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Solvability for a class of evolution equations of fractional order with nonlocal conditions on the half-line

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Abstract

In this paper, we get a new form equivalent integral equation for a class of evolution equations of fractional order with nonlocal conditions on the half-line. With the aid of it, the uniqueness of the mild solution is obtained by the Banach contraction theorem. Also, we present the existence and uniqueness theorem of positive mild solutions by the monotone iterative method without assumption of lower and upper solutions.

Keywords: fractional order; abstract evolution equations; mild solutions; monotone iterative method

1 Introduction

In this paper, we deal with the following functional differential abstract equation of fractional order with nonlocal conditions in the Banach space E :

$$\begin{cases} {}^C D_{0+}^q u(t) = Au(t) + f(t, u(r(t))), & t \in (0, +\infty), \\ u(0) = \sum_{i=1}^{\infty} \sigma_i u(\tau_i), \end{cases} \quad (1)$$

where ${}^C D_{0+}^q$ is the Caputo fractional derivative, $0 < q < 1$, $\sigma_i > 0$, $\tau_i > 0$ ($i = 1, 2, \dots$), A is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t \geq 0}$ of operators on Banach E , $r \in C[0, \infty)$, $r(t) \geq 0$, and $f: [0, +\infty) \times E \rightarrow E$ satisfies certain conditions.

If $r(t) = t$, then problem (1) can be written as

$$\begin{cases} {}^C D_{0+}^q u(t) = Au(t) + f(t, u(t)), & t \in (0, +\infty), \\ u(0) = \sum_{i=1}^{\infty} \sigma_i u(\tau_i). \end{cases} \quad (2)$$

Fractional calculus, a generalization of the ordinary differentiation and integration, has played an important role in science, biology, physics, economy, engineering, and other fields (see [1–3]). A large number of phenomena and processes in the real world are described by differential equation of fractional order, due to the fact of its various applications in many areas. Today, there are a lot of researchers committed to investigating the

fractional differential equations. For more general theory of fractional differential equations, we refer readers to the papers [4–23] and the references given therein.

In [14], El-Sayed obtained the existence and uniqueness results based on an equivalent integral equation for a class of fractional evolution equations, which was described in terms of some probability densities by the method of Laplace transform for the first time. Since then, according to El-Sayed's results, most papers have been working on the evolution equations of fractional order, see [15–23].

Now, one branch of the studies on fractional differential equations is devoted to investigating the fractional evolution equation with nonlocal conditions, which is a valuable tool to describe the physics phenomena. Nonlocal conditions were initiated by Byszewski [24] when he gave the existence and uniqueness results of the mild solutions for the nonlocal Cauchy problems. In [25], Byszewski and Lakshmikantham indicated that the nonlocal condition can be more useful than the standard initial condition to model some physical phenomena.

In [26], Byszewski dealt with the following functional-differential abstract nonlocal Cauchy problem of integer order in a general Banach space:

$$\begin{cases} u'(t) = f(t, u(t), u(a(t))), & t \in [t_0, t_0 + T], \\ u(t_0) + \sum_{k=1}^p c_k u(t_k) = x_0. \end{cases}$$

Applying the Banach contraction theorem and a modified Picard method, the existence and uniqueness of a classical solution is given.

In [27], Chen studied a class of nonlocal evolution equations

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t \geq 0, \\ u(0) = \sum_{k=1}^{\infty} c_k u(t_k). \end{cases}$$

With the help of the monotone iterative method, the existence and uniqueness of the mild positive solutions were obtained in the paper.

As far as we know, recently, evolution equations of fractional order have attracted increasing attention, and we refer to the papers [13–23] and the references therein. However, many of those papers on the existence of solutions of fractional evolution equations are on the finite interval, and the existence results on the half-line are still few.

Motivated by the papers [14, 15, 26, 27], in this paper, we study the fractional differential equations (1) and (2) with nonlocal conditions on the unbounded domains. Here, we give a corrected form of the equivalent integral equation of (1) by similar methods as the ones used in [14, 15], which is different from those obtained in the existing literature. With the aid of the Banach contraction theorem, the uniqueness of the mild solution of problem (1) is given. Besides, employing the monotone iterative method, without the assumption of lower and upper solutions, we present some new results on the existence of positive mild solutions for the abstract fractional differential equations (2).

For the sake of convenience, we denote

$$\chi := \frac{1}{\Gamma(q+1)} \frac{1}{1 - \sum_{i=1}^{\infty} \sigma_i}.$$

Our main results are as follows.

Theorem 1.1 Let E be a Banach space and $\{T(t)\}_{t \geq 0}$ be a uniformly exponentially stable C_0 -semigroup with the growth bound ω_0 ($\omega_0 < 0$), and A is the infinitesimal generator of $\{T(t)\}_{t \geq 0}$. Let $0 < \sum_{i=1}^{\infty} \sigma_i < 1$. Let $r \in C[0, \infty)$ and $r(t) \geq 0$. If $f(t, u)$ satisfies the following condition:

(F) There exist two constants $\omega^* \in (0, |\omega_0|]$ and \mathcal{M} satisfying $\mathcal{M} < \omega^* / \chi$ such that

$$\|f(t, x) - f(t, y)\|_{\omega^*} \leq \mathcal{M} \|x - y\|_{\omega^*}, \quad \forall x, y \in E,$$

then problem (1) has a unique mild solution in $BC(J, E)$. Besides, if $f(t, \theta) \neq \theta$, the unique mild solution in $B(J, E)$ is nontrivial.

