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# Existence and multiple solutions for second-order $p$ -Laplacian difference equations

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## Abstract

In this paper, we investigate the solutions of boundary value problems for second-order  $p$ -Laplacian difference equations. By using the critical point theory, the existence and multiple results are obtained.

**MSC:** 39A10; 47J30; 58E05

**Keywords:** boundary value problems;  $p$ -Laplacian; difference equations; critical point theory

## 1 Introduction

In this paper, we consider the following second-order  $p$ -Laplacian difference equation:

$$\Delta(q_n \varphi_p(\Delta x_{n-1})) + f_n(x_n) = 0, \quad n \in \mathbf{Z}(1, k), \quad (1.1)$$

with boundary value conditions

$$\alpha x_0 - \beta \Delta x_0 = 0 = \gamma x_{k+1} + \sigma \Delta x_k, \quad (1.2)$$

where  $\Delta$  is the forward difference operator  $\Delta x_n = x_{n+1} - x_n$ ,  $\varphi_p(s)$  is the  $p$ -Laplacian operator  $\varphi_p(s) = |s|^{p-2}s$  ( $1 < p < \infty$ ),  $q_n$  is real-valued for each  $n \in \mathbf{Z}$ ,  $k$  is a given positive integer,  $\alpha, \beta, \gamma$  and  $\sigma$  are constants,  $f \in C(\mathbf{R}^2, \mathbf{R})$ . Boundary value problem (1.1) with (1.2) contains the following boundary value conditions:

$$\begin{aligned} x_0 = 0, & \quad x_{k+1} = 0; \\ x_0 = 0, & \quad \Delta x_k = 0; \\ \Delta x_0 = 0, & \quad x_{k+1} = 0; \end{aligned}$$

and

$$\Delta x_0 = 0, \quad \Delta x_k = 0.$$

Eq. (1.1) can be considered as a discrete analogue of the following second-order differential equation:

$$(q(t)\varphi_p(x'))' + f(t, x(t)) = 0, \quad t \in \mathbf{R}. \tag{1.3}$$

Eq. (1.3) includes the following equation:

$$(q(t)\psi(x'))' + f(t, x(t)) = 0, \quad t \in \mathbf{R}, \tag{1.4}$$

which has arose in the study of fluid dynamics, gas diffusion through porous media and adiabatic reactor [1]. Equations similar in structure to (1.4) arise in the study of homoclinic orbits [2–4] of differential equations [5].

In 2007, Chen and Fang [6] obtained a sufficient condition for the existence of periodic and subharmonic solutions of second-order  $p$ -Laplacian difference equation

$$\Delta(\varphi_p(\Delta x_{n-1})) + f_n(x_{n+1}, x_n, x_{n-1}) = 0, \quad n \in \mathbf{Z}, \tag{1.5}$$

by using the critical point theory. Moreover, Shi et al. [7] investigated the existence and multiplicity results for the Dirichlet boundary value problem (BVP) of (1.4) by using the mountain pass lemma in combination with variational techniques.

Liu et al. [8] in 2013 considered the existence of a nontrivial homoclinic orbit of the following second-order  $p$ -Laplacian difference equation:

$$\Delta(\varphi_p(\Delta x_{n-1})) - q_n(\varphi_p(\Delta x_n)) = f_n(x_{n+1}, x_n, x_{n-1}), \quad n \in \mathbf{Z}, \tag{1.6}$$

and gave some new results.

By using critical point theory, Chen and Tang [9] established some existence criteria to guarantee that the second-order discrete  $p$ -Laplacian system

$$\Delta(\varphi_p(\Delta x_{n-1})) - a_n|x_n|^{p-2}x_n + \nabla W_n(x_n) = 0, \quad n \in \mathbf{Z}, \tag{1.7}$$

has at least one homoclinic orbit.

In this paper, we investigate the solutions of boundary value problems (1.1) with (1.2) for a second-order  $p$ -Laplacian difference equation [6–25]. By using the critical point theory [26–28], the existence and multiple results are obtained. The proof is based on the variational methods and linking theorem. The motivation for the present work stems from the recent papers [22, 29].

