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Variable structure control for a singular biological economic model with time delay and stage structure

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Abstract

A singular biological economic model which considers a prey-predator system with time delay and stage structure is proposed in this paper. The local stability at the equilibrium point and the dynamic behavior of the model are studied. Local stability analysis of the model without time delay reveals that there is a phenomenon of singularity-induced bifurcation due to the economic equilibrium. Furthermore, the phenomenon of Hopf bifurcation of the model at the boundary equilibrium point occurs as the time delay satisfies certain conditions. In order to apply variable structure control to eliminate the complex behaviors caused by singularity-induced bifurcation, the singular model is transformed into a single-input and single-output model with parameter varying within definite intervals. Then variable structure control with sliding mode based on a power reaching law is designed to stabilize the model. Numerical simulations are given to verify the effectiveness of the conclusions.

Keywords: predator-prey; singular biological economic model; stage structure; time delay; variable structure control

1 Introduction

The stage structure of the biological population is simply the whole life course of the biological population consisting of some non-overlapping stages [1]. Individuals belonging to the same stage have a wide range of ecological similarities, and individuals belonging to different stages are quite different. When the biological dynamical system is considered, it is commonly assumed that juvenile, adult, and aging populations have the same viability and economic benefits, which may ignore the differences of growth ability at different stages of its growth process. The different physiological characteristics of each life stage affect the persistence and extinction of the biological population to varying degrees [2]. Therefore, it is more practical to consider the population model with stage structure when establishing the mathematical model.

A singular system is also known as differential-algebraic system. Compared with the ordinary differential models, a singular system exhibits more complicated dynamics such as the impulse phenomenon [1]. Recently, many scholars proposed several kinds of biological dynamic systems [1, 3–6] by utilizing the theory of singular system, which studies the dynamic system with capture factors. According to the theory of T-S fuzzy descriptor system control, the control of a class of singular bioeconomic models with stage structure was studied in [5, 7, 8] by constructing a T-S fuzzy descriptor model. [4] studied a single-species fish population logistic model with the invasion of alien species based on the theory of singular system. Whereafter, sufficient conditions for existence of the transcritical bifurcation and the singularity-induced bifurcation were obtained. Then the state feedback control was designed to eliminate the unexpected singularity-induced bifurcation. The bifurcation analysis and control of a class of singularity biological economic models with stage structure were researched in [6]. These ideas are based on the economic theory [9]:

Net Economic Profit = Total Revenue – Total Cost.

The study on dynamics of predator-prey models with time delay has received great attention in recent years, and some complex dynamic behaviors, such as the stability equilibrium point and the Hopf bifurcation, were studied in [10–13]. Whereafter, the dynamic behaviors of delayed predator-prey models with harvesting were discussed in [14, 15], which showed that the equilibrium point switches from stable to unstable and then back to stable as the delay increases.

In this paper, a singular predator-prey bioeconomic model with time delay and stage structure is established and studied. The singular model is often strongly nonlinear and unstable. The complex dynamic behavior, such as the singularity-induced bifurcation, is often exhibited in the singular model. In this case, it is necessary to find an effective method to eliminate the singularity-induced bifurcation.

Variable structure control is often used to deal with some models with internal varying parameters and external disturbances. Furthermore, the system response depends on the gradients of the sliding surface and remains insensitive to parameter variations and external disturbances. Sliding mode variable structure control was first proposed by Emelyanov [16] and elaborated in the 1970s. In their pioneer works, variable structure control was used to handle the second order linear system and then expanded to nonlinear models. In recent decades, variable structure control is successfully applied to a wide variety of engineering solutions such as flight control, satellite attitude control, and flexible space vehicle control [17, 18]. Hence, variable structure control is considered to be used in the paper.

The rest of this paper is organized as follows. In Section 2, a singular biological economic model which considers a prey-predator system with time delay and stage structure is proposed; in Section 3, the local stability analysis is discussed; in Section 4, stability analysis of a boundary equilibrium point and a Hopf bifurcation is performed; in Section 5, the controller is designed. Finally, numerical simulations are given in Section 6 to verify the effectiveness of the conclusions.

2 Modeling

Gordon founded the open or public fishery economic theory in 1954. Sustainable Economic Profit = Sustainable Total Revenue - Sustainable Total Cost, when the harvested effort $E(t)$ is given. Therefore, when harvested effort $E(t)$ switches with time t, we get the following equation:

$$E(t)\bigl(x(t)p - c\bigr) = m(t),$$

where p is the market price of the captive population, c is the cost of a unit of capture effort, and m is the net economic revenue.

Based on [6], the following singular biological economic system is established:

$$\begin{cases} \frac{dx_1(t)}{dt} = ax_2(t) - r_1x_1(t) - \beta x_1(t), \\ \frac{dx_2(t)}{dt} = \beta x_1(t) - r_2x_2(t) - s_1x_2^2(t) - \beta_1x_2(t)y(t), \\ \frac{dy(t)}{dt} = \beta_1x_2(t)y(t) - r_3y(t) - s_2y^2(t) - E(t)y(t), \\ 0 = E(t)(py(t) - c) - m, \end{cases} \tag{1}$$

where $x_1(t), x_2(t), y(t)$ are, respectively, the density of the juveniles, adults of the prey population, and the predator population at time t. It is assumed that all populations are growing in the closed environments. At any time, the birth rate of the juveniles are proportional to the existing adult population with proportionality constant a, the rate of transformation of the adults is proportional to the existing juveniles population with proportionality constant β. r_1, r_2, r_3 are the death rates of the juveniles, adults of the prey population, and the predator population, respectively. The adult preys and the predators compete among themselves for food, s_1, s_2 represent strength of the adult prey and predator intraspecific competition, respectively. To make the model reasonable, it is assumed that the predator catches mainly adult prey, and the number of the juvenile prey population as economic products is much smaller than that of the adult prey population. Therefore, the capture of the predators to the juvenile population can be neglected. In the paper, the predator only consumes the adult prey at the rate β_1.

In general, the reproduction of predator after predating the prey is not instantaneous but will mediate by some discrete time lag required for the gestation of the predator. Let τ represent the discrete time delay, which is the time interval between the moments when an individual prey is killed and when the corresponding biomass to the predator population. In this case, a singular prey-predator bioeconomic model with time delay is given as follows:

$$\begin{cases} \frac{dx_1(t)}{dt} = ax_2(t) - r_1x_1(t) - \beta x_1(t), \\ \frac{dx_2(t)}{dt} = \beta x_1(t) - r_2x_2(t) - s_1x_2^2(t) - \beta_1x_2(t)y(t), \\ \frac{dy(t)}{dt} = \beta_1x_2(t-\tau)y(t-\tau) - r_3y(t) - s_2y^2(t) - E(t)y(t), \\ 0 = E(t)(py(t) - c) - m, \end{cases} \tag{2}$$

where the constants mentioned above are all positive.

For the convenience of calculation, the right-hand side of equation (1) can be expressed in the following form:

$$F(X,E,m) = \begin{pmatrix} ax_2 - r_1x_1 - \beta x_1 \\ \beta x_1 - r_2x_2 - s_1x_2^2 - \beta_1x_2y \\ \beta_1x_2y - r_3y - s_2y^2 - Ey \end{pmatrix},$$

$$G(X,E,m) = (py - c)E - m,$$

where $X = (x_1, x_2, y)$.