Theorem 1.2 Let E be a Banach space, and P is its positive normal with N as the normal constant. Let $\{T(t)\}_{t \geq 0}$ be a uniformly exponentially stable C_0 -semigroup with the growth bound ω_0 ($\omega_0 < 0$), and A is the infinitesimal generator of $\{T(t)\}_{t \geq 0}$. Let $0 < \sum_{i=1}^{\infty} \sigma_i < 1$. Provided that $f(t, u) : J \times E \rightarrow E$ is continuous and $f_0(t) := f(t, \theta) \geq \theta$ is bounded on J , if $f(t, u)$ satisfies the following conditions:

(H1) There exist two constants $d_1 < -\omega_0$ and $d_2 > \max\{-d_1, \omega_0\}$ such that for

$$-d_2(y - x) \leq f(t, y) - f(t, x) \leq d_1(y - x), \quad \theta \leq x \leq y;$$

(H2)

$$0 < \frac{d_1 + d_2}{d_2 - \omega_0} < \frac{1}{\chi},$$

then problem (2) has a unique positive mild solution in $BC(J, E)$.

Remark 1.1 We give a comparison between the two methods. First, the existence of positive mild solution is given in Theorem 1.1. However, Theorem 1.2 only shows the existence of mild solution, and it cannot tell if it is positive. Second, the function $f(t, x)$ should be a monotone one in Theorem 1.2, but this condition is not necessary in Theorem 1.1. Given the above, these two theorems are complementary.

The rest of the paper is organized as follows. In Section 2, we introduce the definitions of the fractional integral and fractional derivative, some results about fractional differential equations and some useful preliminaries. In Section 3, we present the proofs of our main results by the Banach contraction theorem and the monotone iterative method, respectively. Then an example is given in Section 4 to demonstrate the application of our results.

2 Preliminaries

First of all, we present some fundamental facts on the fractional calculus theory which we will use in the next section.

Definition 2.1 ([1–3]) The Riemann-Liouville fractional integral of order $\nu > 0$ of a function $h : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\nu} h(t) = D_{0+}^{-\nu} h(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s) ds, \quad (3)$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([1–3]) The Caputo fractional derivative of order $\nu > 0$ of a continuous function $h : (0, \infty) \rightarrow \mathbb{R}$ is given by

$${}^C D_{0+}^{\nu} h(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} h^{(n)}(s) ds, \quad (4)$$

where $n = [\nu] + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([1, 3]) Assume that ${}^C D_{0+}^{\nu} h(t) \in L^1(0, +\infty)$, $\nu > 0$. Then we have

$$I_{0+}^{\nu} {}^C D_{0+}^{\nu} h(t) = h(t) + C_1 + C_2 t + \cdots + C_N t^{N-1}, \quad t > 0, \quad (5)$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to ν .

If h is an abstract function with values in the Banach space E , then the integrals appearing in Definition 2.1, Definition 2.2 and Lemma 2.1 are taken in Bochner's sense. And a measurable function h is Bochner integrable if the norm of h is Lebesgue integrable.

Now let us recall some definitions and standard facts about the cone.

Let P be a cone in the ordered Banach space E , which defines a partial order on E by $x \leq y$ if and only if $y - x \in P$. P is normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ is the zero element of the Banach space E . The infimum of all N with the property above is called the normal constant of P . For more details on the cone P , we refer the readers to [28, 29].

Throughout the paper, we set E be an ordered Banach space with the norm $\|\cdot\|$ and the partial order ' \leq '. Let $P = \{x \in E | x \geq \theta\}$ be a positive cone, which is normal with normal constant N . Let $J = [0, +\infty)$. Set

$$BC(J, E) = \{u(t) | u(t) \text{ is continuous and bounded on } J\}.$$

Obviously, $BC(J, E)$ is a Banach space with the norm $\|u\|_b = \sup_{t \in J} \|u(t)\|$. Let

$$P_B = \{u \in BC(J, E) | u(t) \geq \theta, t \in J\}.$$

It is easy to see that P_B is also normal with the same normal constant N of the cone P . Besides, $BC(J, E)$ is also an ordered Banach space with the partial order ' \leq ' induced by the positive cone P_B (without confusion, we denote by ' \leq ' the partial order on both E and $BC(J, E)$).

We denote by $[\nu, w]$ the order interval $\{u \in P_B | \nu \leq u \leq w, \nu, w \in BC(J, E)\}$ on $BC(J, E)$, and use $[\nu(t), w(t)]$ to denote the order interval $\{z \in E | \nu(t) \leq z \leq w(t)\}$ on E for $t \in J$.

Next, we give some facts about the semigroups of linear operators. These results can be found in [30, 31].

For a strongly continuous semigroup (i.e., C_0 -semigroup) $\{T(t)\}_{t \geq 0}$, the infinitesimal generator of $\{T(t)\}_{t \geq 0}$ is defined by

$$Ax = \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}, \quad x \in E.$$

We denote by $D(A)$ the domain of A , that is,

$$D(A) = \left\{ x \in E \mid \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

Lemma 2.2 ([30, 31]) *Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup, then there exist constants $C \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Ce^{\omega t}$, $t \geq 0$.*

Lemma 2.3 ([30, 31]) *A linear operator A is the infinitesimal generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ if and only if*

- (i) *A is closed and $\overline{D(A)} = E$.*
- (ii) *The resolvent set $\rho(A)$ of A contains \mathbb{R}^+ and, for every $\lambda > 0$, we have*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda},$$

where

$$R(\lambda, A) := (\lambda I - A)^{-1} = \int_0^{+\infty} e^{-\lambda t} T(t)x \, dt, \quad x \in E.$$

Definition 2.3 ([30, 31]) A C_0 -semigroup $\{T(t)\}_{t \geq 0}$ is said to be uniformly exponentially stable if $\omega_0 < 0$, where ω_0 is the growth bound of $\{T(t)\}_{t \geq 0}$, which is defined by

$$\omega_0 = \inf \left\{ \omega \in \mathbb{R} \mid \exists C \geq 1 \text{ such that } \|T(t)\| \leq Ce^{\omega t}, t \geq 0 \right\}.$$

Definition 2.4 ([27]) A C_0 -semigroup $\{T(t)\}_{t \geq 0}$ is said to be positive on E , if order inequality $T(t)x \geq \theta$, $x \in E$ and $t \geq 0$.