Let

$$F(t, z) = \int_0^z f(t, s) ds \geq 0$$

and

$$q_{\min} = \min\{q_n : n \in \mathbf{Z}(1, k + 1)\}, \quad q_{\max} = \max\{q_n : n \in \mathbf{Z}(1, k + 1)\}.$$

Let  $B_\rho$  denote the open ball in  $E$  about 0 of radius  $\rho$ , and let  $\partial B_\rho$  denote its boundary.

This paper is divided into four parts. First of all, Section 2 states some basic notations and presents variational structure. Second, in Section 3, we shall state and prove our main results by using the variational methods. Finally, in Section 4, we shall give an example to illustrate the applicability of the main results.

### 2 Variational structure

Our main tool is the critical point theory. We shall establish a suitable variational structure to study the existence of boundary value problem (1.1) with (1.2). At first, we shall state some basic notations which will be used in the proofs of our main results.

Let  $\mathbf{R}^k$  be a real Euclidean space with dimension  $k$ . On the one hand, we define the inner product on  $\mathbf{R}^k$  as follows:

$$\langle x, y \rangle = \sum_{j=1}^k x_j y_j, \quad \forall x, y \in \mathbf{R}^k, \tag{2.1}$$

by which the norm  $\| \cdot \|$  can be induced by

$$\|x\| = \left( \sum_{j=1}^k x_j^2 \right)^{\frac{1}{2}}, \quad \forall x \in \mathbf{R}^k. \tag{2.2}$$

On the other hand, we define the norm  $\| \cdot \|_s$  on  $\mathbf{R}^k$  as follows:

$$\|x\|_s = \left( \sum_{j=1}^k |x_j|^s \right)^{\frac{1}{s}}$$

for all  $x \in \mathbf{R}^k$  and  $s > 1$ .

Since  $\|x\|_s$  and  $\|x\|_2$  are equivalent, there exist constants  $k_1, k_2$  such that  $k_2 \geq k_1 > 0$ , and

$$k_1 \|x\|_2 \leq \|x\|_s \leq k_2 \|x\|_2, \quad \forall x \in \mathbf{R}^k. \tag{2.3}$$

For boundary value problem (1.1) with (1.2), consider the functional  $J$  defined on  $\mathbf{R}^k$  as follows:

$$J(x) = \frac{1}{p} \sum_{n=1}^{k+1} q_n |\Delta x_{n-1}|^p - \sum_{n=1}^k F_n(x_n) + \varphi_p \left( \frac{\gamma}{\sigma} \right) \frac{q_{k+1}^p x_{k+1}^p}{p} + \varphi_p \left( \frac{\alpha}{\beta} \right) \frac{q_1^p x_0^p}{p},$$

where

$$x = \{x_n\}_{n=1}^k = (x_1, x_2, \dots, x_k)^*, \quad \alpha x_0 - \beta \Delta x_0 = 0 = \gamma x_{k+1} + \sigma \Delta x_k.$$

It is easy to see that  $J \in C^1(\mathbf{R}^k, \mathbf{R})$  and, for any  $x = \{x_n\}_{n=1}^k = (x_1, x_2, \dots, x_k)^*$ , by using  $\alpha x_0 - \beta \Delta x_0 = 0 = \gamma x_{k+1} + \sigma \Delta x_k$  and the summation by parts

$$\sum_{n=1}^k y_n \Delta x_{n-1} = y_k x_k - y_1 x_0 - \sum_{n=1}^k \Delta y_n x_n,$$

we can compute the partial derivative as

$$\frac{\partial J}{\partial x_n} = -\Delta(q_n \varphi_p(\Delta u_{n-1})) - f_n(x_n), \quad \forall n \in \mathbf{Z}(1, k).$$

Thus,  $x$  is a critical point of  $J$  on  $\mathbf{R}^k$  if and only if

$$\Delta(q_n \varphi_p(\Delta u_{n-1})) + f_n(x_n) = 0, \quad \forall n \in \mathbf{Z}(1, k).$$

We reduce the existence of boundary value problem (1.1) with (1.2) to the existence of critical points of  $J$  on  $\mathbf{R}^k$ . That is, the functional  $J$  is just the variational framework of boundary value problem (1.1) with (1.2).

