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A new result on the existence of periodic solutions for Rayleigh equation with a singularity

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Abstract

In this paper, we study the existence of periodic solutions for Rayleigh equation with a singularity of repulsive type

$$x''(t) + f(x'(t)) + \varphi(t)x(t) - \frac{1}{x^{\alpha}(t)} = p(t),$$

where $\alpha \geq 1$ is a constant, and φ and p are T-periodic functions. The proof of the main result relies on a known continuation theorem of coincidence degree theory. The interesting point is that the sign of the function $\varphi(t)$ is allowed to change for $t \in [0, T]$.

Keywords: second order differential equation; continuation theorem; singularity; periodic solution

1 Introduction

Singular differential equations arise in many disciplines such as physics, fluid dynamics, and ecology (see [1–6] and the references therein). In recent years, the periodic problem of second-order differential equations with singularities has been widely studied. The first study in this area seems to be the paper of Nagumo [7] in 1944. After some works of Forbat and Huaux [8], the interest increased with the pioneering paper of Lazer and Solimini [9]. They considered the existence of periodic solutions suggested by the two fundamental examples ($\alpha > 0$, and $h : R \to R$ is a continuous T-periodic function)

$$x''(t) + \frac{1}{x^{\alpha}(t)} = h(t) \tag{1.1}$$

(the singularity of attractive type) and

$$x''(t) - \frac{1}{x^{\alpha}(t)} = h(t) \tag{1.2}$$

(the singularity of repulsive type). By using topological degree methods they obtained that a necessary and sufficient condition for the existence of positive periodic solutions for equation (1.1) is $\overline{h} > 0$, and if we assume in addition that $\alpha \geq 1$, then a necessary and sufficient condition for the existence of positive periodic solutions for equation (1.2) is $\overline{h} < 0$. After that, some methods associated with nonlinear functional analysis theory have been widely applied to the studied problem in many papers such as the variational methods used in [10–13], fixed point theorems used in [14–19], upper and lower solutions methods used in [20, 21], and continuation theorems of coincidence degree used in [22–31]. For example, Torres [14] studied the periodic problem for the equation with singularity of repulsive type

$$x'' + \varphi(t)x - \frac{b(t)}{x^{\mu}} = h(t), \tag{1.3}$$

where $\varphi, b, h \in L^1[0,T]$, and $\mu > 0$ is a constant. The function φ is required to satisfy

$$\varphi(t) \geq 0 \quad \text{for all } t \in [0,T]. \tag{1.4}$$

This is due to the fact that (1.4), together with some other conditions, can guarantee the Green function $G(t,s)$ associated with the boundary value problem for Hill's equation

$$x'' + \varphi(t)x = h(t), \qquad x(0) = x(T), \qquad x'(0) = x'(T), \tag{1.5}$$

satisfying $G(t,s) \geq 0$ for all $(t,s) \in [0,T] \times [0,T]$; then, the solution to problem (1.5) is given by

$$x(t) = \int_0^T G(t,s)h(s)\,ds. \tag{1.6}$$

Formula (1.6) is crucial in [14–17] for applying some fixed point theorems on cones. Wang [25] studied the problem of periodic solutions for the singular delay Liénard equation of repulsive type

$$x''(t) + f\bigl(x(t)\bigr)x'(t) + \varphi(t)x(t-\tau) - \frac{1}{x^{\mu}(t-\tau)} = h(t), \tag{1.7}$$

where $f : [0,+\infty) \to R$ is continuous, $\varphi : R \to R$ is continuous T-periodic, and $\tau > 0$ and $\mu \geq 1$ are constants. To balance the forces of $\varphi(t)x$ at $x = +\infty$ and $\frac{1}{x^{\mu}}$ at $x = 0$, φ is also required to satisfy

$$\varphi(t) \geq 0 \quad \text{for all } t \in [0,T]. \tag{1.8}$$

In [26, 28], the authors studied the periodic problem of the equation

$$x'' + f(x)x' + \varphi(t)x - \frac{1}{x^{\mu}} = h(t). \tag{1.9}$$

In (1.9), the function φ is required to satisfy $\int_0^T \varphi(s)\,ds > 0$, which means that the sign of the function φ is allowed to change. Now, the question is that how to investigate the existence of T-periodic solutions for a Rayleigh equation with a singularity of repulsive type

$$x''(t) + f\bigl(x'(t)\bigr) + \varphi(t)x(t) - \frac{1}{x^{\alpha}(t)} = p(t), \tag{1.10}$$

where $f : R \to R$ is continuous with $f(0) = 0$, $\alpha \geq 1$, and $\varphi, p : R \to R$ are continuous and T-periodic.