Considering the biological significance, the model is discussed in the following interval:

$$R_+^4 = \{\chi = (x_1 \geq 0, x_2 \geq 0, y \geq 0, E \geq 0)\}.$$

Due to the limitation of the environment, the density of the juveniles, adults of the prey population, and the predator population cannot exceed the environment maximum carrying capacity. So the state variables and the parameters satisfy the following conditions:

$$0 < x_1 < x_{1\max}, \qquad 0 < x_2 < x_{2\max}, \qquad 0 < y < y_{\max}, \qquad 0 < E < \beta_1,$$

where $x_{1\max}, x_{2\max}, y_{\max}$ are, respectively, the maximum environment carrying capacity of the juveniles, adults of the prey population, and the predator population.

3 Local stability analysis

In the section, the local stability of the differential-algebraic system (2) without discrete time delay and economic profit at the interior equilibrium will be investigated. In general, we can pay more attention to local dynamic characteristics near the positive equilibrium of the system in the actual situation of biological economics.

When $m = 0$, system (1) can be written as:

$$\begin{cases} \frac{dx_1(t)}{dt} = ax_2(t) - r_1x_1(t) - \beta x_1(t), \\ \frac{dx_2(t)}{dt} = \beta x_1(t) - r_2x_2(t) - s_1x_2^2(t) - \beta_1x_2(t)y(t), \\ \frac{dy(t)}{dt} = \beta_1x_2(t)y(t) - r_3y(t) - s_2y^2(t) - E(t)y(t), \\ 0 = E(t)(py(t) - c). \end{cases} \tag{3}$$

It is clear that (3) has one equilibrium point $P^*(x_1^*, x_2^*, y^*, E^*)$, where

$$x_1^* = \frac{a}{r_1 + \beta}x_2^*, \qquad x_2^* = \frac{pa\beta - pr_2(r_1 + \beta) - \beta_1c(r_1 + \beta)}{s_1(r_1 + \beta)p},$$
$$y^* = \frac{c}{p}, \qquad E^* = \beta x_2^* - r_3 - \frac{s_2c}{p}.$$

In order to guarantee that each component exists at P^*, that is, the juveniles, adults of the prey population, the predator population, and harvested effort all exist, the following inequalities need to be satisfied:

$$\begin{cases} p\beta a - pr_2(r_1 + \beta) - \beta_1c(r_1 + \beta) > 0 \\ \beta(p\beta a - pr_2(r_1 + \beta) - \beta_1c(r_1 + \beta)) - s_1p(r_1 + \beta)(r_3 + \frac{s_2c}{p}) > 0. \end{cases}$$

By analysis, we know there is a bifurcation at the positive equilibrium point for model (3), which is shown in the following theorem.

Theorem 3.1 *System* (1) *without discrete time delay will show the phenomenon of singularity-induced bifurcation at* $P^*(x_1^*, x_2^*, y^*, E^*)$, *and m is a bifurcation parameter. Furthermore, a stability switch occurs as m increases through* 0.

Proof Let m be a bifurcation parameter for model (3). Due to $y^* = \frac{c}{p}$, $\Delta = \det[D_EG] = py - c = 0$, the following three conditions are satisfied:

(1)

$$\begin{aligned}\operatorname{trace}\left(D_EFadj(D_EG)D_XG\right)_{P^*} &= \operatorname{trace}\begin{pmatrix}0\\0\\-y\end{pmatrix}\begin{pmatrix}0 & 0 & pE\end{pmatrix}\\ &= -pE^*y = -cE^* \neq 0.\end{aligned}$$

(2)

$$\begin{aligned}&\begin{vmatrix}D_XF & D_EF\\ D_XG & D_EG\end{vmatrix}_{P^*}\\ &\quad= \begin{vmatrix}-r_1-\beta & a & 0 & 0\\ \beta & -r_2-2s_1x_2-\beta_1 y & -\beta_1 x_2 & 0\\ 0 & \beta_1 y & \beta_1 x_2 - r_3 - 2s_2 y - E & -y\\ 0 & 0 & pE & py-c\end{vmatrix}_{P^*}\\ &\quad= cE^*\left(\beta a - r_2(r_1+\beta) - \frac{\beta_1 c}{p}(r_1+\beta)\right)\\ &\quad= cE^*(r_1+\beta)s_1x_2^* \neq 0.\end{aligned}$$

(3)

$$\begin{aligned}&\begin{vmatrix}D_XF & D_EF & D_mF\\ D_XG & D_EG & D_mG\\ D_X\Delta & D_E\Delta & D_m\Delta\end{vmatrix}_{P^*}\\ &\quad= \begin{vmatrix}-r_1-\beta & a & 0 & 0 & 0\\ \beta & -r_2-2s_1x_2-\beta_1 y & -\beta_1 x_2 & 0 & 0\\ 0 & \beta_1 y & \beta_1 x_2 - r_3 - 2s_2 y - E & -y & 0\\ 0 & 0 & pE & py-c & -1\\ 0 & 0 & p & p & 0\end{vmatrix}_{P^*}\\ &\quad= py^*\left(\beta a - r_2(r_1+\beta) - \frac{c\beta_1}{p}(r_1+\beta)\right)\\ &\quad= c(r_1+\beta)\left(\frac{\beta a}{r_1+\beta} - r_2 - \frac{\beta_1 c}{p}\right) \neq 0.\end{aligned}$$

(4)

$$\begin{aligned}&\begin{vmatrix}D_XF & D_EF & D_mF\\ D_XG & D_EG & D_mG\\ D_X\Delta & D_E\Delta & D_m\Delta\end{vmatrix}_{P^*}\\ &\quad= \begin{vmatrix}-r_1-\beta & a & 0 & 0 & 0\\ \beta & -r_2-2s_1x_2-\beta_1 y & -\beta_1 x_2 & 0 & 0\\ 0 & \beta_1 y & \beta_1 x_2 - r_3 - 2s_2 y - E & -y & 0\\ 0 & 0 & pE & py-c & -1\\ 0 & 0 & p & p & 0\end{vmatrix}_{P^*}\end{aligned}$$

$$= py^*\left(\beta a - r_2(r_1+\beta) - \frac{c\beta_1}{p}(r_1+\beta)\right)$$

$$= c(r_1+\beta)\left(\frac{\beta a}{r_1+\beta} - r_2 - \frac{\beta_1 c}{p}\right) \neq 0.$$

Thus, we can conclude that there exists a smooth curve in R^4 which passes through the positive equilibrium point P^*, and it is transversal to the singular surface at the positive equilibrium point P^*. It can be calculated that we can get the following equations:

$$B = -\operatorname{trace}\left(D_E F adj(D_E G) D_X G\right)_{P^*}$$

$$= -\operatorname{trace}\begin{pmatrix} 0 \\ 0 \\ -y \end{pmatrix}(0 \quad 0 \quad pE)_{P^*}$$

$$= pE^* y = cE^*,$$

$$C = \left(D_m\Delta - (D_X\Delta \quad D_E\Delta)\begin{pmatrix} D_X F & D_E F \\ D_X G & D_E G \end{pmatrix}\right)_{P^*} = \frac{1}{pE^*},$$

$$\frac{B}{C} = pc\left(\beta x_2^* - r_3 - s_2\frac{c}{p}\right)^2 > 0.$$

Based on Theorem 3 in [19], system (1) has a singularity-induced bifurcation at the positive equilibrium point P^* when bifurcation parameter $m = 0$. If m increases through zero, one eigenvalue of system (1) will move from C^- (the open complex left half plane) to C^+ (the open complex right half plane) along the real axis by diverging into ∞, and thus the stability of the positive equilibrium point P^* changes from stable to unstable. Hence the conclusion follows. □

Remark 3.1 Theorem 3.1 reveals that model (3) will undergo a singularity-induced bifurcation when $m = 0$. From the ecological prospective, the singularity-induced bifurcation will lead to a surge in a population density within a short period of time and eventually break the ecological balance. These are disastrous for the real-world biological model. Therefore a proper control strategy is necessary to regulate the biological model (3).