According to Lemma 2.2 and Definition 2.3, if $\{T(t)\}_{t \geq 0}$ is a uniformly exponentially stable C_0 -semigroup with the growth bound ω_0 , then for any $\omega \in (0, |\omega_0|]$, there exists a constant $C \geq 1$ such that $\|T(t)\| \leq Ce^{\omega t}$, $t \geq 0$. Now, we define a norm in E by

$$\|x\|_\omega = \sup_{t \geq 0} \|e^{\omega t} T(t)x\|.$$

Evidently, we have $\|x\| \leq \|x\|_\omega \leq C\|x\|$, that is to say, the norm $\|\cdot\|_\omega$ and $\|\cdot\|$ are equivalent. We denote by $\|T(t)\|_\omega$ the norm of $T(t)$ induced by the norm $\|\cdot\|_\omega$. Then we obtain

$$\|T(t)\|_\omega \leq e^{-\omega t}, \quad t \geq 0. \quad (6)$$

Also, we can define the equivalent norm on $BC(J, E)$ by

$$\|u\|_{b\omega} = \sup_{t \in J} \|u(t)\|_{\omega}, \quad u \in BC(J, E).$$

Obviously, if $u(t) \equiv u_0$, $t \in J$, $u_0 \in E$, then we have

$$\|u\|_{b\omega} = \|u_0\|_{b\omega} = \|u\|_{\omega}.$$

Consider the one-sided stable probability density [15, 16, 32]

$$\psi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad \theta \in (0, \infty),$$

where $0 < q < 1$. By Remark 2.8 in [16], for $0 \leq \gamma \leq 1$, one has

$$\int_0^{\infty} \theta^{-q\gamma} \psi_q(\theta) d\theta = \frac{\Gamma(1+\gamma)}{\Gamma(1+q\gamma)}. \quad (7)$$

From [15, 16, 32], the Laplace transform of the one-sided stable probability density $\psi_q(\theta)$ is given by

$$\mathcal{L}[\psi_q(\theta)] = \int_0^{\infty} e^{-\lambda\theta} \psi_q(\theta) d\theta = e^{-\lambda^q}, \quad 0 < q < 1. \quad (8)$$

In the following, we assume that $\{T(t)\}_{t \geq 0}$ is a uniformly exponentially stable C_0 -semigroup with the growth bound ω_0 , and $\omega \in (0, |\omega_0|)$.

Lemma 2.4 Define an operator

$$\begin{aligned} (\Upsilon h)(t) &:= \int_0^1 \int_0^{\infty} \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} \tau^{-q} (1-\tau)^{q-1} T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right) h(s) d\theta d\tau, \\ h &\in BC(J, E). \end{aligned} \quad (9)$$

Then $\Upsilon : BC(J, E) \longrightarrow BC(J, E)$ and

$$\|\Upsilon h\|_{b\omega} \leq \|h\|_{b\omega}.$$

In particular, if $h(t) \equiv x$, $t \in J$, $x \in E$, then

$$\|\Upsilon x\|_{b\omega} \leq \|x\|_{\omega}.$$

Proof Since

$$\begin{aligned} &\|(\Upsilon h)(t)\|_{\omega} \\ &\leq \int_0^1 \int_0^{\infty} \frac{q}{\Gamma(1-q)} \tau^{-q} (1-\tau)^{q-1} \frac{\psi_q(\theta)}{\theta^q} \left\| T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right) h(s) \right\|_{\omega} d\theta d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 \int_0^\infty \frac{q}{\Gamma(1-q)} \tau^{-q} (1-\tau)^{q-1} \frac{\psi_q(\theta)}{\theta^q} \left\| T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right) \right\|_\omega \|h(s)\|_\omega d\theta d\tau \\
&\leq \int_0^1 \int_0^\infty \frac{q}{\Gamma(1-q)} \tau^{-q} (1-\tau)^{q-1} \frac{\psi_q(\theta)}{\theta^q} e^{-\omega(\frac{t^q(1-\tau)^q}{\theta^q})} \|h\|_{b\omega} d\theta d\tau \\
&\leq \frac{q}{\Gamma(1-q)} \|h\|_{b\omega} \int_0^1 \tau^{-q} (1-\tau)^{q-1} \left(\int_0^\infty \frac{\psi_q(\theta)}{\theta^q} d\theta \right) d\tau \\
&\leq \|h\|_{b\omega},
\end{aligned}$$

then the proof is finished. \square

Lemma 2.5 Define a linear operator $\mathcal{R} : BC(J, E) \longrightarrow BC(J, E)$ as

$$\begin{aligned}
\mathcal{R}h &:= \sum_{i=1}^{\infty} \sigma_i [(\Upsilon h)(\tau_i)] \\
&= \sum_{i=1}^{\infty} \sigma_i \int_0^1 \int_0^\infty \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} \tau^{-q} (1-\tau)^{q-1} T\left(\tau_i^q \frac{(1-\tau)^q}{\theta^q}\right) x d\theta d\tau.
\end{aligned}$$

Then \mathcal{R} is bounded and $\|\mathcal{R}\|_\omega \leq \sum_{i=1}^{\infty} \sigma_i$. Besides, if $0 < \sum_{i=1}^{\infty} \sigma_i < 1$, then $(I - \mathcal{R})^{-1}$ is a linear bounded operator and

$$\|(I - \mathcal{R})^{-1}\|_\omega \leq \frac{1}{1 - \sum_{i=1}^{\infty} \sigma_i}.$$