Motivated by this, the aim of this paper is to search for positive T-periodic solutions for (1.10). Using a known continuation theorem of theorem of coincidence degree theory (see [32, 33], and [34]), we obtain a new result on the existence of positive periodic solutions for equation (1.10). In present paper, the sign of φ in (1.10) is allowed to change for $t \in [0, T]$. Although this condition is the same as that in [26, 28], for studying the periodic problem of (1.9), the methods used in [26, 28] for estimating a priori bounds of positive T-periodic solutions to (1.9) cannot be directly applied to (1.10). This is due to the fact that mechanism of the first-order derivative term $f(x'(t))$ influencing a priori bounds of positive T-periodic solutions to (1.10) is different from the corresponding ones of $f(x(t))x'(t)$ in (1.10). For example, if $x(t)$ is a positive T-periodic function such that $x \in C^1(R, R)$, then $\int_0^T f(x(t))x'(t)\,dt = 0$, but, generally, $\int_0^T f(x'(t))\,dt \neq 0$.

2 Preliminary lemmas

Let $C_T = \{x \in C(R, R) : x(t + T) = x(t), \forall t \in R\}$ with the norm $|x|_\infty = \max_{t\in[0,T]} |x(t)|$, and let $C_T^1 = \{x' \in C^1(R, R) : x'(t + T) = x'(t), \forall t \in R\}$ with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, C_T and C_T^1 are both Banach spaces. For any T-periodic solution $\varphi(t)$ with $\varphi \in C_T$, by $\varphi_+(t)$ and $\varphi_-(t)$ we denote $\max\{\varphi(t), 0\}$ and $-\min\{\varphi(t), 0\}$, respectively, and $\overline{\varphi} = \frac{1}{T}\int_0^T \varphi(s)\,ds$. Then $\varphi(t) = \varphi_+(t) - \varphi_-(t)$ for all $t \in R$, and $\overline{\varphi} = \overline{\varphi_+} - \overline{\varphi_-}$. Furthermore, for each $u \in C_T$, let $\|u\|_p := (\int_0^T |u(s)|^p\,ds)^{1/p}$, $p \in [1, +\infty)$.

The following result can be easily obtained by using Theorem 4 in [32], Chapter 6 of [33], and Theorem 3.1 in [34].

Lemma 2.1 *Assume that there exist positive constants N_0, N_1, and N_2 with $0 < N_0 < N_1$ such that the following conditions hold.*

1. *For each $\lambda \in (0, 1]$, each possible positive T-periodic solution x to the equation*

$$u'' + \lambda f(u') + \lambda\varphi(t)u - \frac{\lambda}{u^\alpha} = \lambda p(t)$$

   satisfies the inequalities $N_0 < x(t) < N_1$ and $|x'(t)| < N_2$ for all $t \in [0, T]$.

2. *Each possible solution c to the equation*

$$\frac{1}{c^\alpha} - c\overline{\varphi} + \overline{p} = 0$$

   satisfies the inequality $N_0 < c < N_1$.

3. *The inequality*

$$\left(\frac{1}{N_0^\alpha} - N_0\overline{\varphi} + \overline{p}\right)\left(\frac{1}{N_1^\alpha} - N_1\overline{\varphi} + \overline{p}\right) < 0$$

   holds.

Then equation (1.10) *has at least one positive T-periodic solution u such that $N_0 < u(t) < N_1$ for all $t \in [0, T]$.*

Now, we list the following assumptions, which will be used in Section 3 for investigating the existence of positive T-periodic solutions to (1.10).

$[H_1]$ There exist constants $L > 0$, $\sigma > 0$, and $n \geq 1$ such that

$$\left|\int_0^T f\bigl(x'(t)\bigr)\,dt\right| \leq L\int_0^T \bigl|x'(t)\bigr|\,dt, \quad \forall x \in C_T^1 \tag{2.1}$$

and

$$yf(y) \geq \sigma|y|^{n+1}, \quad \forall y \in R. \tag{2.2}$$

$[H_2]$ The function φ satisfies $\overline{\varphi_+} > \overline{\varphi_-}$;
$[H_3]$ $\|\varphi\|_2 < \sigma T^{-\frac{1}{2}}$ and $(LT^{-\frac{1}{2}} + T^{\frac{1}{2}}\overline{\varphi_+})\|\varphi\|_2 < \sigma(\overline{\varphi_+} - \overline{\varphi_-})$.