4 Stability analysis of a boundary equilibrium point and a Hopf bifurcation

In the section, the local stability of differential-algebraic system (2) with discrete time delay τ at the boundary equilibrium is investigated in the case of economic equilibrium. Furthermore, the phenomenon of Hopf bifurcation at one boundary equilibrium is also investigated.

When $m = 0$, (2) can be written as

$$\begin{cases} \frac{dx_1(t)}{dt} = ax_2(t) - r_1x_1(t) - \beta x_1(t), \\ \frac{dx_2(t)}{dt} = \beta x_1(t) - r_2x_2(t) - s_1x_2^2(t) - \beta_1 x_2(t)y(t), \\ \frac{dy(t)}{dt} = \beta_1 x_2(t-\tau)y(t-\tau) - r_3y(t) - s_2y^2(t) - E(t)y(t), \\ 0 = E(t)(py(t) - c). \end{cases} \tag{4}$$

It is clear that (4) has three boundary equilibrium points: $P_1(0,0,0,0)$, $P_2(x_{11},x_{21},0,0)$, $P_2(x_{13},x_{23},y_3,0)$, where $x_{11} = \frac{a}{r_1+\beta}x_{21}$, $x_{21} = \frac{\beta a - r_2(r_1+\beta)}{(r_1+\beta)s_1}$, $x_{13} = \frac{a}{r_1+\beta}x_{23}$, $x_{23} = \frac{s_2(a\beta - r_1r_2 - r_2\beta)+\beta r_3(r_1+\beta)}{(r_1+\beta)(s_1s_2+\beta\beta_1)}$.

For $P_2(x_{11},x_{21},0,0)$, in order to guarantee that the juveniles and adults of the prey population all exist, the following inequality needs to be satisfied: $\beta a - r_2(r_1+\beta) > 0$. Similarly, for $P_3(x_{13},x_{23},y_3,0)$, the following inequality is satisfied: $\beta_1(a\beta - r_1r_2 - r_2\beta) - r_3(r_1+\beta)s_1 > 0$. Next, we can consider the stability in the neighborhood of each boundary equilibrium.

Theorem 4.1 *Equation* (4) *is unstable at the equilibrium point* $P_1(0,0,0,0)$.

Proof The Jacobian matrix of model (4) is given by

$$\begin{aligned} J &= D_XF - D_EF(D_EG)^{-1}D_XG \\ &= \begin{bmatrix} -r_1-\beta & a & 0 \\ \beta & -r_2-2s_1x_2-\beta_1 y & -\beta_1 x_2 \\ 0 & \beta_1 y e^{-\lambda\tau} & \beta_1 x_2 e^{-\lambda\tau} - r_3 - 2s_2 y - E + \frac{pEy}{py-c} \end{bmatrix}. \end{aligned}$$

Then the characteristic polynomial at P_1 is

$$\det(\lambda A - J_{P_1}) = (\lambda + r_3)\bigl(\lambda^2 + (r_1+r_2+\beta)\lambda + (r_1+\beta)r_2 - a\beta\bigr) = 0,$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Based on the condition of the existence of the adult prey population: $r_1 + r_2 + \beta > 0$, $(r_1+\beta)r_2 - a\beta < 0$. It can be judged that $P_1(0,0,0,0)$ is an unstable equilibrium. □

Similarly, in order to study the stability of P_2, we firstly obtain the Jacobian matrix of model (4) at P_2:

$$J_{P_2} = \begin{bmatrix} -r_1-\beta & a & 0 \\ \beta & -r_2-2s_1x_{21} & -\beta_1 x_{21} \\ 0 & 0 & \beta_1 x_{21} e^{-\lambda\tau} - r_3 \end{bmatrix}.$$

Then the characteristic polynomial at P_2 is

$$\begin{aligned} &\det(\lambda A - J_{P_2}) \\ &\quad = \bigl(\lambda + r_3 - \beta_1 x_{21} e^{-\lambda\tau}\bigr)\bigl(\lambda^2 + (r_1+r_2+\beta+2s_1x_{21})\lambda + (r_1+\beta)(r_2+2s_1x_{21}) - a\beta\bigr) \\ &\quad = 0. \end{aligned}$$

Based on the above analysis, we obtain the following:

$$(r_1+\beta)(r_2+2s_1x_{21}) - a\beta = a\beta - r_2(r_1+\beta) > 0,$$

$$r_1 + r_2 + \beta + 2s_1x_{21} = a\beta - r_2(r_1 + \beta) > 0.$$

So the other characteristic of system (4) at P_2 is determined in the following formula:

$$\lambda + r_3 - \frac{\beta_1(a\beta - r_2(r_1 + \beta))}{(r_1 + \beta)s_1}e^{-\lambda\tau} = 0.$$

By the analysis of roots for the above formula, we obtain that the equilibrium point P_2 is a stable focus or node if $s_1r_3(r_1 + \beta) > \beta_1(a\beta - r_1r_2 - r_2\beta)$; otherwise P_2 is a saddle.

Similarly, the Jacobian matrix of model (4) at the equilibrium point P_3 is given by

$$J_{P_3} = \begin{bmatrix} -r_1 - \beta & a & 0 \\ \beta & -r_2 - 2s_1x_{23} - \beta_1y_3 & -\beta_1x_{23} \\ 0 & \beta_1y_3e^{-\lambda\tau} & \beta_1x_{23}e^{-\lambda\tau} - r_3 - 2s_2y_3 \end{bmatrix},$$

$$\begin{aligned} D(\lambda,\tau) &= \det\begin{pmatrix} \lambda + r_1 + \beta & -a & 0 \\ -\beta & \lambda + r_2 + 2s_1x_{23} + \beta_1y_3 & \beta_1x_{23} \\ 0 & -\beta_1x_{23}e^{-\lambda\tau} & \lambda + r_3 + 2s_2y_3 - \beta_1x_{23}e^{-\lambda\tau} \end{pmatrix} \\ &= P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0. \end{aligned}$$

It can be simplified as follows:

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 + \bigl(q_1\lambda^2 + q_2\lambda + q_3\bigr)e^{-\lambda\tau} = 0, \tag{5}$$

where

$$\begin{aligned} &P(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3, \\ &Q(\lambda) = q_1\lambda^2 + q_2\lambda + q_3, \\ &p_1 = r_1 + \beta + r_2 + 2s_1x_{23} + \beta_1y_3 + r_3 + 2s_2y_3, \\ &p_2 = (r_1 + \beta)(r_2 + 2s_1x_{23} + \beta_1y_3) + (r_3 + 2s_2y_3)(r_1 + \beta + r_2 + 2s_1x_{23} + \beta_1y_3) - a\beta, \\ &p_3 = \bigl((r_1 + \beta)(r_2 + 2s_1x_{23} + \beta_1y_3) - a\beta\bigr)(r_3 + 2s_2y_3), \\ &q_1 = -\beta_1x_{23}, \qquad q_2 = -\bigl(r_2\beta_1x_{23} + 2s_1\beta_1x_{23}^2 + (r_1 + \beta)\beta_1x_{23}\bigr), \\ &q_3 = a\beta\beta_1x_{23} - (r_1 + \beta)r_2\beta_1x_{23} - 2s_1(r_1 + \beta)\beta_1x_{23}^2. \end{aligned}$$