Proof In view of Lemma 2.4, one can get

$$\begin{aligned}
\|\mathcal{R}h\|_\omega &= \left\| \sum_{i=1}^{\infty} \sigma_i [(\Upsilon h)(\tau_i)] \right\|_\omega \leq \sum_{i=1}^{\infty} \sigma_i \|[(\Upsilon h)(\tau_i)]\|_\omega \leq \sum_{i=1}^{\infty} \sigma_i \|\Upsilon h\|_{b\omega} \\
&\leq \left(\sum_{i=1}^{\infty} \sigma_i \right) \|h\|_\omega.
\end{aligned}$$

Hence, \mathcal{R} is bounded and $\|\mathcal{R}\|_\omega \leq \sum_{i=1}^{\infty} \sigma_i$. \square

Lemma 2.6 Set

$$(\mathcal{W}h)(t) = \int_0^t \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q} (t-s)^{q-1} T\left(\frac{(t-s)^q}{\theta^q}\right) h(s) d\theta ds, \quad h \in BC(J, E).$$

Then $\mathcal{W} : BC(J, E) \longrightarrow BC(J, E)$ and

$$\|(\mathcal{W}h)(t)\|_\omega \leq \frac{1}{\omega \Gamma(q+1)} \|h\|_{b\omega}; \quad \|(\mathcal{W}h)\|_{b\omega} \leq \frac{1}{\omega \Gamma(q+1)} \|h\|_{b\omega}.$$

Proof Since

$$\begin{aligned}
(\mathcal{W}h)(t) &= \int_0^t \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q} (t-s)^{q-1} T\left(\frac{(t-s)^q}{\theta^q}\right) h(s) d\theta ds \\
&= \int_0^1 \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q} t^q (1-\tau)^{q-1} T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right) h(t\tau) d\theta d\tau,
\end{aligned}$$

then

$$\begin{aligned}
 & \|(\mathcal{W}h)(t)\|_{\omega} \\
 & \leq \int_0^1 \int_0^{\infty} q \frac{\psi_q(\theta)}{\theta^q} t^q (1-\tau)^{q-1} \left\| T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right) \right\|_{\omega} \|h(t\tau)\|_{\omega} d\theta d\tau \\
 & \leq \int_0^1 \int_0^{\infty} q \frac{\psi_q(\theta)}{\theta^q} t^q (1-\tau)^{q-1} e^{-\omega \frac{t^q(1-\tau)^q}{\theta^q}} \|h(t\tau)\|_{\omega} d\theta d\tau \\
 & \leq \frac{1}{\omega} \|h\|_{b\omega} \int_0^{\infty} \left[\left(\int_0^1 e^{-\omega \left(\frac{t^q(1-\tau)^q}{\theta^q} \right)} d\left(-\omega \frac{t^q(1-\tau)^q}{\theta^q}\right) \right) \frac{\psi_q(\theta)}{\theta^q} \right] d\theta \\
 & = \frac{1}{\omega} \|h\|_{b\omega} \int_0^{\infty} (1 - e^{-\omega \frac{t^q}{\theta^q}}) \frac{\psi_q(\theta)}{\theta^q} d\theta \\
 & \leq \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega}.
 \end{aligned}$$

Therefore,

$$\|(\mathcal{W}h)\|_{b\omega} \leq \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega}. \quad \square$$

Lemma 2.7 Let $h \in BC(J, E)$ and $u_0 \in D(A)$. Assume that $0 < \sum_{i=1}^{\infty} \sigma_i < 1$. Then the linear fractional evolution equation

$$\begin{cases} {}^C D_{0+}^q u(t) = Au(t) + h(t), & t \in (0, +\infty), \\ u(0) = u_0, \end{cases} \quad (10)$$

has a unique solution $u \in BC(J, E)$ of the following form:

$$\begin{aligned}
 u(t) &= (\Upsilon u_0)(t) + (\mathcal{W}h)(t) \\
 &= \int_0^1 \int_0^{\infty} \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} \tau^{-q} (1-\tau)^{q-1} T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right) u_0 d\theta d\tau \\
 &\quad + \int_0^t \int_0^{\infty} q \frac{\psi_q(\theta)}{\theta^q} (t-s)^{q-1} T\left(\frac{(t-s)^q}{\theta^q}\right) h(s) d\theta ds.
 \end{aligned} \quad (11)$$

Proof In view of Definitions 2.1, 2.2 and Lemma 2.1, equation (10) can be rewritten by the equivalent integral equation as follows:

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Au(s) + h(s)] ds. \quad (12)$$

Denote by $U(\lambda)$ and $H(\lambda)$ the Laplace transform of $u(t)$ and $h(t)$, respectively, using a similar method as that in [13, 14], then with the Laplace transform, we can rewrite the above equation as

$$U(\lambda) = \frac{1}{\lambda} u_0 + \frac{1}{\lambda^q} AU(\lambda) + \frac{1}{\lambda^q} H(\lambda), \quad \lambda > 0. \quad (13)$$

Then one has

$$(\lambda^q I - A)U(\lambda) = \lambda^{q-1} u_0 + H(\lambda).$$

By virtue of (8) and Lemma 2.3, we obtain

$$\begin{aligned}
 U(\lambda) &= (\lambda^q I - A)^{-1} \lambda^{q-1} u_0 + (\lambda^q I - A)^{-1} H(\lambda) \\
 &= \lambda^{q-1} \int_0^\infty e^{-\lambda^q s} T(s) u_0 ds + \int_0^\infty e^{-\lambda^q s} T(s) H(\lambda) ds \\
 &= \lambda^{q-1} \int_0^\infty \int_0^\infty e^{-\lambda s^{1/q} \theta} \psi_q(\theta) T(s) u_0 d\theta ds \\
 &\quad + \int_0^\infty \int_0^\infty e^{-\lambda s^{1/q} \theta} \psi_q(\theta) T(s) H(\lambda) d\theta ds \\
 &= \lambda^{q-1} \int_0^\infty e^{-\lambda t} \left[\int_0^\infty q \frac{t^{q-1}}{\theta^q} \psi_q(\theta) T\left(\frac{t^q}{\theta^q}\right) u_0 d\theta \right] dt \\
 &\quad + \int_0^\infty e^{-\lambda t} \left[\int_0^t \int_0^\infty q \frac{(t-s)^{q-1}}{\theta^q} \psi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) h(s) d\theta ds \right] dt.
 \end{aligned}$$