Remark 2.1 If assumption $[H_2]$ holds, then there are constants D_1 and D_2 with $0 < D_1 < D_2$ such that

$$\frac{1}{x^\alpha} - \overline{\varphi}x + \overline{p} > 0 \quad \text{for all } x \in (0, D_1)$$

and

$$\frac{1}{x^\alpha} - \overline{\varphi}x + \overline{p} < 0 \quad \text{for all } x \in (D_2, \infty).$$

Now, we embed equation (1.10) into the following equations family with parameter $\lambda \in (0,1]$:

$$x'' + \lambda f\bigl(x'\bigr) + \lambda\varphi(t)x - \frac{\lambda}{x^\alpha} = \lambda p(t), \quad \lambda \in (0,1]. \tag{2.3}$$

Let

$$\Omega = \left\{x \in C_T : x'' + \lambda f\bigl(x'\bigr) + \lambda\varphi(t)x - \frac{\lambda}{x^\alpha} = \lambda p(t), \lambda \in (0,1]; x(t) > 0, \forall t \in [0,T]\right\}, \tag{2.4}$$

and let

$$M_0 = \max\left\{1, \frac{LT^{\frac{-1}{n+1}} + T^{\frac{n}{n+1}}\overline{\varphi_+}}{\overline{\varphi_+} - \overline{\varphi_-}}B + \frac{1 + \overline{p}}{\overline{\varphi_+} - \overline{\varphi_-}}\right\}, \tag{2.5}$$

where B will be determined by (2.13). Clearly, M_0 is independent of $(\lambda, x) \in (0,1] \times \Omega$.

Lemma 2.2 *Assume that assumptions* $[H_1]$-$[H_3]$ *hold. Then for each function* $x \in \Omega$, *there exists a point* $t_0 \in [0,T]$ *such that*

$$x(t_0) \leq M_0,$$

where M_0 *is defined by* (2.5)

Proof If the conclusion does not hold, then there is a function $x_0 \in \Omega$ satisfying

$$x_0(t) > M_0 \quad \text{for all } t \in [0,T]. \tag{2.6}$$

From (2.4) we get

$$x_0'' + \lambda f\big(x_0'\big) + \lambda\varphi(t)x_0 - \frac{\lambda}{x_0^\alpha} = \lambda p(t). \tag{2.7}$$

Integrating (2.7) over the interval $[0, T]$, we get

$$\begin{aligned}&\int_0^T f\big(x_0'(t)\big)\,dt + \int_0^T \varphi_+(t)x_0(t)\,dt\\ &\quad= \int_0^T \varphi_-(t)x_0(t)\,dt + \int_0^T \frac{1}{x_0^\alpha(t)}\,dt + \int_0^T p(t)\,dt,\end{aligned}$$

that is,

$$\begin{aligned}&\int_0^T \varphi_+(t)x_0(t)\,dt\\ &\quad= -\int_0^T f\big(x_0'(t)\big)\,dt + \int_0^T \varphi_-(t)x_0(t)\,dt + \int_0^T \frac{1}{x_0^\alpha(t)}\,dt + \int_0^T p(t)\,dt.\end{aligned}$$

Since $\varphi_+(t) \geq 0$ and $\varphi_-(t) \geq 0$ for all $t \in [0, T]$, it follows from the integral mean value theorem and condition (2.1) in $[H_1]$ that there are two points $\xi, \zeta \in [0, T]$ such that

$$x_0(\xi)T\overline{\varphi_+} \leq L\int_0^T \big|x_0'(t)\big|\,dt + x_0(\zeta)T\overline{\varphi_-} + M_0^{-\alpha}T + T\overline{p},$$

which, together with the fact of $M_0 \geq 1$ in (2.5), yields

$$x_0(\xi)T\overline{\varphi_+} \leq L\int_0^T \big|x_0'(t)\big|\,dt + |x_0|_\infty T\overline{\varphi_-} + T + T\overline{p},$$

that is,

$$x_0(\xi) \leq \frac{LT^{\frac{-1}{n+1}}}{\overline{\varphi_+}}\left(\int_0^T \big|x_0'(t)\big|^{n+1}\,dt\right)^{\frac{1}{n+1}} + \frac{\overline{\varphi_-}}{\overline{\varphi_+}}|x_0|_\infty + \frac{1+\overline{p}}{\overline{\varphi_+}}. \tag{2.8}$$

