & 0 \\ 0 & 0 & 0 & -0.1 & 0.6 & -0.6 \end{pmatrix}.$$

Obviously, $f_j(\cdot)$ $(j = 1, 2, 3)$ satisfies the Lipschitz condition with $\Pi_j = 0.6$.

We choose nodes 1, 2 and 3 as pinned nodes. Take $S(t) = (0, 0, 0) \in \mathbb{R}^3$. $k_1 = 0.2$, $k_2 = 0.2$, $k_3 = 0.2$, $k_4 = k_5 = k_6 = 0$.

Case 1 By exploiting the MATLAB LMI Toolbox, we can get the matrices P_1, P_2 and K satisfying (21), $P_1 = \mathrm{diag}(0.6164, 0.3794, 0.7305)$, $P_2 = \mathrm{diag}(0.5164, 0.7861, 0.5532)$, $K = \mathrm{diag}(0.0567, 0.7613, 0.0387)$.

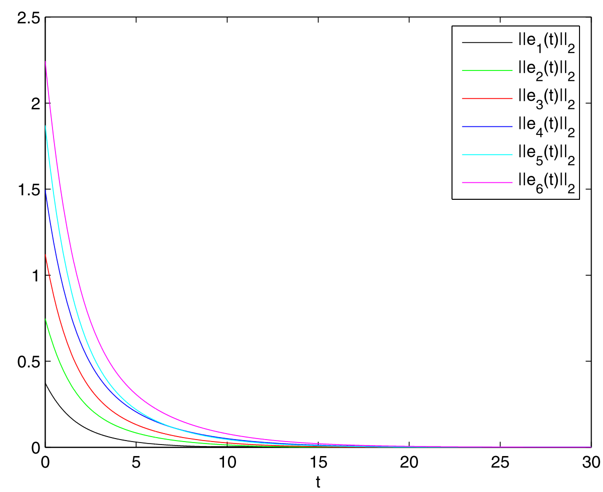

Figure 5 The synchronized error $\|e_i(t)\|_2$, $i = 1, 2, 3, 4, 5, 6$, of system (33) under pinning control (18).

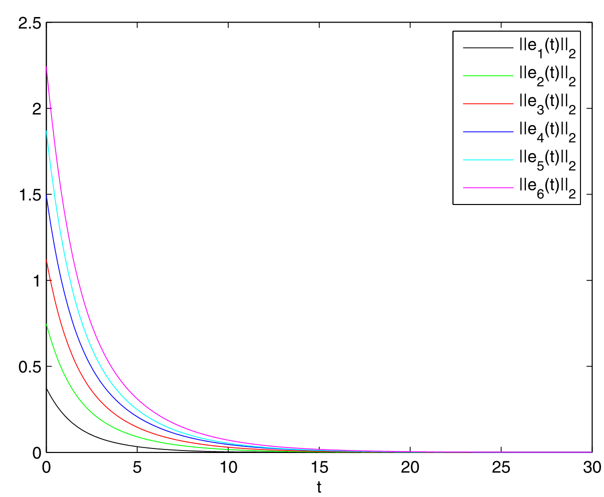

Figure 6 The synchronized error $\|e_i(t)\|_2$, $i = 1, 2, 3, 4, 5, 6$, of system (33) under adaptive control (26).

According to Theorem 4.1, system (33) is synchronized. The simulation results are given in Figure 5.

Case 2 Take $\beta_2 = 0.06$, we can easily get the matrices P_1, P_2 and k^* satisfying (29), $P_1 = \mathrm{diag}(2.6117, 3.5616, 3.5498)$, $P_2 = \mathrm{diag}(2.8325, 2.8051, 3.5430)$, $k^* = \mathrm{diag}(0.1671, 0.5675, 0.0653)$.

By Theorem 4.2, it is obvious that system (33) is synchronized by using pinning feedback controllers. The simulation results are given in Figures 6 and 7.

Example 5.3 Consider complex networks with 10 nodes, the fractional-order dynamical equation of each node is described by the following fractional-order chaotic Lorenz

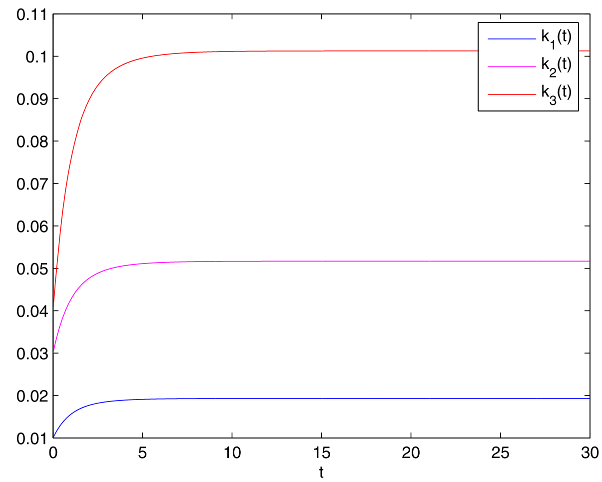

Figure 7 Adaptive feedback gains $k_1(0) = 0.01$, $k_2(0) = 0.03$, $k_3(0) = 0.04$.