By the definition of Laplace transforms and the convolution theorem, applying Lemma 2.6 and the inverse Laplace transforms on the above equations, then one can derive that

$$\begin{aligned}
 u(t) &= \mathcal{L}^{-1}[\lambda^{q-1}] * \mathcal{L}^{-1} \left[\int_0^\infty e^{-\lambda t} \left[\int_0^\infty q \frac{t^{q-1}}{\theta^q} \psi_q(\theta) T\left(\frac{t^q}{\theta^q}\right) u_0 d\theta \right] dt \right] \\
 &\quad + \mathcal{L}^{-1} \left[\int_0^\infty e^{-\lambda t} \left(\int_0^t \int_0^\infty q \frac{(t-s)^{q-1}}{\theta^q} \psi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) h(s) d\theta ds \right) dt \right] \\
 &= \frac{t^{-q}}{\Gamma(1-q)} * \left[\int_0^\infty q \frac{t^{q-1}}{\theta^q} \psi_q(\theta) T\left(\frac{t^q}{\theta^q}\right) u_0 d\theta \right] \\
 &\quad + \int_0^t \int_0^\infty q \frac{(t-s)^{q-1}}{\theta^q} \psi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) h(s) d\theta ds \\
 &= \int_0^t \int_0^\infty \frac{q}{\Gamma(1-q)} s^{-q} \frac{(t-s)^{q-1}}{\theta^q} \psi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) u_0 d\theta ds + (\mathcal{W}h)(t) \\
 &= (\Upsilon u_0)(t) + (\mathcal{W}h)(t).
 \end{aligned}$$

Since

$$\|(\Upsilon u_0)(t) + (\mathcal{W}h)(t)\|_\omega \leq \|(\Upsilon u_0)(t)\|_\omega + \|(\mathcal{W}h)(t)\|_\omega \leq \|u_0\|_\omega + \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega},$$

then

$$\|\Upsilon u_0 + \mathcal{W}h\|_{b\omega} \leq \|u_0\|_\omega + \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega}.$$

Hence, $u \in BC(J, E)$. Then we complete the proof. \square

3 Main results

In this section, we will present the existence theorem for the abstract fractional differential equation on the half-line. In order to prove our main result, we need the following facts and lemmas.

Lemma 3.1 *Let $h \in BC(J, E)$ and $u_0 \in D(A)$. Let $0 < \sum_{i=1}^{\infty} \sigma_i < 1$. Then the linear fractional evolution equation*

$$\begin{cases} {}^C D_{0+}^q u(t) = Au(t) + h(t), & t \in (0, +\infty), \\ u(0) = \sum_{i=1}^{\infty} \sigma_i u(\tau_i), \end{cases} \quad (14)$$

has a unique solution $u \in BC(J, E)$ of the following form:

$$\begin{aligned} u(t) &= (\Theta_A h)(t) \\ &:= \left(\Upsilon \left[(I - \mathcal{R})^{-1} \sum_{i=1}^{\infty} \sigma_i (\mathcal{W}h)(\tau_i) \right] \right)(t) + (\mathcal{W}h)(t) \\ &= \int_0^1 \int_0^\infty \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} \tau^{-q} (1-\tau)^{q-1} T \left(t^q \frac{(1-\tau)^q}{\theta^q} \right) \\ &\quad \times \left[(I - \mathcal{R})^{-1} \left(\sum_{i=1}^{\infty} \sigma_i (\mathcal{W}h)(\tau_i) \right) \right] d\theta d\tau \\ &\quad + \int_0^t \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q} (t-s)^{q-1} T \left(\frac{(t-s)^q}{\theta^q} \right) h(s) d\theta ds. \end{aligned} \quad (15)$$

Also, Θ_A is a linear operator on the Banach space $BC(J, E)$ and

$$\|\Theta_A\|_{bw} \leq \frac{\chi}{\omega}.$$

Proof In view of Lemma 2.7, one can obtain

$$u(\tau_i) = \int_0^{\tau_i} \int_0^\infty \left[q \frac{(\tau_i - s)^{q-1}}{\theta^q} \psi_q(\theta) T \left(\frac{(\tau_i - s)^q}{\theta^q} \right) \left(\frac{s^{-q}}{\Gamma(1-q)} u(0) + h(s) \right) \right] d\theta ds.$$

From $u(0) = \sum_{i=1}^{\infty} \sigma_i u(\tau_i)$, we have

$$\begin{aligned} u(0) &= \sum_{i=1}^{\infty} \sigma_i \int_0^{\tau_i} \int_0^\infty \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} s^{-q} (\tau_i - s)^{q-1} T \left(\frac{(\tau_i - s)^q}{\theta^q} \right) u(0) d\theta ds \\ &\quad + \sum_{i=1}^{\infty} \sigma_i (\mathcal{W}h)(\tau_i) \\ &= \sum_{i=1}^{\infty} \sigma_i \int_0^1 \int_0^\infty \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} \tau^{-q} (1-\tau)^{q-1} T \left(\tau_i^q \frac{(1-\tau)^q}{\theta^q} \right) u(0) d\theta d\tau \\ &\quad + \sum_{i=1}^{\infty} \sigma_i (\mathcal{W}h)(\tau_i) \\ &= \mathcal{R}u(0) + \sum_{i=1}^{\infty} \sigma_i (\mathcal{W}h)(\tau_i). \end{aligned}$$

Therefore,

$$(I - \mathcal{R})u(0) = \sum_{i=1}^{\infty} \sigma_i(\mathcal{W}h)(\tau_i).$$

So, we obtain

$$u(0) = (I - \mathcal{R})^{-1} \left(\sum_{i=1}^{\infty} \sigma_i(\mathcal{W}h)(\tau_i) \right).$$

Then (15) follows.

