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Parameter estimation for nonergodic Ornstein-Uhlenbeck process driven by the weighted fractional Brownian motion

Panhong Cheng¹, Guangjun Shen^{2*} and Qin Chen²

*Correspondence: gjshen@163.com
²Department of Mathematics, Anhui Normal University, Wuhu, 241000, China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the nonergodic Ornstein-Uhlenbeck process

$$X_0 = 0, \quad dX_t = \theta X_t dt + dB_t^{a,b}, \quad t \geq 0,$$

driven by the weighted fractional Brownian motion $B_t^{a,b}$ with parameter a and b . Our goal is to estimate the unknown parameter $\theta > 0$ based on the discrete observations of the process. We construct two estimators $\hat{\theta}_n$ and $\check{\theta}_n$ of θ and show their strong consistency and the rate consistency.

Keywords: parameter estimation; Ornstein-Uhlenbeck process; weighted fractional Brownian motion; discrete observations

1 Introduction

The fractional Brownian motion (fBm for short) has already been widely applied in hydrology, traffic volume prediction, estimation of Hurst exponent of seismic signal, finance, and various other areas due to its properties such as long-range dependence, self-similarity, and stationarity of its increments. However, fBm is not sufficient for some random phenomena, so many researchers have chosen more general stochastic processes to construct stochastic models. For instance, Azzaoui and Clavier [1] studied impulse response of the 60-GHz channel by using α -stable processes. Lin and Lin [2] studied pricing debt value in stochastic interest rate by using Lévy processes. Meanwhile, the weighted fractional Brownian motion (wfBm), which is a kind of generalizations of the fBm, can be also used for modeling.

We recall that the wfBm $B_t^{a,b}$ with parameters (a, b) such that $a > -1$, $|b| < 1$, and $|b| < a + 1$ and long/short-range dependence is a centered self-similar Gaussian process with covariance function

$$R^{a,b}(t, s) := E[B_t^{a,b} B_s^{a,b}] = \int_0^{s \wedge t} u^a [(t-u)^b + (s-u)^b] du, \quad s, t \geq 0. \quad (1)$$

Obviously, when $a = b = 0$, $B_t^{a,b}$ is the standard Brownian motion B_t . When $a = 0$, we have

$$E[B_t^{a,b} B_s^{a,b}] = \frac{1}{b+1} [t^{b+1} + s^{b+1} - |s-t|^{b+1}],$$

which is the covariance function of the fBm with Hurst index $\frac{b+1}{2}$ when $-1 < b < 1$. Therefore, the wfBm is an extension of the fBm. The wfBm has some applications in various areas. It is known that the process $B_t^{a,b}$ introduced in Bojdecki et al. [3] is neither a semimartingale nor a Markov process unless $a = 0$ and $b = 0$, so many useful technical tools of stochastic analysis are ineffective when researchers deal with $B_t^{a,b}$. More studies on the wfBm can be found in Bojdecki et al. [4, 5], Garzón [6], Shen, Yan, and Cui [7], Yan, Wang, and Jing [8], and the references therein.

Kleptsyna and LeBreton [9] first studied the maximum likelihood estimator (MLE) of the fractional Ornstein-Uhlenbeck (fOU) process X_t and proved the convergence of the MLE. Hu and Nualart [10] studied the drift parameter estimation by a least squares approach and obtained the consistency and the asymptotic properties of the estimator based on continuous observations $\{X_t, t \in [0, T]\}$. In the ergodic case, Azmoodeh and Morlanes [11], Azmoodeh and Viitasaari [12], Hu and Song [13], and Jiang and Dong [14] studied the statistical inference for several models. In addition, Belfadli, Es-Sebaiy, and Ouknine [15], El Machkouri, Es-Sebaiy, and Ouknine [16], Es-Sebaiy and Ndiaye [17], Shen, Yin, and Yan [18] studied the parameter estimation in the nonergodic case for fOU processes. Liu and Song [19] considered minimum distance estimation for fOU processes. Xiao et al. [20] considered the fOU processes with discretization procedure of an integral transform.