Since

$$|x_0|_\infty \leq x_0(\xi) + T^{\frac{n}{n+1}}\left(\int_0^T \big|x_0'(s)\big|^{n+1}\,ds\right)^{\frac{1}{n+1}}, \tag{2.9}$$

it follows from (2.8), (2.9), and $[H_2]$ that

$$|x_0|_\infty \leq \frac{LT^{\frac{-1}{n+1}} + T^{\frac{n}{n+1}}\overline{\varphi_+}}{\overline{\varphi_+} - \overline{\varphi_-}}\left(\int_0^T \big|x_0'(s)\big|^{n+1}\,ds\right)^{\frac{1}{n+1}} + \frac{1+\overline{p}}{\overline{\varphi_+} - \overline{\varphi_-}}. \tag{2.10}$$

On the other hand, multiplying both sides of (2.7) by $x_0'(t)$ and integrating it over the interval $[0, T]$, we get

$$\lambda\int_0^T f\big(x_0'(t)\big)x_0'(t)\,dt = -\lambda\int_0^T \varphi(t)x_0(t)x_0'(t)\,dt + \lambda\int_0^T p(t)x_0'(t)\,dt.$$

From condition (2.2) in $[H_1]$ we have

$$\begin{aligned}
&\sigma \int_0^T \left|x_0'(t)\right|^{n+1} dt \\
&\leq -\int_0^T \varphi(t)x_0(t)x_0'(t)\,dt + \int_0^T p(t)x_0'(t)\,dt \\
&\leq |x_0|_\infty \int_0^T \left|\varphi(t)\right|\left|x_0'(t)\right| dt + \int_0^T \left|p(t)\right|\left|x_0'(t)\right| dt \\
&\leq |x_0|_\infty \left(\int_0^T \left|x_0'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}} \left(\int_0^T |\varphi|^{\frac{n+1}{n}} dt\right)^{\frac{n}{n+1}} \\
&\quad + \left(\int_0^T \left|x_0'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}} \left(\int_0^T \left|p(t)\right|^{\frac{n+1}{n}} dt\right)^{\frac{n}{n+1}},
\end{aligned}$$

that is,

$$\begin{aligned}
\int_0^T \left|x_0'(t)\right|^{n+1} dt &\leq \sigma^{-1}|x_0|_\infty \|\varphi\|_{\frac{n+1}{n}} \left(\int_0^T \left|x_0'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}} \\
&\quad + \sigma^{-1}\|p\|_{\frac{n+1}{n}} \left(\int_0^T \left|x_0'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}}. \qquad (2.11)
\end{aligned}$$

We infer from (2.10) and (2.11) that

$$\begin{aligned}
&\int_0^T \left|x_0'(t)\right|^{n+1} dt \\
&\leq \frac{LT^{\frac{-1}{n+1}} + T^{\frac{n}{n+1}}\overline{\varphi_+}}{\sigma(\overline{\varphi_+} - \overline{\varphi_-})} \|\varphi\|_{\frac{n+1}{n}} \left(\int_0^T \left|x_0'(t)\right|^{n+1} dt\right)^{\frac{2}{n+1}} \\
&\quad + \sigma^{-1}\left(\frac{1+\overline{p}}{\overline{\varphi_+} - \overline{\varphi_-}} \|\varphi\|_{\frac{n+1}{n}} + \|p\|_{\frac{n+1}{n}}\right) \left(\int_0^T \left|x_0'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}}. \qquad (2.12)
\end{aligned}$$

According to (2.12), we list two cases.

Case 1: If $n > 1$, then we see that there exists $B_0 > 0$ such that $(\int_0^T |x_0'(t)|^{n+1}\,dt)^{\frac{1}{n+1}} \leq B_0$;
Case 2: If $n = 1$, then by assumption $[H_3]$ there exists $B_1 > 0$ such that $(\int_0^T |x_0'(t)|^2\,dt)^{\frac{1}{2}} \leq B_1$.