By (15), one has

$$\begin{aligned} & \|(\Theta_A h)(t)\|_{\omega} \\ & \leq \int_0^1 \int_0^{\infty} \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} \tau^{-q} (1-\tau)^{q-1} \left\| T \left(t^q \frac{(1-\tau)^q}{\theta^q} \right) \right\|_{\omega} \\ & \quad \times \left\| (I - \mathcal{R})^{-1} \left(\sum_{i=1}^{\infty} \sigma_i(\mathcal{W}h)(\tau_i) \right) \right\|_{\omega} d\theta d\tau + \|(\mathcal{W}h)(t)\|_{\omega} \\ & \leq \int_0^1 \int_0^{\infty} \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} \tau^{-q} (1-\tau)^{q-1} e^{-\omega \frac{t^q(1-\tau)^q}{\theta^q}} \|(I - \mathcal{R})^{-1}\|_{\omega} \\ & \quad \times \left\| \sum_{i=1}^{\infty} \sigma_i(\mathcal{W}h)(\tau_i) \right\|_{\omega} d\theta d\tau + \|(\mathcal{W}h)(t)\|_{\omega} \\ & \leq \frac{1}{1 - \sum_{i=1}^{\infty} \sigma_i} \frac{\sum_{i=1}^{\infty} \sigma_i}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega} + \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega} \\ & = \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \frac{1}{1 - \sum_{i=1}^{\infty} \sigma_i} \|h\|_{b\omega} \\ & = \frac{\chi}{\omega} \|h\|_{b\omega}. \end{aligned}$$

Therefore,

$$\|\Theta_A h\|_{b\omega} \leq \frac{\chi}{\omega} \|h\|_{b\omega}.$$

□

In consequence, we have the following lemma.

Lemma 3.2 *Let $0 < \sum_{i=1}^{\infty} \sigma_i < 1$. Then problem (1) can be written as the following equivalent integral equation:*

$$u(t) = Tu(t) := (\Theta_A f)(t) = \left(\Upsilon \left[(I - \mathcal{R})^{-1} \sum_{i=1}^{\infty} \sigma_i(\mathcal{W}f)(\tau_i) \right] \right)(t) + (\mathcal{W}f)(t). \quad (16)$$

Now, we give the proofs of the main result on the existence of positive solutions to problem (1) and (2) in the following.

Proof of Theorem 1.1 Take two arbitrary elements u and v in Banach $BC(J, E)$. From Lemmas 2.4, 2.6 and 3.2, by the condition (F), we get

$$\begin{aligned}
 & \|Tu(t) - Tv(t)\|_{\omega^*} \\
 &= \left\| \left(\Upsilon \left[(I - \mathcal{R})^{-1} \sum_{i=1}^{\infty} \sigma_i \mathcal{W}[f(\tau_i, u(r(\tau_i))) - f(\tau_i, v(r(\tau_i)))] \right] \right)(t) \right. \\
 &\quad \left. + \mathcal{W}[f(t, u(r(t))) - f(t, v(r(t)))] \right\|_{\omega^*} \\
 &\leq \left\| \left(\Upsilon \left[(I - \mathcal{R})^{-1} \sum_{i=1}^{\infty} \sigma_i \mathcal{W}[f(\tau_i, u(r(\tau_i))) - f(\tau_i, v(r(\tau_i)))] \right] \right)(t) \right\|_{\omega^*} \\
 &\quad + \|\mathcal{W}[f(t, u(r(t))) - f(t, v(r(t)))]\|_{\omega^*} \\
 &\leq \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \left[\frac{1}{1 - \sum_{i=1}^{\infty} \sigma_i} \sum_{i=1}^{\infty} \sigma_i \|f(\tau_i, u(r(\tau_i))) - f(\tau_i, v(r(\tau_i)))\|_{\omega^*} \right. \\
 &\quad \left. + \|f(t, u(r(t))) - f(t, v(r(t)))\|_{\omega^*} \right] \\
 &\leq \frac{\mathcal{M}}{\omega} \frac{1}{\Gamma(q+1)} \left[\frac{\sum_{i=1}^{\infty} \sigma_i \|u(r(\tau_i)) - v(r(\tau_i))\|_{\omega^*}}{1 - \sum_{i=1}^{\infty} \sigma_i} + \|u(r(t)) - v(r(t))\|_{\omega^*} \right] \\
 &\leq \frac{\mathcal{M}}{\omega} \frac{1}{\Gamma(q+1)} \left[\frac{\sum_{i=1}^{\infty} \sigma_i}{1 - \sum_{i=1}^{\infty} \sigma_i} + 1 \right] \|u - v\|_{b\omega^*} \\
 &= \frac{\mathcal{M}}{\omega} \frac{1}{\Gamma(q+1)} \frac{1}{1 - \sum_{i=1}^{\infty} \sigma_i} \|u - v\|_{b\omega^*}.
 \end{aligned}$$

Thus,

$$\|Tu - Tv\|_{b\omega^*} \leq \mathcal{M} \frac{\chi}{\omega} \|u - v\|_{b\omega^*},$$

which implies that problem (1) has a unique mild solution. \square

Proof of Theorem 1.2 In the following, similar to the methods used in [27], we will deduce the result of Theorem 1.2 by the monotone iterative method. The proof is divided into four steps.