It is assumed that for some values of $\tau > 0$, there exists a real such that $\lambda = i\omega$ is a root of the characteristic equation (5). Now, substituting $\lambda = i\omega$ into equation (5), we have

$$-i\omega^3 - n_1\omega^2 + ip_2\omega + n_3 + \bigl(-n_4\omega^2 + in_5\omega + n_6\bigr)e^{-i\omega\tau} = 0.$$

Then, by separating real and imaginary parts of $D = 0$, we obtain that

$$\omega^3 - p_2\omega = q_2\omega\cos(\omega\tau) - \bigl(q_3 - q_1\omega^2\bigr)\sin(\omega\tau), \tag{6}$$

$$p_1\omega^2 - p_3 = \bigl(q_3 - q_1\omega^2\bigr)\cos(\omega\tau) + q_2\omega\sin(\omega\tau). \tag{7}$$

By squaring and adding (6) and (7), it can be shown that

$$\begin{aligned}
&\omega^6 + \left({p_1}^2 - 2p_2 - {q_1}^2\right)\omega^4 + \left({p_2}^2 - {q_2}^2 - 2p_1p_3 + 2q_1q_3\right)\omega^2 + {p_3}^2 - {q_3}^2 = 0,\\
&\omega^6 + C_1\omega^4 + C_2\omega^2 + C_3 = 0,
\end{aligned}\tag{8}$$

where parameters C_1, C_2, C_3 can be expressed as follows:

$$\begin{aligned}
C_1 &= {p_1}^2 - 2p_2 - {q_1}^2,\\
C_2 &= {p_2}^2 - 2p_1p_3 - {q_2}^2 + 2q_1q_3,\\
C_3 &= {p_3}^2 - {q_3}^2.
\end{aligned}$$

According to the Routh-Hurwitz criteria [20], in order to guarantee equation (8) has at least one real root ω_0, C_1, C_2, C_3 need to satisfy any one of which is less than zero. A simple assumption that equation (8) has a positive real root ω_0 is $C_3 < 0$. Hence, under the assumption, equation (8) will have a pair of purely imaginary roots of the form $\pm i\omega_0$.

By computing, it can be obtained that τ_n corresponding to ω_0 is as follows:

$$\tau_n = \frac{1}{\omega_0}\arccos\left(\frac{(p_1{\omega_0}^2 - p_3)(q_3 - q_1{\omega_0}^2) + q_2\omega_0({\omega_0}^3 - p_2\omega_0)}{(q_3 - q_1{\omega_0}^2)^2 + (q_2\omega_0)^2}\right) + \frac{2n\pi}{\omega_0},\tag{9}$$

where $n = 0, 1, 2, \ldots$.

Based on Butler's lemma in [21], the conclusion can be expressed as follows: system (4) is stable at the boundary equilibrium point P_3 for $\tau < \tau_0$.

Theorem 4.2 *System* (4) *undergoes a Hopf bifurcation at the equilibrium point* P_3 *when* $\tau = \tau_0$ *if the inequality* ${p_1}^2 - 2p_2 - {q_1}^2 > 0$ *is satisfied.*

Proof Let

$$\begin{aligned}
\Theta &= \operatorname{sign}\left[\frac{d(\operatorname{Re}\lambda)}{d\tau}\right]_{\lambda=i\omega_0}\\
&= \operatorname{sign}\left[\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0}.
\end{aligned}$$

By differentiating (5) with respect to τ, we can obtain

$$\begin{aligned}
&\left(3\lambda^2 + 2p_1 + p_2\right)\frac{d\lambda}{d\tau} + (2q_1\lambda + q_2)e^{-\lambda\tau}\frac{d\lambda}{d\tau} - \left(q_1\lambda^2 + q_2\lambda + q_3\right)\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau}\\
&\quad = \left(q_1\lambda^2 + q_2\lambda + q_3\right)e^{-\lambda\tau}\lambda.
\end{aligned}$$

Then we get

$$\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{3\lambda^2 + 2p_1\lambda + p_2}{\lambda e^{-\lambda\tau}(q_1\lambda^2 + q_2\lambda + q_3)} + \frac{2q_1\lambda + q_2\lambda}{\lambda(q_1\lambda^2 + q_2\lambda + q_3)} - \frac{\tau}{\lambda}\\
&= \frac{2\lambda^3 + p_1\lambda^2 - p_3}{-\lambda^2(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3)} + \frac{q_1\lambda^2 - q_3}{\lambda^2(q_1\lambda^2 + q_2\lambda + q_3)} - \frac{\tau}{\lambda}.
\end{aligned}$$

By computing the real part of $(\frac{d\lambda}{d\tau})^{-1}$ for $\lambda = i\omega_0$, we can obtain

$$\begin{aligned}
&\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} \\
&= \left[\mathrm{Re}\left(\frac{2\lambda^3 + p_1\lambda^2 - p_3}{-\lambda^2(\lambda^3 + p_1\lambda^2 + p_2\lambda + p_3)}\right) + \frac{q_1\lambda^2 - q_3}{\lambda^2(q_1\lambda^2 + q_2\lambda + q_3)} - \frac{\tau}{\lambda}\right]_{|\lambda=i\omega_0} \\
&= \mathrm{Re}\left(\frac{-2i{\omega_0}^3 - p_1{\omega_0}^2 - p_3}{{\omega_0}^2(-i{\omega_0}^3 - p_1{\omega_0}^2 + p_2 i\omega_0 + p_3)} + \frac{-q_1{\omega_0}^2 - q_3}{-{\omega_0}^2(-q_1{\omega_0}^2 + q_2 i\omega_0 + q_3)}\right) \\
&= \frac{1}{{\omega_0}^2}\mathrm{Re}\left(\frac{(-p_1{\omega_0}^2 - p_3) - 2{\omega_0}^3 i}{(p_3 - p_1{\omega_0}^2) + (p_2\omega_0 - {\omega_0}^3)i} + \frac{-q_1{\omega_0}^2 - q_3}{(q_3 - q_1{\omega_0}^2) + q_2 i\omega_0}\right) \\
&= \frac{1}{{\omega_0}^2}\mathrm{Re}\left[\frac{2{\omega_0}^6 + ({p_1}^2 - 2p_2 - {q_1}^2){\omega_0}^4 - ({p_3}^2 - {q_3}^2)}{(q_3 - q_1{\omega_0}^2)^2 + (q_2\omega_0)^2}\right].
\end{aligned}$$

According to the assumption for value of C_1, furthermore, based on the assumption that equation (8) has a positive real root, it follows that $C_3 < 0$. Hence, the conclusion can be expressed as follows:

$$\Theta = \mathrm{sign}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} > 0.$$

Hence, system (4) has at least one eigenvalue with a positive real part; furthermore, system (4) undergoes a Hopf bifurcation [22] at the equilibrium point P_3 when $\tau = \tau_0$, where τ_0 is the bifurcation value. Thus the proof is completed. □

Remark 4.1 It follows from Theorem 4.2 that the juvenile prey, the adult prey and the predator are existent in the case of $\tau = \tau_0$, while in the absence of harvested effort, system (4) will undergo a Hopf bifurcation. When the delay varies in a certain interval, system (4) is stable, which means the biological system is sustainable. But when the delay is beyond the threshold, it will become unstable.