Letting $B = \max\{B_0, B_1\}$, it follows from Case 1 or Case 2 that

$$\left(\int_0^T \left|x_0'(t)\right|^{n+1} dt\right)^{\frac{1}{n+1}} \leq B. \qquad (2.13)$$

Substituting (2.13) into (2.10), we have

$$|x_0|_\infty \leq \frac{LT^{\frac{-1}{n+1}} + T^{\frac{n}{n+1}}\overline{\varphi_+}}{\overline{\varphi_+} - \overline{\varphi_-}} B + \frac{1+\overline{p}}{\overline{\varphi_+} - \overline{\varphi_+}}.$$

By the definition of M_0 in (2.5) we have

$$|x_0|_\infty \leq M_0,$$

that is,

$$x_0(t) \le M_0 \quad \text{for all } t \in [0, T],$$

which contradicts (2.6). This contradiction proves Lemma 2.2. □

Lemma 2.3 *Assume that* $[H_2]$ *holds. Then there exists a positive constant* $\gamma > 0$ *such that, for each* $x \in \Omega$, *there is a point* $t_1 \in [0, T]$ *satisfying*

$$x(t_1) \ge \gamma.$$

Proof Let $x(t_1) = \max_{t\in[0,T]} x(t)$. Then $x''(t_1) \le 0$ and $x'(t_1) = 0$, which, together with (2.3), yields

$$\lambda f(0) + \lambda\varphi(t_1)x(t_1) - \frac{\lambda}{x^\alpha(t_1)} \ge \lambda p(t_1).$$

Since $f(0) = 0$, we have

$$x(t_1) \max_{t\in[0,T]} \varphi(t) - \frac{1}{x^\alpha(t_1)} \ge p(t_1) \ge -|p|_\infty. \tag{2.14}$$

Multiplying both sides of (2.14) by $x^\alpha(t_1)$, we get

$$x^{\alpha+1}(t_1) \max_{t\in[0,T]} \varphi(t) + x^\alpha(t_1)|p|_\infty - 1 \ge 0. \tag{2.15}$$

Set $S(u) = u^{\alpha+1} \max \varphi(t) + u^\alpha |p|_\infty - 1$ for $u \in [0, +\infty)$. By assumption $[H_2]$ we have

$$S(0) = -1 < 0,$$
$$\lim_{u\to+\infty} S(u) = +\infty.$$

So $S(u)$ has zero points on $(0, +\infty)$. Let γ be the minimum zero point of $S(u)$ on $(0, +\infty)$. Then $S(\gamma) = 0$. It follows from (2.15) that

$$x(t_1) \ge \gamma.$$

The proof is complete. □

3 Main result

Theorem 3.1 *Assume that* $[H_1]$-$[H_3]$ *hold. Then equation* (1.10) *has at least one positive* T*-periodic solution.*

Proof Firstly, we will show that there exist $N_1 > 0$ and $N_2 > 0$ such that each positive T-periodic solution $x(t)$ of equation (2.3) satisfying

$$x(t) < N_1 \quad \text{and} \quad \left|x'(t)\right| < N_2 \quad \text{for all } t \in [0, T]. \tag{3.1}$$

Suppose that x is an arbitrary positive T-periodic solution of equation (2.3). Then

$$x'' + \lambda f(x') + \lambda\varphi(t)x - \frac{\lambda}{x^\alpha} = \lambda p(t), \quad \lambda \in (0,1]. \tag{3.2}$$

This implies that $x \in \Omega$. So by Lemma 2.2 there exists a point $t_0 \in [0,T]$ such that

$$x(t_0) \le M_0,$$

and then

$$|x|_\infty \le M_0 + T^{\frac{n}{n+1}} \left(\int_0^T |x'(s)|^{n+1}\, ds \right)^{\frac{1}{n+1}}. \tag{3.3}$$

Integrating (3.2) over the interval $[0,T]$, we get

$$\int_0^T f(x'(t))\, dt + \int_0^T \varphi(t)x(t)\, dt - \int_0^T \frac{1}{x^\alpha(t)}\, dt = \int_0^T p(t)\, dt. \tag{3.4}$$

On the other hand, similarly to the proof of (2.11), we have

$$\begin{aligned} \int_0^T |x'(t)|^{n+1}\, dt &\le \sigma^{-1}|x|_\infty \|\varphi\|_{\frac{n+1}{n}} \left(\int_0^T |x'(t)|^{n+1}\, dt \right)^{\frac{1}{n+1}} \\ &\quad + \sigma^{-1}\|p\|_{\frac{n+1}{n}} \left(\int_0^T |x'(t)|^{n+1}\, dt \right)^{\frac{1}{n+1}}. \end{aligned} \tag{3.5}$$

Substituting (3.3) into (3.5), we have

$$\begin{aligned} &\int_0^T |x'(t)|^{n+1}\, dt \\ &\quad \le \sigma^{-1}\|\varphi\|_{\frac{n+1}{n}} T^{\frac{n}{n+1}} \left(\int_0^T |x'(t)|^{n+1}\, dt \right)^{\frac{2}{n+1}} \\ &\qquad + \left(\sigma^{-1}\|\varphi\|_{\frac{n+1}{n}} M_0 + \sigma^{-1}\|p\|_{\frac{n+1}{n}}\right) \left(\int_0^T |x'(t)|^{n+1}\, dt \right)^{\frac{1}{n+1}}. \end{aligned} \tag{3.6}$$

According to (3.6), we list two cases.