Step 1: First of all, consider an abstract fractional differential equation as follows:

$$\begin{cases} {}^c D_{0+}^q u(t) = Au(t) + d_1 u(t) + f_0(t), & t \in (0, +\infty), \\ u(0) = \sum_{i=1}^{\infty} \sigma_i u(\tau_i). \end{cases} \quad (17)$$

Obviously, $\{e^{d_1 t} T(t)\}_{t \geq 0}$ is a uniformly exponentially stable C_0 -semigroup on Banach E generated by $A + d_1 I$, which is positive with the growth bound $d_1 + \omega_0$ ($d_1 + \omega_0 < 0$). Applying Lemma 3.1, equation (17) has a unique mild solution $\phi_0 \in BC(J, E)$ and $\phi_0 \geq \theta$ due to $f_0(t) \geq \theta$, $t \in J$.

Step 2: Letting $g \in BC(J, E)$, consider the following abstract fractional differential equation:

$$\begin{cases} {}^C D_{0+}^q u(t) + d_2 u(t) = Au(t) + g(t), & t \in (0, +\infty). \\ u(0) = \sum_{i=1}^{\infty} \sigma_i u(\tau_i). \end{cases} \quad (18)$$

It is easy to see that $A - d_2 I$ generates a uniformly exponentially stable C_0 -semigroup $\{e^{-d_2 t} T(t)\}_{t \geq 0}$ on Banach E . Also, it is positive with the growth bound $-d_2 + \omega_0$ ($-d_2 + \omega_0 < 0$).

According to Lemma 3.1, we can get that problem (18) has a unique mild solution $u = \Theta_{A-d_2 I} g$, where $\Theta_{A-d_2 I} : BC(J, E) \rightarrow BC(J, E)$ is a positive bounded linear operator (similar to the operator Θ_A) with the property that

$$\|\Theta_{A-d_2 I}\|_{bw} \leq \frac{\chi}{d_2 - \omega_0}, \quad \omega = d_2 - \omega_0.$$

Then ϕ_0 is the mild solution of problem (18) for $g = f_0 + d_1 \phi_0 + d_2 \phi_0$, thus

$$\phi_0 = \Theta_{A-d_2 I}(f_0 + d_1 \phi_0 + d_2 \phi_0). \quad (19)$$

Step 3: We will prove the existence of a mild positive solution of problem (2).

Take $F(u) = f(t, u) + d_2 u$. Evidently, $F(\theta) = f(t, \theta) = f_0(t) \geq \theta$ and $F : BC(J, E) \rightarrow BC(J, E)$ is continuous due to condition (H1), (H1) and the normality of the cone P_B .

By condition (H1), then

$$F(y) - F(x) = f(t, y) + d_2 y - f(t, x) - d_2 x = f(t, y) - f(t, x) + d_2(y - x) \geq \theta, \quad \theta \leq x \leq y,$$

which implies that F is an increasing operator on the positive cone P .

Let $\phi_0 = \theta$ and $\mathcal{O} = \Theta_{A-d_2 I} \circ F$. It is easy to notice that the fixed point of \mathcal{O} is the mild solution of problem (2). Now, our next step is to prove that the operator \mathcal{O} has at least one fixed point.

Define two sequences

$$\phi_n = \mathcal{O}(\phi_{n-1}), \quad n = 1, 2, 3, \dots, \quad (20)$$

and

$$\varphi_n = \mathcal{O}(\varphi_{n-1}), \quad n = 1, 2, 3, \dots \quad (21)$$

From condition (H1), one can obtain

$$f(t, \phi_0(t)) - f(t, \theta) \leq d_1 \phi_0(t),$$

then

$$f(t, \phi_0(t)) \leq d_1 \phi_0(t) + f_0(t).$$

Thus,

$$F(\phi_0) = f(t, \phi_0(t)) + d_2\phi_0(t) \leq d_1\phi_0(t) + d_2\phi_0(t) + f_0(t).$$

Therefore, we can get

$$\theta \leq f_0(t) = F(\theta) \leq F(\phi_0) \leq d_1\phi_0 + d_2\phi_0 + f_0. \quad (22)$$

For the positivity of the linear bounded operator Θ_{A-d_2I} , by (19) and (22), one can get

$$\theta \leq \Theta_{A-d_2I} \circ F(\theta) = \mathcal{O}(\theta) \leq \Theta_{A-d_2I} \circ F(\phi_0) = \mathcal{O}(\phi_0) \leq \Theta_{A-d_2I}(d_1\phi_0 + d_2\phi_0 + f_0) = \phi_0,$$

which shows that

$$\theta = \varphi_0 \leq \phi_1 \leq \phi_0. \quad (23)$$

Since \mathcal{O} is an increasing operator on the order interval $[\theta, \phi_0]$, in view of the definition of \mathcal{O} and (23), we can get two sequences $\{\phi_n\}$ and $\{\varphi_n\}$ ($n = 0, 1, 2, 3, \dots$) such that

$$\theta = \varphi_0 \leq \varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_n \leq \dots \leq \phi_n \leq \dots \leq \phi_2 \leq \phi_1 \leq \phi_0.$$

From condition (H1), we have

$$\begin{aligned} \theta \leq \phi_n - \varphi_n &= \mathcal{O}(\phi_{n-1}) - \mathcal{O}(\varphi_{n-1}) \\ &= \Theta_{A-d_2I} \circ F(\phi_{n-1}) - \Theta_{A-d_2I} \circ F(\varphi_{n-1}) \\ &= \Theta_{A-d_2I} [f(\cdot, \phi_{n-1}) + d_2\phi_{n-1} - f(\cdot, \varphi_{n-1}) - d_2\varphi_{n-1}] \\ &\leq (d_1 + d_2)\Theta_{A-d_2I}(\phi_{n-1} - \varphi_{n-1}). \end{aligned}$$

Then

$$\theta \leq \phi_n - \varphi_n \leq (d_1 + d_2)^n \Theta_{A-d_2I}(\phi_0 - \varphi_0) = (d_1 + d_2)^n \Theta_{A-d_2I}^n(\phi_0).$$