5 Variable structure control

By the conclusion analyzed above in Section 3, the singularity-induced bifurcation can be obtained for system (1) at the positive equilibrium point when the economic profit $m = 0$. From the ecological prospective, the singularity-induced bifurcation and the system instability will cause the following effects on the ecological model. The collapse which the singularity-induced bifurcation leads to will be embodied by a surge in a population density within a short period in the ecological system and loss of the stability of the system. Hence, populations in the inferior position will face the extinction due to the shortage of resources, which is not beneficial to the maintenance of the population diversity in the ecological model.

In order to guarantee the sustainable development of the population, it is necessary to find an effective method to eliminate the singularity-induced bifurcation of (1) at the positive equilibrium point P^* when the economic profit is zero.

Variable structure control [23] is often used to deal with some models with internal varying parameters and external disturbances; furthermore, the system response depends on the gradients of the sliding surface and remains insensitive to parameter variations and external disturbances. Hence, in this paper, variable structure control is introduced to eliminate bifurcation behavior and ensure the system stability. In order to facilitate the controller design, we differentiate the second differential equation in model (1) and substitute the other three equations into it as follows:

$$\frac{dx_2^2}{dt} + a_1\frac{dx_2}{dt} + a_0x_2 = b\frac{du}{dt}, \tag{10}$$

where

$$\begin{aligned}
&a_1 = r_2 + 2s_1x_2 + \frac{\beta_1(m+Ec)}{pE},\\
&a_0 = \beta_1\left(\beta_1x_2\frac{m+Ec}{pE} - r_3\frac{m+Ec}{pE} - s_2\left(\frac{m+Ec}{pE}\right)^2 - \frac{m+Ec}{p}\right),\\
&b = \beta, \qquad x_1 = u.
\end{aligned}$$

Equation (10) can be written as a single-input and single-output model with the parameters varying within definite intervals. In system (10), u can be regarded as an input and x_2 is an output. Therefore, we can get the varying intervals of the coefficients a_0, a_1 as follows:

$$\begin{aligned}
&\frac{\beta_1c}{p} + r_2 < a_1 < r_2 + 2s_1x_{2\max} + \frac{m+\beta_1c}{p},\\
&-\frac{s_2c^2}{p^2} - \beta_1r_3\frac{c}{p} - \frac{m}{p} < a_0 < \beta_1{}^2x_{2\max}\frac{m+\beta_1c}{p\beta_1}.
\end{aligned}$$

Suppose that the system is locally stable at $\bar{x}_2$. In order to make the destiny of adult prey population reach the fixed value $\bar{x}_2$, let

$$e = \bar{x}_2 - x_2, \tag{11}$$

where is the error e between x_2 and $\bar{x}_2$.

Differentiating formula (11) twice and considering model (10), the following equation is obtained:

$$\frac{d^2e}{dt^2} + a_1\frac{de}{dt} + a_0e = a_0\bar{x}_2 - b\frac{du}{dt}. \tag{12}$$

For the differential equation (12), the set-point signal $0 < \bar{x}_2 < x_{2\max}$ is regarded as an external disturbance. According to the transformation, model (10) is considered as a linear uncertain system with the control input.

To guarantee the invariance of the variable-structure system in sliding mode to the parameter uncertainties and external disturbances, it is required that certain matching contains should hold. And model (12) is transformed into

$$\begin{cases}
\frac{de_1}{dt} = e_2,\\
\frac{de_2}{dt} = -a_0e_1 - a_1e_2 - b\frac{du}{dt} + a_0\bar{x}_2,
\end{cases}$$

where $e = e_1$.

Define the auxiliary variable

$$\eta = \frac{du}{dt}.$$

Then we can get the following extended system:

$$\begin{bmatrix} \frac{de_1}{dt} \\ \frac{de_2}{dt} \\ \frac{du}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -a_0 & -a_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ -b \\ 1 \end{bmatrix} \eta + \begin{bmatrix} 0 \\ a_0 \\ 0 \end{bmatrix} \bar{x}_2, \tag{13}$$

where η can be regarded as an auxiliary variable and the original input variable u is now an added state to the system.

The sliding surface is defined as follows:

$$s(e) = fe_1 + e_2, \tag{14}$$

where f is a strictly positive constant.

A converging law is needed to make the trajectories of system (13) reach the sliding surface $s(e) = 0$ at a fast speed. The power reaching law [24] is designed as follows:

$$\frac{ds(e)}{dt} = -\varepsilon \left|s(e)\right|^r \operatorname{sgn}\bigl(s(e)\bigr),$$

where $\varepsilon > 0, 0 < r < 1$, and r determine the speed at which the trajectories approach the sliding surface.

On the basis of the variable structure system theory, it can be shown that

$$\frac{ds(e)}{dt} = f\frac{de_1}{dt} + \frac{de_2}{dt} = -\varepsilon \left|s(e)\right|^r \operatorname{sgn}\bigl(s(e)\bigr). \tag{15}$$

Solving equation (15), we get

$$\eta = \frac{\varepsilon |s(e)|^r \operatorname{sgn}(s(e)) + fe_2 - a_0 e_1 - a_1 e_2 + a_0 \bar{x}_2}{b}.$$

Hence, based on the power reaching law, the variable structure controller is obtained as follows:

$$\eta = \begin{cases} \eta^+ = \frac{\varepsilon |s(e)|^r + fe_2 - a_0 e_1 - a_1 e_2 + a_0 \bar{x}_2}{b}, & s(e) > 0, \\ \eta^- = \frac{-\varepsilon |s(e)|^r + fe_2 - a_0 e_1 - a_1 e_2 + a_0 \bar{x}_2}{b}, & s(e) < 0. \end{cases} \tag{16}$$

This designed variable structure controller can make system (13) approach the sliding surface and reach it, which is shown in the next theorem.

Theorem 5.1 *The variable structure controller η guarantees that system* (13) *reaches the sliding surface* (14).

Proof When $s(e) > 0$, the controller η^+ is used to stabilize system (13), the inequality can be obtained as follows:

$$\begin{aligned}\frac{ds(e)}{dt} &= f\frac{de_1}{dt} + \frac{de_2}{dt} \\ &= fe_2 - a_0e_1 - a_1e_2 - b\eta^+ + a_0\bar{x}_2 \\ &= -\varepsilon\left|s(e)^r\right| < 0.\end{aligned}$$

The reachable condition $\frac{ds(e)}{dt}s(e) < 0$ is satisfied.

When $s(e) < 0$, the controller η^- is used to stabilize system (13), we can get the following inequality:

$$\begin{aligned}\frac{ds(e)}{dt} &= f\frac{de_1}{dt} + \frac{de_2}{dt} \\ &= fe_2 - a_0e_1 - a_1e_2 - b\eta^- + a_0\bar{x}_2 \\ &= \varepsilon\left|s(e)^r\right| > 0.\end{aligned}$$

Obviously, the reachable condition $\frac{ds(e)}{dt}s(e) < 0$ is also satisfied.

Therefore, the variable structure controller η guarantees that system (13) reaches the sliding surface (14). □

Remark 5.1 When the variable structure control is used, the singular biological economic model is transformed into a linear system with parameters varying within finite time intervals. The power reaching law is designed to weaken the chattering on the sliding surface. The control method can stabilize the nonlinear system effectively. Indeed, the variable structure controller is designed to eliminate the singularity-induced bifurcation arising in the biological system (3). By appropriately regulating the population density of biological resources, the conversion among the population is restrained, and the stability of the biological system is eventually ensured. This may be an alternative way to maintain the sustainable development of the biological resources.

6 Numerical simulations

(1) Numerical simulation of a Hopf bifurcation.