Case 1: If $n > 1$, then there exists $\rho_0 > 0$ such that $(\int_0^T |x'(t)|^{n+1}\, dt)^{\frac{1}{n+1}} \le \rho_0$;
Case 2: If $n = 1$, then by assumption $[H_3]$ there exists $\rho_1 > 0$ such that $(\int_0^T |x'(t)|^2\, dt)^{\frac{1}{2}} \le \rho_1$.

Letting $\rho = \max\{\rho_0, \rho_1\}$, it follows from Case 1 or Case 2 that

$$\left(\int_0^T |x'(t)|^{n+1}\, dt \right)^{\frac{1}{n+1}} \le \rho, \tag{3.7}$$

and according to (3.3), we have

$$x(t) \le M_0 + T^{\frac{n}{n+1}}\rho := N_1 \quad \text{for all } t \in [0,T]. \tag{3.8}$$

Clearly, there is a point $t_2 \in [0, T]$ such that $x'(t_2) = 0$. Multiplying both sides of (3.2) by $x'(t)$ and integrating it over the interval $[t_2, t]$, we get

$$\begin{aligned}
&\int_{t_2}^{t} x''(t)x'(t)\,dt \\
&\quad = \lambda \int_{t_2}^{t} \left[-f\bigl(x'(t)\bigr)x'(t) - \varphi(t)x(t)x'(t) + \frac{x'(t)}{x^{\alpha}(t)} + p(t)x'(t) \right] dt \\
&\qquad \text{for all } t \in [t_2, t_2 + T],
\end{aligned}$$

and then

$$\begin{aligned}
\frac{|x'(t)|^2}{2} &\le \lambda \bigl|x'\bigr|_\infty \left[|x|_\infty \int_{t_2}^{t_2+T} \bigl|\varphi(t)\bigr|\,dt + \int_{t_2}^{t_2+T} \frac{1}{x^{\alpha}(t)}\,dt + \int_{t_2}^{t_2+T} \bigl|p(t)\bigr|\,dt \right] \\
&= \lambda \bigl|x'\bigr|_\infty \left[|x|_\infty \int_0^T \bigl|\varphi(t)\bigr|\,dt + \int_0^T \frac{1}{x^{\alpha}(t)}\,dt + \int_0^T \bigl|p(t)\bigr|\,dt \right] \\
&= \lambda \bigl|x'\bigr|_\infty \left[N_1 T\overline{|\varphi|} + \int_0^T \frac{1}{x^{\alpha}(t)}\,dt + T\overline{|p|} \right] \quad \text{for all } t \in [t_2, t_2 + T]. \qquad (3.9)
\end{aligned}$$

Since

$$\bigl|x'\bigr|_\infty = \max_{t\in[0,T]} \bigl|x'(t)\bigr| = \max_{t\in[t_2,t_2+T]} \bigl|x'(t)\bigr|,$$

it follows from (3.9) that

$$\frac{|x'|_\infty^2}{2} \le \lambda \bigl|x'\bigr|_\infty \left[N_1 T\overline{|\varphi|} + \int_0^T \frac{1}{x^{\alpha}(t)}\,dt + T\overline{|p|} \right],$$

that is,

$$\frac{|x'|_\infty}{2} \le \lambda \left[N_1 T\overline{|\varphi|} + \int_0^T \frac{1}{x^{\alpha}(t)}\,dt + T\overline{|p|} \right],$$

which implies that

$$\frac{|x'(t)|}{2} \le \frac{|x'|_\infty}{2} \le \lambda \left[N_1 T\overline{|\varphi|} + \int_0^T \frac{1}{x^{\alpha}(t)}\,dt + T\overline{|p|} \right] \quad \text{for all } t \in [0, T]. \qquad (3.10)$$