Denote

$$W = \{(x_1, x_2, \dots, x_k)^* \in \mathbf{R}^k \mid x_n = c, c \in \mathbf{R}, n \in \mathbf{Z}(1, k)\},$$

and let  $V$  be the direct orthogonal complement of  $\mathbf{R}^k$  to  $W$ , i.e.,  $\mathbf{R}^k = V \oplus W$ .

Let

$$M = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

be a  $(k + 1) \times (k + 1)$  matrix.

It is easy to see that 0 is an eigenvalue of  $M$  and  $\xi = \frac{1}{\sqrt{k+1}}(1, 1, \dots, 1)^* \in \mathbf{R}^{k+1}$  is an eigenvector of  $P$  corresponding to 0. Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  be the other eigenvalues of  $M$ . Applying matrix theory, it is obvious that  $\lambda_j > 0$  for all  $j \in \mathbf{Z}(1, k)$ .

Let

$$\lambda_{\max} = \max\{\lambda_j \mid j = 1, 2, \dots, k\}, \tag{2.4}$$

$$\lambda_{\min} = \min\{\lambda_j \mid j = 1, 2, \dots, k\}. \tag{2.5}$$

For convenience, we identify  $x \in \mathbf{R}^k$  with  $x = (x_1, x_2, \dots, x_k)^*$ .

### 3 Main results

In this section, we shall state and prove our main results by using the variational methods.

**Theorem 3.1** *Suppose that the following assumptions are satisfied:*

- (B<sub>1</sub>)  $\alpha \geq 0, \beta > 0, \gamma \geq 0$  and  $\sigma > 0$ ;
- (q<sub>1</sub>) For any  $n \in \mathbf{Z}(1, k + 1), q_n > 0$ ;
- (F<sub>1</sub>) There exist constants  $c_1 > 0, c_2 > 0$  and  $\tau > p$  such that

$$F(t, y) \geq c_1 |y|^\tau - c_2, \quad \forall (t, y) \in \mathbf{R}^2.$$

Then boundary value problem (1.1) with (1.2) possesses at least one solution.

*Proof* For any  $x = (x_1, x_2, \dots, x_k)^* \in \mathbf{R}^k$ , combining with  $(F_1)$ , it is easy to see that

$$\begin{aligned} J(x) &= \frac{1}{p} \sum_{n=1}^{k+1} q_n |\Delta x_{n-1}|^p - \sum_{n=1}^k F_n(x_n) + \varphi_p \left( \frac{\gamma}{\sigma} \right) \frac{q_{k+1} x_{k+1}^p}{p} + \varphi_p \left( \frac{\alpha}{\beta} \right) \frac{q_1 x_0^p}{p} \\ &\leq \frac{q_{\max} 2^p}{p} \sum_{n=1}^{k+1} (|x_n|^p + |x_{n-1}|^p) - c_1 \sum_{n=1}^k |x_n|^\tau + c_2 k + \varphi_p \left( \frac{\gamma}{\sigma} \right) \frac{q_{\max}}{p} \|x\|_p^p \\ &\quad + \varphi_p \left( \frac{\alpha}{\beta} \right) \frac{q_{\max}}{p} \|x\|_p^p \\ &\leq \frac{3q_{\max} 2^p}{p} \|x\|_p^p - c_1 k_2^\tau \|x\|_\tau^\tau + c_2 k + \varphi_p \left( \frac{\gamma}{\sigma} \right) \frac{q_{\max}}{p} \|x\|_p^p + \varphi_p \left( \frac{\alpha}{\beta} \right) \frac{q_{\max}}{p} \|x\|_p^p \\ &\leq \left[ \frac{3q_{\max} 2^p}{p} + \varphi_p \left( \frac{\gamma}{\sigma} \right) \frac{q_{\max}}{p} + \varphi_p \left( \frac{\alpha}{\beta} \right) \frac{q_{\max}}{p} \right] k_2^p \|x\|^p - c_1 k_1^\tau \|x\|^\tau + c_2 k \rightarrow -\infty \end{aligned}$$

as  $\|x\| \rightarrow +\infty$ . By the continuity of  $J(x)$ , we have from the above inequality that there exist upper bounds of values of functional  $J$ . Classical calculus shows that  $J$  attains its maximal value at some point which is just the critical point of  $J$ , and the result follows. The proof of Theorem 3.1 is finished.  $\square$