With the help of MATLAB, numerical simulations are provided to substantiate the theoretical results which have been established in the previous sections of this paper. In order to substantiate the Hopf bifurcation theory of above results, the values of parameters are taken as follows:

$$\begin{aligned}&r_1 = 0.2, \quad r_2 = 0.1, \quad r_3 = 0.3, \\ &a = 1.8, \quad \beta = 1.5, \quad \beta_1 = 0.8, \\ &s_1 = 0.3, \quad s_2 = 0.2, \quad p = 2.5, \quad c = 1.5.\end{aligned}$$

By virtue of the given values of parameters, it can be computed that $C_1 = 9.8200 > 0$, $C_3 = -0.6059 < 0$. Furthermore, the boundary equilibrium point P_3 can be obtained as $P_3(6.2304, 0.5934, 0.8735, 0)$. Based on the analysis in Section 4, it satisfies the assumption

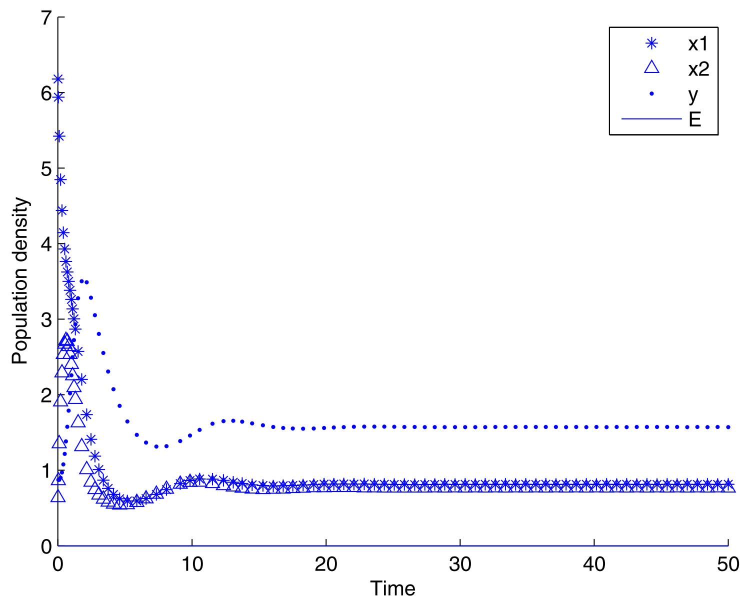

Figure 1 The state response of system (4) with $\tau = 0.7234$ at P_3. The state response of system (4) with $\tau = 0.7234$ at P_3.

that equation (8) has a positive root $\omega_0 = 0.3289$, and then the corresponding $\tau_0 = 0.7234$ can be calculated by solving equation (10). Hence, the interior equilibrium P_3 remains stable for $\tau < 0.7234$. It is shown that $\tau = 0.1$ in Figure 1 is randomly selected in the interval $(0, 0.7234)$, which is enough to merit the above mathematical study. As τ increases through τ_0, the phenomenon of Hopf bifurcation occurs for $\tau_0 = 0.7234$, which is shown in Figure 2. Figure 3 shows a bifurcation diagram of the model for $\tau_0 = 0.7234$. Figures 4 and 5 show that system (4) remains unstable for sufficiently large τ, but show complex structures with increasing oscillations.

(2) Numerical simulation of the controller.

In order to substantiate the variable structure control theory of above results, the values of parameters are taken as follows:

$$\begin{cases} \frac{dx_1(t)}{dt} = 0.88x_2(t) - 0.06x_1(t) - 0.56x_1(t). \\ \frac{dx_2(t)}{dt} = 0.56x_1(t) - 0.02x_2(t) - 0.1x_2^2(t) - 0.42x_2(t)y(t). \\ \frac{dy(t)}{dt} = 0.42x_2(t)y(t) - 0.02y(t) - 0.1y^2(t) - E(t)y(t). \\ 0 = E(t)(2.5y(t) - 1.5) - m. \end{cases} \tag{17}$$

When the economic profit m varies, there are some complex dynamic behaviors for model (17) such as the singularity-induced bifurcation. When the economic profit $m = 0$, model (17) has a positive equilibrium point $P^*(6.9951, 4.9284, 0.6, 1.9899)$. When economic profit $m = -0.02$, there are three eigenvalues for the system -0.1677, -1.7101, and -2970.14. The eigenvalues become 2969.53, -0.1677, and -1.7096 when the parameter $m = 0.02$. The state response of system (17) is shown in Figure 6 when economic profit $m = 0.02$. Furthermore, it is obvious that two eigenvalues remain basically constant, and the other eigenvalue moves from C^- to C^+ along the real axis by diverging through ∞. It is

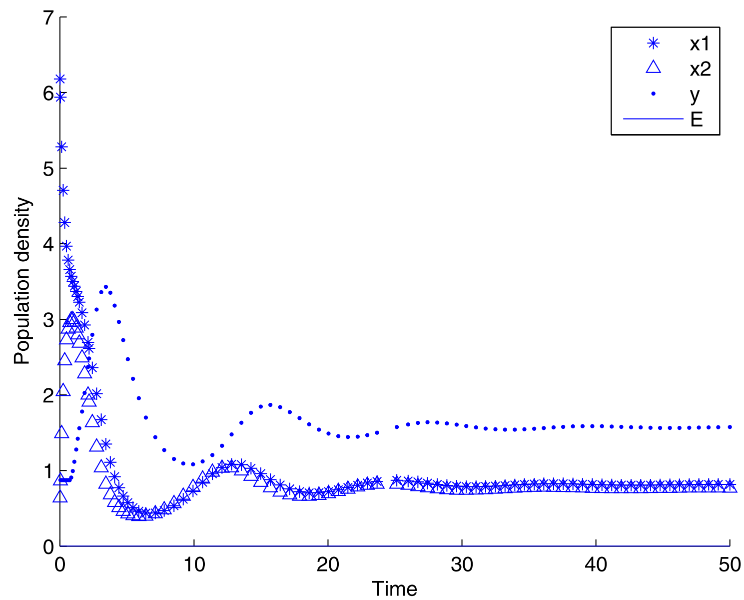

Figure 2 The state response of system (4) with $\tau = 0.1$ at P_3. The state response of system (4) with $\tau = 0.1$ at P_3.

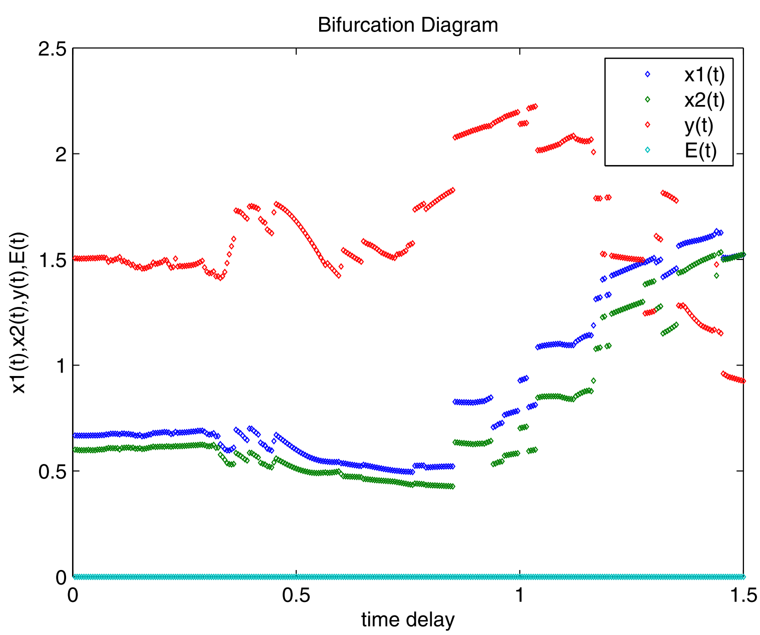

Figure 3 The bifurcation diagram of system (4) with $\tau = 0.7234$ at P_3. The bifurcation diagram of system (4) with $\tau = 0.7234$ at P_3.

called the over exploitation, and it causes the extinction of the population. Figure 7 shows a bifurcation diagram of the model for $m = 0$.