On the other hand, from (3.4) and condition (2.1) in $[H_1]$ we have

$$\begin{aligned}
\int_0^T \frac{1}{x^{\alpha}(t)}\,dt &= \int_0^T f\bigl(x'(t)\bigr)\,dt + \int_0^T \varphi(t)x(t)\,dt - \int_0^T p(t)\,dt \\
&\le L \int_0^T \bigl|x'(t)\bigr|\,dt + N_1 T\overline{|\varphi|} + T\overline{|p|} \\
&\le L\rho T^{\frac{n}{n+1}} + N_1 T\overline{|\varphi|} + T\overline{|p|},
\end{aligned}$$

where ρ is determined in (3.7). Substituting this formula into (3.10), we obtain

$$\bigl|x'(t)\bigr| \le \lambda \bigl[2L\rho T^{\frac{n}{n+1}} + 4N_1 T\overline{|\varphi|} + 4T\overline{|p|} \bigr] := \lambda N_2 \quad \text{for all } t \in [0, T]. \qquad (3.11)$$

So we have

$$\left|x'(t)\right| \le N_2 \quad \text{for all } t \in [0,T]. \tag{3.12}$$

We further show that there exists a constant $\gamma_0 \in (0,\gamma)$ such that each positive T = periodic solution of (2.3) satisfies

$$x(t) > \gamma_0 \quad \text{for all } t \in [0,T]. \tag{3.13}$$

In fact, suppose that $x(t)$ is an arbitrary positive T-periodic solution of (2.3). Then

$$x'' + \lambda f\left(x'\right) + \lambda\varphi(t)x - \frac{\lambda}{x^\alpha} = \lambda p(t), \quad \lambda \in (0,1]. \tag{3.14}$$

By Lemma 2.3 we see that there is a point $t_1 \in [0,T]$ such that

$$x(t_1) \ge \gamma.$$

For $t \in [t_1, t_1 + T]$, multiplying both sides of (3.14) with $x'(t)$ and integrating it over the interval $[t_1, t]$ (or $[t, t_1]$), we get

$$\frac{|x'(t)|^2}{2} - \frac{|x'(t_1)|^2}{2} + \lambda\int_{t_1}^t f\left(x'\right)x'\,dt = \lambda\int_{t_1}^t \frac{1}{x^\alpha}x'\,dt - \lambda\int_{t_1}^t \varphi(t)xx'\,dt + \lambda\int_{t_1}^t p(t)x'\,dt,$$

which results in

$$\begin{aligned}\lambda&\int_{x(t_1)}^{x(t)} \frac{1}{s^\alpha}\,ds \\ &= \frac{|x'(t)|^2}{2} - \frac{|x'(t_1)|^2}{2} + \lambda\int_{t_1}^t f\left(x'(s)\right)x'(s)\,ds + \lambda\int_{t_1}^t \varphi(s)x(s)x'(s)\,ds - \lambda\int_{t_1}^t p(s)x'(s)\,ds,\end{aligned}$$

that is,

$$\begin{aligned}\lambda\int_{x(t)}^{x(t_1)} \frac{1}{s^\alpha}\,ds = {}& -\frac{|x'(t)|^2}{2} + \frac{|x'(t_1)|^2}{2} - \lambda\int_{t_1}^t f\left(x'(s)\right)x'(s)\,ds \\ & - \lambda\int_{t_1}^t \varphi(s)x(s)x'(s)\,ds + \lambda\int_{t_1}^t p(s)x'(s)\,ds.\end{aligned}$$

According to (2.2) in $[H_1]$, we get $\int_{t_1}^t f(x'(s))x'(s)\,ds \ge 0$. Thus, it follows from the last formula that

$$\begin{aligned}\lambda\int_{x(t)}^{x(t_1)} \frac{1}{s^\alpha}\,ds &\le -\frac{|x'(t)|^2}{2} + \frac{|x'(t_1)|^2}{2} - \lambda\int_{t_1}^t \varphi(s)x(s)x'(s)\,ds + \lambda\int_{t_1}^t p(s)x'(s)\,ds \\ &\le \left|x'\right|_\infty^2 + \lambda\int_0^T \left|\varphi(s)x(s)x'(s)\right|\,ds + \lambda\int_0^T \left|p(s)x'(s)\right|\,ds,\end{aligned}$$

which, together with (3.8) and (3.11), yields

$$\lambda\int_{x(t)}^{x(t_1)} \frac{1}{s^\alpha}\,ds \le \lambda^2 N_2^2 + \lambda^2 N_1 N_2 T\overline{|\varphi|} + \lambda^2 N_2 T\overline{|p|},$$

that is,

$$\int_{x(t)}^{x(t_1)} \frac{1}{s^\alpha}\,ds \le N_2^2 + N_1N_2T\overline{|\varphi|} + N_2T\overline{|p|} := N_3. \tag{3.15}$$

Since $\alpha \ge 1$, it follows that there exists $\gamma_0 \in (0,\gamma)$ such that

$$\int_\eta^\gamma \frac{1}{x^\alpha(t)}\,dt > N_3 \quad \text{for all } \eta \in (0,\gamma_0),$$

which, together with (3.15), implies that

$$x(t) > \gamma_0 \quad \text{for all } t \in [0,T].$$

So (3.13) holds.