**Theorem 3.2** *Suppose that the following hypotheses are satisfied:*

$(B_2)$   $\alpha = 0, \beta > 0, \gamma = 0$  and  $\sigma > 0$ ;

$(q_2)$  For any  $n \in \mathbf{Z}(1, k), q_n > 0$ ;

$(F_2)$  There exist constants  $\eta_1 > 0, a_1 \in (0, \frac{q_{\min} \lambda_{\min}^{\frac{p}{2}}}{p} (\frac{k_1}{k_2})^p)$  such that

$$F(t, y) \leq a_1 |y|^p, \quad \forall |y| \leq \eta_1;$$

$(F_3)$  There exist constants  $\eta_2 > 0, a_2 \in (\frac{\lambda_{\max}^{\frac{p}{2}} q_{\max}}{p} (\frac{k_2}{k_1})^p, +\infty), a_3 > 0$  such that

$$F(t, y) \geq a_2 |y|^p - a_3, \quad \forall |y| \geq \eta_2,$$

where  $\lambda_{\min}$  and  $\lambda_{\max}$  are constants which can be referred to in (2.4) and (2.5).

Then BVP (1.1) with (1.2) possesses at least two nontrivial solutions.

**Remark 3.1** By  $(F_3)$  it is easy to see that there exists a constant  $a_4 > 0$  such that

$$(F'_3) \quad F(t, y) \geq a_2 |y|^p - a_4, \quad \forall (t, y) \in \mathbf{R}^2.$$

**Lemma 3.1** (Linking theorem [27, 28]) *Let  $E$  be a real Banach space,  $E = E_1 \oplus E_2$ , where  $E_1$  is finite dimensional. Suppose that  $J \in C^1(E, \mathbf{R})$  satisfies the P.S. condition and*

$(J_1)$  There exist constants  $a > 0$  and  $\rho > 0$  such that  $J|_{\partial B_\rho \cap E_2} \geq a$ ;

$(J_2)$  There exist  $e \in \partial B_1 \cap E_2$  and a constant  $R_0 \geq \rho$  such that  $J|_{\partial Q} \leq 0$ , where  $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{se | 0 < s < R_0\}$ .

Then  $J$  possesses a critical value  $c \geq a$ , where

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

and  $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = I\}$ , where  $I$  denotes the identity operator.

**Lemma 3.2** (Clark theorem [27, 28]) *Let  $E$  be a real Banach space,  $J \in C^1(E, \mathbf{R})$ , with  $J$  being even, bounded from below and satisfying the P.S. condition. Suppose  $J(\theta) = 0$ , there is a set  $G \subset E$  such that  $G$  is homeomorphic to  $S^{n-1}$  ( $n-1$  dimension unit sphere) by an odd map, and  $\sup_G J < 0$ . Then  $J$  has at least  $n$  distinct pairs of nonzero critical points.*

**Lemma 3.3** *Suppose that  $(B_2)$ ,  $(q_2)$ ,  $(F_2)$  and  $(F_3)$  are satisfied. Then the functional  $J$  satisfies the P.S. condition.*

*Proof* It follows from  $(B_2)$  that

$$J(x) = \frac{1}{p} \sum_{n=1}^{k+1} q_n |\Delta x_{n-1}|^p - \sum_{n=1}^k F_n(x_n).$$