Now, we consider using variable structure control to eliminate the singularity-induced bifurcation. Without loss of generality, it is assumed that the densities of the juvenile prey

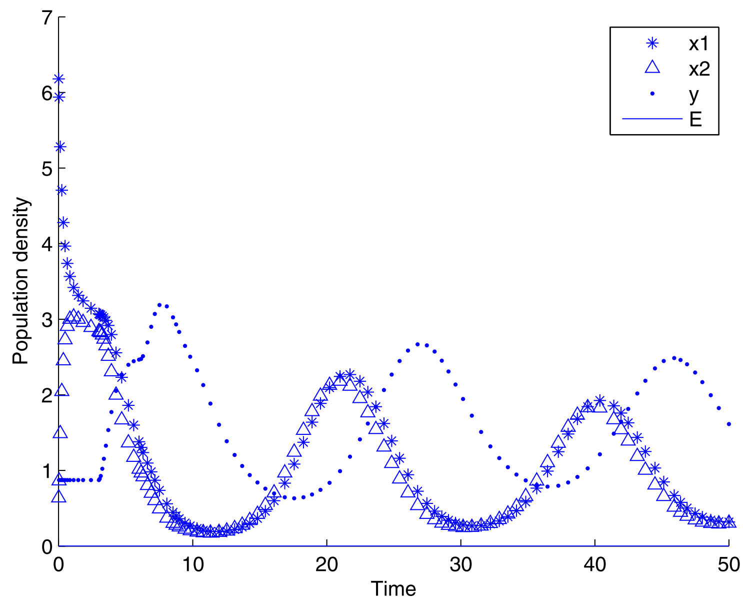

Figure 4 The state response of system (4) with $\tau = 3$ at P_3. The state response of system (4) with $\tau = 3$ at P_3.

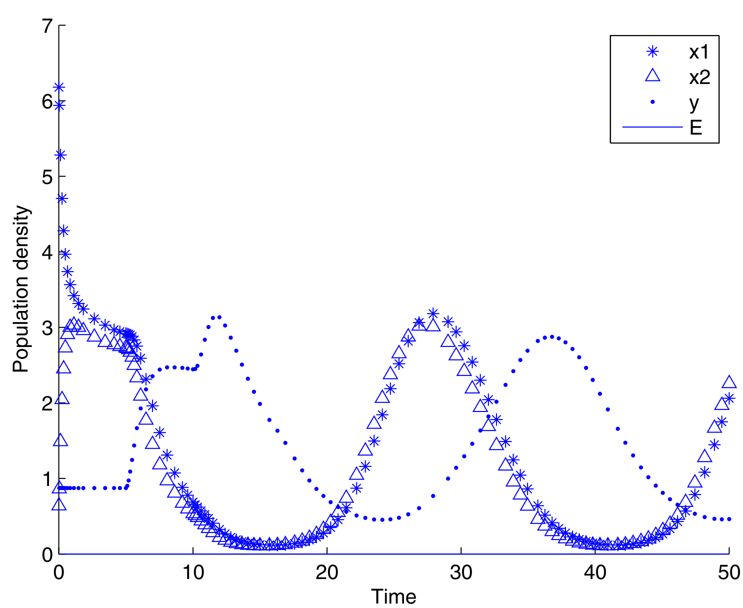

Figure 5 The state response of system (4) with $\tau = 5$ at P_3. The state response of system (4) with $\tau = 5$ at P_3.

population, the adult prey population, the predator population, and harvested effort satisfy the following conditions:

$$0 < x_1 < 10, \qquad 0 < x_2 < 15, \qquad 0 < y < 20, \qquad 0 < E < 0.42.$$

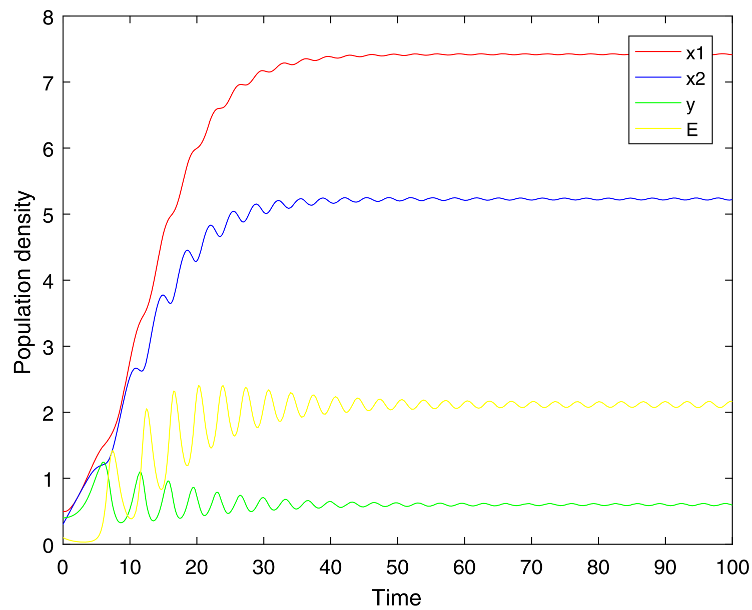

Figure 6 The state response of system (17) with $m = 0.02$ at P^*. The state response of system (17) with $m = 0.02$ at P^*.

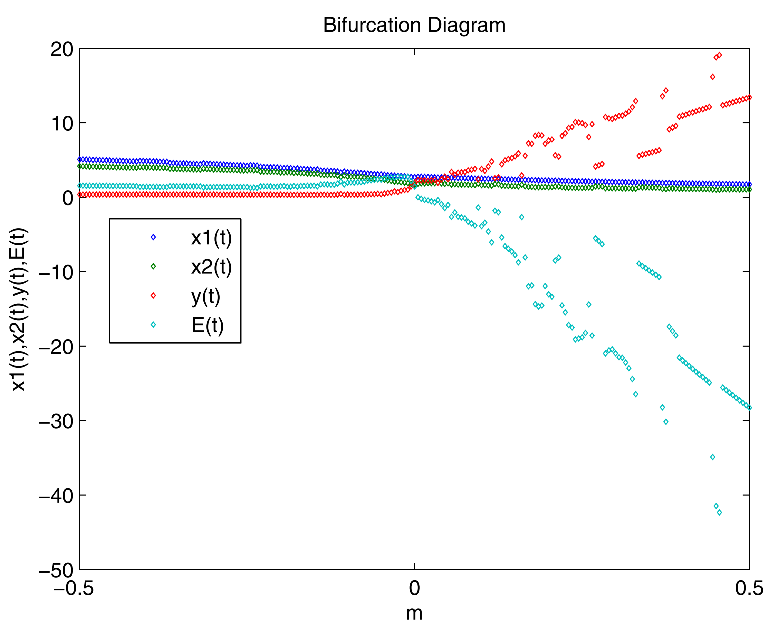

Figure 7 The bifurcation diagram of system (17) with $m = 0$ at P^*. The bifurcation diagram of system (17) with $m = 0$ at P^*.