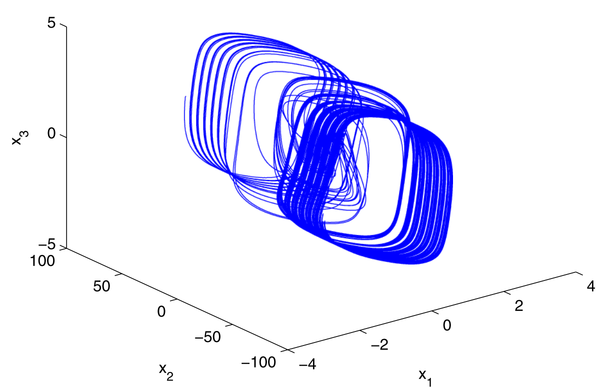

Figure 8 Chaotic attractor of a fractional-order chaotic Lorenz system with order $\alpha = 0.98$.

system:

$$\begin{aligned} D^{\alpha}_{t_0,t}x_{i1}(t) &= \varrho\big(x_{i1}(t) - x_{i2}(t)\big), \\ D^{\alpha}_{t_0,t}x_{i2}(t) &= \omega x_{i1}(t) - x_{i1}(t)x_{i3}(t) - x_{i2}(t), \\ D^{\alpha}_{t_0,t}x_{i3}(t) &= x_{i1}(t)x_{i2}(t) - \nu x_{i3}(t), \end{aligned} \tag{34}$$

where $i = 1, \dots, 10$. When the parameters are chosen as $\varrho = 1$, $\omega = 2.8$, $\nu = \frac{8}{3}$ and $\alpha = 0.98$, system (34) displays a chaotic attractor in Figure 8.

System (34) can be rewritten as system (5) consisting of ten nodes ($N = 10$) with the following parameters:

$$A = \begin{pmatrix} -\varrho & \varrho & 0 \\ \omega & -1 & 0 \\ 0 & 0 & -\nu \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix},$$

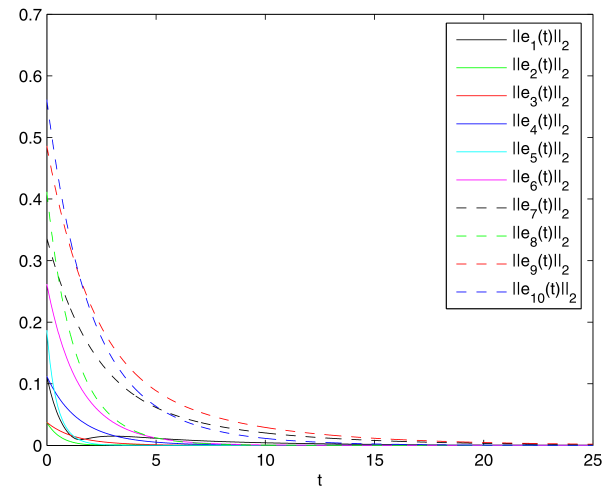

Figure 9 The synchronized error $\|e_i(t)\|_2$, $i = 1, \ldots, 10$, of system (32) under pinning control (3).

$$G = \begin{pmatrix} -0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0.03 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0 & 0 & 0 \\ 0 & 0.09 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.10 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2 \end{pmatrix}.$$

$f(x_i) = (0, -x_{i1}x_{i3}, x_{i1}x_{i2})^T$, $J = (0,0,0)^T$, $A = \mathrm{diag}(0.5, 0.4, 0.4)$, $\Gamma = \mathrm{diag}(0.5, 0.6, 0.5)$, $c = 2$; obviously, $f_j(\cdot)$ ($j = 1,2,3$) satisfies the Lipschitz condition with $\Pi_j = 0.6$. We select nodes 1 and 2 as pinned nodes. Take $S(t) = (0,0,0) \in \mathbb{R}^3$. By exploiting the MATLAB LMI Toolbox, we can get the matrices P_1 and P_2 satisfying (7), $P_1 = \mathrm{diag}(0.2872, 0.4371, 0.4351)$, $P_2 = \mathrm{diag}(2.8063, 1.3935, 0.7430)$. According to Theorem 3.1, system (32) is synchronized. The simulation results are given in Figure 9.

6 Conclusions

In this paper, synchronization of fractional-order complex dynamical networks with and without time-varying delay has been studied by applying pinning adaptive control. First, by using the fractional Lyapunov method and generalized Barbalat's lemma, several sufficient conditions have been derived to realize synchronization of fractional-order complex networks without time-varying delay. Second, by using Razumikhin-type stability theory and fractional integral inequality, some sufficient conditions have been derived to realize synchronization of fractional-order complex networks with time-varying delay. Moreover, several adaptive control strategies to tune the coupling strength and pinning feedback gain have been proposed, and by using the designed adaptive laws, several criteria for synchronization have been established. In the future, it is very interesting to study the multi-synchronization of coupled fractional-order complex dynamical networks with and without time-varying delay.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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