Let  $\{x^{(l)}\}_{l \in \mathbf{N}} \subset \mathbf{R}^k$  be such that  $\{J(x^{(l)})\}_{l \in \mathbf{N}}$  is bounded and  $J'(x^{(l)}) \rightarrow 0$  as  $l \rightarrow \infty$ . Then there exists a positive constant  $K$  such that

$$-K \leq |J(x^{(l)})| \leq K, \quad \forall l \in \mathbf{N}.$$

By  $(F'_3)$ , for any  $\{x^{(l)}\}_{l \in \mathbf{N}} \subset \mathbf{R}^k$ ,

$$\begin{aligned} -K &\leq J(x^{(l)}) = \frac{1}{p} \sum_{n=1}^{k+1} q_n |\Delta x_{n-1}^{(l)}|^p - \sum_{n=1}^k F_n(x_n^{(l)}) \\ &\leq \frac{q_{\max} k_2^p}{p} \left[ \sum_{n=1}^{k+1} (\Delta x_{n-1}^{(l)})^2 \right]^{\frac{p}{2}} - \sum_{n=1}^k [a_2 |x_n^{(l)}|^p - a_4] \\ &\leq \frac{q_{\max} k_2^p}{p} \|x^{(l)}\|^p - a_2 k_1^p \|x^{(l)}\|^p + a_4 k \\ &= \left( \frac{q_{\max} k_2^p \lambda_{\max}^{\frac{p}{2}}}{p} - a_2 k_1^p \right) \|x^{(l)}\|^p + a_4 k. \end{aligned}$$

Thus, we have

$$\left( \frac{q_{\max} k_2^p \lambda_{\max}^{\frac{p}{2}}}{p} - a_2 k_1^p \right) \|x^{(l)}\|^p \leq K + a_4 k.$$

From  $a_2 \in (\frac{p}{\lambda_{\max} q_{\max}} (\frac{k_2}{k_1})^p, +\infty)$  and the above inequality, it is easy to see that  $\{x^{(l)}\}_{l \in \mathbf{N}}$  is a bounded sequence in  $\mathbf{R}^k$ . Therefore,  $\{x^{(l)}\}_{l \in \mathbf{N}}$  possesses a convergent subsequence in  $\mathbf{R}^k$  and the result follows. Lemma 3.3 is proved.  $\square$

*Proof of Theorem 3.2* By Lemma 3.3, we have that  $J$  is bounded from above on  $\mathbf{R}^k$ . Let  $c_0 = \sup_{x \in \mathbf{R}^k} J(x)$ . It comes from the proof of Lemma 3.3 that

$$\lim_{\|x\| \rightarrow +\infty} J(x) = -\infty.$$

This implies that  $-J(x)$  is coercive. From the continuity of  $J(x)$ , there exists  $\bar{x} \in \mathbf{R}^k$  such that  $J(\bar{x}) = c_0$ . It is easy to see that  $\bar{x}$  is a critical point of  $J$ .

For any  $x \in V, \|x\| \leq \rho$ , it follows from  $(F_2)$  that

$$\begin{aligned} J(x) &= \frac{1}{p} \sum_{n=1}^{k+1} q_n |\Delta x_{n-1}|^p - \sum_{n=1}^k F_n(x_n) \\ &\geq \frac{q_{\min} k_1^p}{p} \left[ \sum_{n=1}^{k+1} (\Delta x_{n-1})^2 \right]^{\frac{p}{2}} - a_1 \sum_{n=1}^k |x_n|^p \\ &\geq \frac{q_{\min} k_1^p \lambda_{\min}^{\frac{p}{2}}}{p} \|x\|^p - a_1 k_2^p \|x\|^p \\ &= \left( \frac{q_{\min} k_1^p \lambda_{\min}^{\frac{p}{2}}}{p} - a_1 k_2^p \right) \|x\|^p. \end{aligned}$$

Thus,  $c_0 > 0$ . Taking

$$a = \left( \frac{q_{\min} k_1^p \lambda_{\min}^{\frac{p}{2}}}{p} - a_1 k_2^p \right) \rho^p,$$

we have

$$J(x) \geq a, \quad \forall x \in V \cap \partial B_\rho. \tag{3.1}$$

This implies that  $J$  satisfies the condition  $(J_1)$  of the linking theorem.