By calculating, we can get the following inequality based on the previous conclusion:

$$0.2720 < a_1 < 5.2720, \qquad 0 < a_0 < 2.6460.$$

In order to avoid the phenomenon of over exploitation, system (17) is stable when the number of the adult population $\bar{x}_2 = 9.4$. The following variable structure control is de-

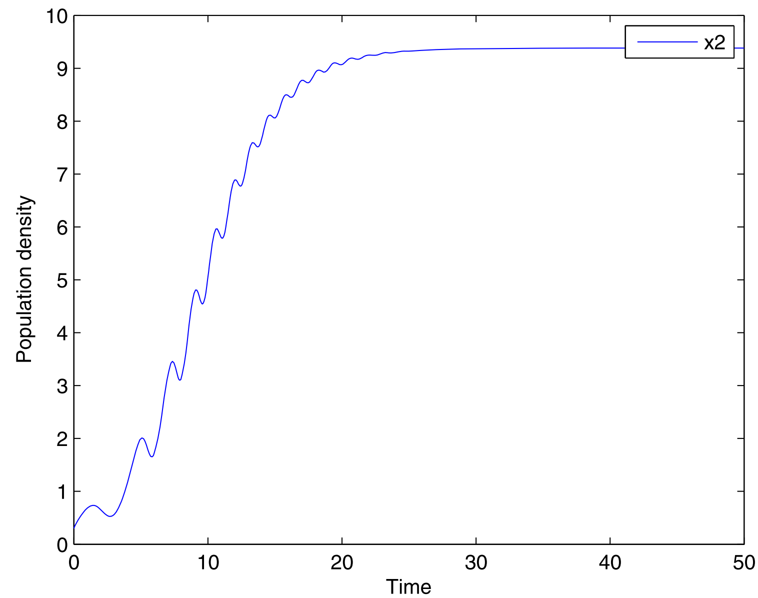

Figure 8 Changing of the adult prey x_2 with variable structure control. Changing of the adult prey x_2 with variable structure control.

signed:

$$\eta = \begin{cases} \eta^+ = \frac{3|s(e)|^{10}+9.96e_2-0.6e_1-0.56e_2+0.6\bar{x}_2}{b}, & s(e) > 0, \\ \eta^- = \frac{-3|s(e)|s(e)|^{10}+9.96e_2-0.6e_1-0.56e_2+0.6\bar{x}_2}{b}, & s(e) < 0, \end{cases}$$

where the sliding surface is defined as $s(e) = 9.96e_1 + e_2$. In order to illustrate the control result, numerical simulation is given in Figure 8. In Figure 8, the state variable stays in a stable situation, and the bifurcation behavior is eliminated by the controller. Therefore, variable structure control can stabilize the system effectively.

7 Conclusions

In the paper, the singular biological economic model with time delay and stage structure was established to investigate the effects of harvesting and gestation delay on the dynamic behavior of the model. The local stability at the boundary equilibrium point and the dynamic behavior for the model in Theorem 4.1 and Theorem 4.2 were discussed. Furthermore, as analyzed in Theorem 3.1, a singularity-induced bifurcation and a stability switch occurred in the case of positive economic interest of harvesting. A direct damage done by the singularity-induced bifurcation to the model was impulse phenomenon. As the parameters changed, the singular model underwent the singularity-induced bifurcation. In order to apply variable structure control to eliminate this complex behavior, the singular model was transformed into a linear single-input and single-output model with parameters varying within definite intervals. Variable structure control with sliding mode was designed to stabilize the model. Some simulations showed the effectiveness of the conclusions.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References
1. Zhang, Q, Liu, C, Zhang, X: Complexity, Analysis and Control of Singular Biological Systems. Springer, London (2012)
2. Niu, H, Zhang, Q: Generalized predictive control for difference-algebraic biological economic systems. Int. J. Biomath. **6**, 1-16 (2013)
3. Jiang, Y, Zhang, Q, Wang, H: Modeling boyciana-fish-human interaction with partial differential algebraic equations. Math. Biosci. **277**, 141-152 (2016)
4. Zhang, Y, Zhang, Q, Li, J: The bifurcation and control of a single-species fish population logistic model with the invasion of alien species. Discrete Dyn. Nat. Soc. **2014**, Article ID 548384 (2014). doi:10.1155/2014/548384
5. Zhang, Y, Zhang, Q, Zhang, T: H_∞ control of descriptor bioeconomic systems. J. Northeast. Univ. **32**, 1369-1373 (2011)
6. Zhang, Y, Zhang, Q, Zhao, L: Bifurcation and control in singular biological economic model with stage structure. J. Syst. Eng. **22**, 233-238 (2007)
7. Li, L, Zhang, Q, Zhu, B: Fuzzy stochastic optimal guaranteed cost control of bio-economic singular Markovian jump systems. IEEE Trans. Cybern. **45**, 2512-2521 (2015)
8. Zhang, Y, Zhang, Q, Zhang, G: H_∞ control of T-S fuzzy fish population logistic model with the invasion of alien species. Neurocomputing **173**, 724-733 (2016)
9. Clark, CW: Mathematical Bioeconomics: The Optimal Management of Renewable Resources. Wiley, New York (1990)
10. Beretta, E, Kuang, Y: Global analyses in some delayed ratio-dependent predator-prey systems. Nonlinear Anal. **32**, 381-408 (1998)
11. Chakraborty, K, Chakraborty, M, Kar, TK: Bifurcation and control of a bioeconomic model of a prey-predator system with a time delay. Nonlinear Anal. Hybrid Syst. **5**, 613-625 (2011)
12. Song, Y, Yuan, S: Bifurcation analysis in the delayed Leslie-Gower predator-prey system. Appl. Math. Model. **33**, 4049-4061 (2009)
13. Yuan, S, Song, Y: Stability and Hopf bifurcations in a delayed Leslie-Gower predator-prey system. J. Math. Anal. Appl. **355**, 82-100 (2009)
14. Kar, TK, Ghorai, A: Dynamic behavior of a delayed predator-prey model with harvesting. Appl. Math. Comput. **217**, 9085-9104 (2011)
15. Martin, A, Ruan, S: Predator-prey models with delay and prey harvesting. J. Math. Biol. **43**, 247-267 (2001)
16. Emelyanov, SV: Variable Structure Control Systems. Nauka, Moscow (1967)
17. Gao, W, Cheng, M: Variable structure control of flexible spacecraft. J. Beihang Univ. **9**, 274-280 (1988)
18. Zhang, Q, Li, L, Yan, XG, Spurgeon, SK: Sliding mode control for singular stochastic Markovian jump systems with uncertainties. Automatica **79**, 27-34 (2017)
19. Venkatasubramanian, V, Schattler, H, Zaborszky, J: Local bifurcations and feasibility regions in differential-algebraic systems. IEEE Trans. Autom. Control **40**, 1992-2013 (1995)
20. Freedman, H, Rao, VSH: The trade-off between mutual interference and time lags in predator-prey systems. Bull. Math. Biol. **45**, 991-1004 (1983)
21. Erbe, LH, Freedman, H, Rao, VSH: Three-species food-chain models with mutual interference and time delays. Math. Biosci. **80**, 57-80 (1986)
22. Hale, JK: Theory of Functional Differential Equations. Springer, New York (1997)
23. Li, J, Zhang, Q, Yan, X, Spurgeon, SK: Integral sliding mode control for Markovian jump T-S fuzzy descriptor systems based on the super-twisting algorithm. IET Control Theory Appl. **11**, 1134-1143 (2017)
24. Zlateva, P: Variable structure control of nonlinear systems. Control Eng. Pract. **4**, 1023-1028 (1996)