Thus, motivated by all these studies, the present paper is concerned with the parameter estimation problem for nonergodic O-U process driven by the wfBm

$$X_0 = 0, \quad dX_t = \theta X_t dt + dB_t^{a,b}, \quad t \geq 0, \tag{2}$$

where $\theta > 0$ is an unknown parameter. Hu and Nualart [10] used the LSE technique to define the estimation of the unknown parameter as follows:

$$\hat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \tag{3}$$

where the integral $\int_0^t X_s dX_s$ is interpreted in the Young sense (see Young [21]). In fact, when $\hat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}$, the quadratic function of θ

$$\int_0^t |\dot{X}_s - \theta X_s|^2 ds = \int_0^t \dot{X}_s^2 ds - 2\theta \int_0^t X_s dX_s + \theta^2 \int_0^t X_s^2 ds$$

has minimum.

Note that ElOnsy, Es-Sebaiy, and Ndiaye [22] studied the parameter estimation for nonergodic fOU processes of the second kind with discrete observations. They proved the strong consistency of the estimators and obtained the rate consistency of those estimators. In this paper, we study parameter estimation problems for nonergodic OU processes driven by the weighted fractional Brownian motion with discrete observations. We construct two estimators, then prove the consistency of the estimators and get their rate consistency. By comparing ElOnsy, Es-Sebaiy, and Ndiaye [22] with our paper, we obtain both

the consistency of the estimators and their rate consistency. However, there is some difference of the processes we study.

From a practical standpoint, it is more realistic and amusing to consider asymptotic properties of the estimator based on discrete observations of X_t . We suppose that an Ornstein-Uhlenbeck process X_t is observed in equidistant times with step size $\Delta_n : t_1 = \Delta_n, t_2 = 2\Delta_n, \dots, t_n = T_n = n\Delta_n$, where we denote by T_n the length of the ‘observation window.’ The goal is to construct two estimators for θ that converge at rate $\sqrt{n\Delta_n}$ based on the observational data $X_{t_i}, i = 0, 1, \dots, n$.

Since $\int_0^t X_s dX_s$ is a Young integral (pathwise sense), we have $\int_0^t X_s dX_s = \frac{1}{2}X_t^2$. Thus, we obtain

$$\tilde{\theta}_{T_n} = \frac{\int_0^{T_n} X_s dX_s}{\int_0^{T_n} X_s^2 ds} = \frac{X_{T_n}^2}{2 \int_0^{T_n} X_s^2 ds}. \tag{4}$$

In the following, we construct two different discrete version estimators of $\tilde{\theta}_{T_n}$. In (4), let us replace $\int_0^{T_n} X_s dX_s$ by $\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})$ and $\int_0^{T_n} X_s^2 ds$ by $\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2$.

Then the estimators of θ are as follows:

$$\hat{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2} \tag{5}$$

and

$$\check{\theta}_n = \frac{X_{T_n}^2}{2\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}. \tag{6}$$

Denote $S_n = \Delta_n \sum_{i=1}^n X_{t_{i-1}}^2$. Then (5) and (6) can be rewritten respectively as

$$\hat{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{S_n} \tag{7}$$

and

$$\check{\theta}_n = \frac{X_{T_n}^2}{2S_n}. \tag{8}$$

The paper is organized as follows. In Section 2, some preliminaries for the wfbm $B_t^{a,b}$ and main lemmas are provided. In Section 3 the strong consistency of $\hat{\theta}_n$ and $\check{\theta}_n$ are proved. In Section 4, we show the rate consistency of $\hat{\theta}_n$ and $\check{\theta}_n$. Finally, we make simulations to show the performance of two estimators $\hat{\theta}_n$ and $\check{\theta}_n$.