For all  $x \in W$ , it comes from  $\sum_{n=1}^{k+1} |\Delta x_{n-1}|^p = 0$  that

$$J(x) = \frac{1}{p} \sum_{n=1}^{k+1} |\Delta x_{n-1}|^p - \sum_{n=1}^k F_n(x_n) = - \sum_{n=1}^k F_n(x_n) \leq 0.$$

Therefore, the critical point  $\bar{x}$  of  $J$  corresponding to the critical value  $c_0$  is a nontrivial solution of BVP (1.1) with (1.2).

It follows from Lemma 3.3 that  $J$  satisfies the P.S. condition on  $\mathbf{R}^k$ . So it remains to verify the condition  $(J_2)$ .

Take  $e \in \partial B_1 \cap V$ , for any  $z \in W$  and  $r \in \mathbf{R}$ , let  $x = re + z$ . Then

$$\begin{aligned} J(x) &= \frac{1}{p} \sum_{n=1}^{k+1} q_n |\Delta(re_{n-1} + z_{n-1})|^p - \sum_{n=1}^k F_n(re_n + z_n) \\ &= \frac{1}{p} \sum_{n=1}^{k+1} q_n |\Delta(re_{n-1})|^p - \sum_{n=1}^k (a_2 |re_n + z_n|^p - a_4) \\ &\leq \frac{q_{\max} k_2^p}{p} \left( \sum_{n=1}^{k+1} |\Delta(re_{n-1})|^2 \right)^{\frac{p}{2}} - a_2 k_1^p \left( \sum_{n=1}^k |re_n + z_n|^2 \right)^{\frac{p}{2}} + a_4 k \\ &\leq \frac{q_{\max} k_2^p \lambda_{\max}^{\frac{p}{2}}}{p} r^p - a_2 k_1^p r^p - a_2 k_1^p \|z\|^p + a_4 k \\ &= \left( \frac{q_{\max} \lambda_{\max}^{\frac{p}{2}} k_2^p}{p} - a_2 k_1^p \right) r^p - a_2 k_1^p \|z\|^p + a_4 k \\ &\leq -a_2 k_1^p \|z\|^p + a_4 k. \end{aligned}$$

Therefore, for any  $x \in \partial Q$ , where  $Q = (\bar{B}_{K_1} \cap W) \oplus \{re | 0 < r < K_1\}$ , there exists a constant  $K_1 > \rho > 0$  such that  $J(x) \leq 0$ . By the linking theorem,  $J$  possesses a critical value  $c \geq \rho > 0$ , where  $c = \inf_{h \in \Gamma} \sup_{x \in Q} J(h(x))$  and  $\Gamma = \{h \in C(\bar{Q}, \mathbf{R}^k) \mid h|_{\partial Q} = I\}$ .

The rest of the proof is similar to that of [7], Theorem 1.1; for the sake of simplicity, we omit its proof. □

**Theorem 3.3** *Suppose that  $(B_2), (q_2), (F_2), (F_3)$  and the following hypothesis are satisfied:*

(f)  $f(t, -y) = -f(t, y)$  for all  $(t, y) \in \mathbf{R}^2$ .

*Then BVP (1.1) with (1.2) possesses at least  $s$  distinct pairs of nontrivial solutions, where  $s$  is the dimension of  $V$  which can be referred to in Section 2.*

**Remark 3.2** The proof of Theorem 3.3 is similar to that of Theorem 4.4 in [12], and we omit its proof.

#### 4 An example

As an application of Theorem 3.2, we give an example to illustrate our main result.

**Example 4.1** Consider the BVP

$$\Delta(\varphi_p(\Delta x_{n-1})) + \mu x_n^{\mu-1} = 0, \quad n \in \mathbf{Z}(1, k) \tag{4.1}$$

with boundary value conditions (1.2), where  $\mu > p$ . We have

$$q_n \equiv 1$$

and

$$F_n(x_n) = x_n^\mu.$$

It is easy to verify that all the conditions of Theorem 3.2 are satisfied and then BVP (4.1) with (1.2) possesses at least two nontrivial solutions.

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#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors read and approved the final manuscript.

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