Since the cone P_B is normal with the normal constant N , by virtue of condition (H2), one can get

$$\begin{aligned} \|\phi_n - \varphi_n\|_{b\omega} &\leq N(d_1 + d_2)^n \|\Theta_{A-d_2I}^n(\phi_0)\|_{b\omega} \\ &\leq N(d_1 + d_2)^n \|\Theta_{A-d_2I}^n\|_{b\omega} \|\phi_0\|_{b\omega} \\ &\leq N(d_1 + d_2)^n \|\Theta_{A-d_2I}\|_{b\omega}^n \|\phi_0\|_{b\omega} \\ &\leq N(d_1 + d_2)^n \left(\frac{\chi}{d_2 - \omega_0} \right)^n \|\phi_0\|_{b\omega} \\ &= N \left(\frac{\chi(d_1 + d_2)}{d_2 - \omega_0} \right)^n \|\phi_0\|_{b\omega} \rightarrow 0, \quad n \rightarrow +\infty. \end{aligned} \quad (24)$$

Therefore, by a method similar to the nested interval one, by (24), there exists unique $u^* \in \bigcap_{n=1}^{\infty} [\varphi_n, \phi_n]$ such that $u^* = \lim_{n \rightarrow \infty} \phi_n = \lim_{n \rightarrow \infty} \varphi_n$.

Taking limit of $n \rightarrow \infty$ both on (20) and (21), we can obtain that

$$u^* = \mathcal{O}(u^*),$$

which shows that u^* is the fixed point of \mathcal{O} . Hence, u^* is a mild positive solution of problem (2).

Step 4: In the last step, we will prove the uniqueness of the mild solution for problem (2).

By reduction to absurdity, assume that u_1^* and u_2^* are two different positive mild solutions for the fractional evolution equation (2), so $\|u_1^* - u_2^*\|_{b\omega} > 0$.

Replace ϕ_0 by u_1^* and u_2^* in (20), respectively. Following the same steps as above, for each u_i^* ($i = 1, 2$), we can get that $u_i^* = \mathcal{O}(u_i^*)$, $\|u_i^* - \varphi_n\|_{b\omega} \rightarrow 0$ ($n \rightarrow \infty$) and $\phi_n = u_i^*$ for each $n \in \mathbb{N}$ ($i = 1, 2$). Therefore,

$$0 < \|u_1^* - u_2^*\|_{b\omega} \leq \|u_1^* - \varphi_n\|_{b\omega} + \|u_2^* - \varphi_n\|_{b\omega} \rightarrow 0, \quad n \rightarrow \infty.$$

which is a contradiction.

Hence, problem (2) has a unique positive solution. The proof is completed. \square

4 Examples

To illustrate our main result, we will present an example. Consider the following partial fractional differential equation.

Example 1

$$\begin{cases} \partial_t^q z(t, x) = \partial_x^2 z(t, x) + F(t, z(t, x)), & t \in [0, +\infty), \\ z(t, 0) = z(t, \pi) = 0, & t \in [0, +\infty), \\ z(0, x) = \sum_{i=1}^{\infty} \sigma_i z(\tau_i, x), & x \in [0, \pi], \end{cases} \quad (25)$$

where ∂_t^q is the Caputo fractional partial derivative of order $q \in (0, 1)$.

Set $E = L^2([0, \pi], \mathbb{R})$ and $Az = \partial_x^2 z$, according to [33], then $A : D(A) \rightarrow E$ is a linear operator with domain $D(A) = \{u \in E | u' \in E, u(0) = u(\pi) = 0\}$. Besides, the operator A generates a uniformly exponentially stable C_0 -semigroup $\{T(t)\}_{t \geq 0}$ with the growth bound $\omega_0 \leq -1$.

Let $u(t) = z(t, \cdot)$, $f(t, u(t)) = F(t, z(t, \cdot))$, then problem (1) can be written as

$$\begin{cases} {}^c D_{0+}^q u(t) = Au(t) + f(t, u(t)), & t \in (0, +\infty), \\ u(0) = \sum_{i=1}^{\infty} \sigma_i u(\tau_i). \end{cases} \quad (26)$$

Take $q = 1/2$, $\sum_{i=1}^{\infty} \sigma_i = 5/6$, $\tau_i > 0$, $i = 1, 2, \dots$, then we can get

$$\chi = \frac{1}{\Gamma(q+1)} \frac{1}{1 - \sum_{i=1}^{\infty} \sigma_i} = \frac{12}{\sqrt{\pi}}.$$

Consider the following function:

$$f(t, x) = \left(-d_2 + \frac{-\omega_0}{(1 + a^2(t))(\chi - 1)} \right) x,$$

where $a \in C[0, +\infty)$ is bounded and

$$d_1 = 2\omega_0, \quad d_2 = -\left(2 + \frac{2}{\chi - 1}\right)\omega_0.$$

It is easy for us to certify that

$$0 < \frac{d_1 + d_2}{d_2 - \omega_0} = \frac{2}{3\chi - 1} < \frac{1}{\chi}.$$

Since

$$-d_2 - \frac{\omega_0}{\chi - 1} = \left(2 + \frac{2}{\chi - 1}\right)\omega_0 - \frac{\omega_0}{\chi - 1} \leq 2\omega_0 = d_1,$$

then, for $\theta \leq x \leq y$,

$$\begin{aligned} -d_2(y - x) \leq f(t, y) - f(t, x) &= \left[-d_2 + \frac{-\omega_0}{(1 + a^2(t))(\chi - 1)}\right](y - x) \\ &\leq \left(-d_2 - \frac{\omega_0}{\chi - 1}\right)(y - x) \leq d_1(y - x). \end{aligned}$$

Noting that $f(t, \theta) = \theta$. Thereby, f satisfies the conditions of Theorem 1.2. We can conclude that problem (1) has a unique positive mild solution.

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Authors' contributions

All of the authors contributed equally in writing this paper. All authors read and approved the final manuscript.

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