Let $n_0 = \min\{D_1,\gamma_0\}$ and $n_1 \in (N_1 + D_2, +\infty)$ be two constants. Then from (3.8), (3.12), and (3.13) we see that each possible positive T-periodic solution x to (2.3) satisfies

$$n_0 < x(t) < n_1, \qquad \left|x'(t)\right| < N_2.$$

This implies that condition 1 and condition 2 of Lemma 2.1 hold. In addition, from Remark 2.1 we can infer that

$$\frac{1}{c^\alpha} - c\overline{\varphi} + \overline{p} > 0 \quad \text{for } c \in (0,n_0]$$

and

$$\frac{1}{c^\alpha} - c\overline{\varphi} + \overline{p} < 0 \quad \text{for } c \in [n_1,+\infty),$$

which results in

$$\left(\frac{1}{n_0^\alpha} - n_0\overline{\varphi} + \overline{p}\right)\left(\frac{1}{n_1^\alpha} - n_1\overline{\varphi} + \overline{p}\right) < 0.$$

Therefore, condition 3 of Lemma 2.1 holds. Thus, by Lemma 2.1 we see that equation (1.10) has at least one positive T-periodic solution. The proof is complete. □

Example 3.1 Consider the equation

$$x''(t) + 10x'(t) - \frac{(x'(t))^3}{1 + (x'(t))^2} + a(1 + 2\sin t)x(t) - \frac{1}{x^2(t)} = \cos t, \tag{3.16}$$

where $a \in (0,\infty)$. Corresponding to (1.10), we see that $f(x) = 10x - \frac{x^3}{1+x^2}$, $\varphi(t) = a(1+2\sin t)$, $p(t) = \cos t$, and $T = 2\pi$.

Firstly, from (3.16) we see that $f(0) = 0$ and

$$\overline{\varphi_+} = \frac{1}{T}\int_0^T \varphi_+(t)\,dt = \frac{\frac{2\pi}{3} + \sqrt{3}}{\pi}a, \qquad \overline{\varphi_-} = \frac{1}{T}\varphi_-(t)\,dt = \frac{-\frac{\pi}{3} + \sqrt{3}}{\pi}a.$$

Obviously, $[H_2]$ is satisfied. Secondly, integrating $f(x')$ over the internal $[0, T]$, we get

$$\begin{aligned}
\left|\int_0^T f(x')\,dt\right| &= \left|\int_0^T \left[10x'(t) - \frac{(x'(t))^3}{1+(x'(t))^2}\right] dt\right| \\
&= \left|-\int_0^T \frac{(x'(t))^3}{1+(x'(t))^2}\,dt\right| \\
&= \left|\int_0^T \frac{|x'(t)|^3}{1+(x'(t))^2}\,dt\right| \\
&\le \int_0^T \left|x'(t)\right| dt,
\end{aligned}$$

which implies that we can chose $L = 1$ such that assumption $[H_1]$ holds. Besides, from

$$yf(y) = 10y^2 - \frac{y^4}{1+y^2} \ge 9y^2$$

we see that the constant σ can be chosen as $\sigma = 9$ such that assumption $[H_1]$ is satisfied. Last, let $L = 1$, $\sigma = 9$, $n = 1$. Then we get

$$\begin{aligned}
&1 - \frac{LT^{\frac{-1}{2}} + T^{\frac{1}{2}}\overline{\varphi_+}}{\sigma(\overline{\varphi_+} - \overline{\varphi_-})}\|\varphi\|_2 = 1 - \frac{\sqrt{3}}{9} - \frac{18+4\sqrt{3}\pi}{27}a > 0, \\
&1 - \sigma^{-1}\|\varphi\|_2 T^{\frac{1}{2}} = 1 - \frac{2\pi}{3\sqrt{3}}a > 0.
\end{aligned}$$

If

$$a < \frac{27 - 3\sqrt{3}}{18 + 4\sqrt{3}\pi},$$

then $[H_3]$ holds. Thus, by Theorem 3.1 we have that equation (3.16) has at least one positive 2π-periodic solution.

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