2 Preliminaries and main lemmas

Let $B_t^{a,b}$ be a wfbm defined on a complete probability space (Ω, \mathcal{F}, P) with parameters a, b ($a > -1, 0 < b < 1, b < a + 1$). It is possible for researchers to construct a variety of stochastic calculus with respect to the wfbm $B_t^{a,b}$ associated with the Malliavin calculus. More studies and references can be found in Nualart [23] and the references therein. Here, we need to review the basic concepts and some results of the Malliavin calculus.

The crucial ingredient is the canonical Hilbert space \mathcal{H} (also called the reproducing kernel Hilbert space) associated with the wfbm $B_t^{a,b}$ defined as the closure of the linear

space \mathcal{E} generated by the indicator functions $\{\mathbf{1}_{[0,t]}, t \in [0, T]\}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R^{a,b}(t, s).$$

The mapping $\mathcal{E} \ni \varphi \mapsto B^{a,b}(\varphi) = \int_0^T \varphi(s) dB_s^{a,b}$ ($B^{a,b}(\varphi)$ is a Gaussian process on \mathcal{H} , and $E[B^{a,b}(\varphi)B^{a,b}(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}}$ for all $\varphi, \psi \in \mathcal{H}$) is an isometry from the space \mathcal{E} to the Gaussian space generated by the wfbm $B_t^{a,b}$, and it can be extended to the Hilbert space \mathcal{H} .

We can find a linear space of functions contained in \mathcal{H} in the following way. Let $|\mathcal{H}|$ be the linear space of measurable functions φ on $[0, T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := \int_0^T \int_0^T |\varphi(s)| |\varphi(r)| \phi(s, r) ds dr < \infty \tag{9}$$

with $\phi(s, r) = b(s \wedge r)^a (s \vee r - s \wedge r)^{b-1}$.

It can be proved that $(|\mathcal{H}|, \langle \cdot, \cdot \rangle_{|\mathcal{H}|})$ is a Banach space (see Shen et al. [18] and Pipiras and Taqqu [24]). Moreover,

$$L^2([0, T]) \subset L^{\frac{2}{1+a+b}} \subset |\mathcal{H}| \subset \mathcal{H}.$$

Furthermore, for all $\varphi, \phi \in |\mathcal{H}|$ (see Shen et al. [18]),

$$\begin{aligned} & E\left(\int_0^T \varphi_s dB_s^{a,b} \int_0^T \phi_s dB_s^{a,b}\right) \\ &= b \int_0^T \int_0^T \varphi_u \phi_v (u \wedge v)^a (u \vee v - u \wedge v)^{b-1} du dv. \end{aligned} \tag{10}$$

For every $n \geq 1$, we denote by \mathcal{H}_n the n th Wiener chaos of $B_t^{a,b}$. Namely, \mathcal{H}_n is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(B_t^{a,b}(f)), f \in \mathcal{H}, \|f\|_{\mathcal{H}} = 1\}$, where H_n is the n th Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n!H_n(B_t^{a,b}(f))$ gives a linear isometry between the symmetric tensor product $\mathcal{H}^{\otimes n}$ and \mathcal{H}_n , where the symmetric tensor product $\mathcal{H}^{\otimes n}$ is equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\otimes n}} = \frac{1}{\sqrt{n!}} \|\cdot\|_{\mathcal{H}^{\otimes n}}$, where $\mathcal{H}^{\otimes n}$ denotes the tensor product, $h^{\otimes n} \in \mathcal{H}^{\otimes n}$. For every $f, g \in \mathcal{H}^{\otimes n}$, we have the following formula:

$$E(I_n(f)I_n(g)) = n! \langle f, g \rangle_{\mathcal{H}^{\otimes n}},$$

where $I_n(f)$ is the multiple stochastic integral of a function f . It has the following property:

$$(E[|I_q(f)|^p])^{\frac{1}{p}} \leq C_{p,q} (E[|I_q(f)|^2])^{\frac{1}{2}} \quad \text{for any } p \geq 2.$$

Naturally, for any $F \in \oplus_{i=1}^q \mathcal{H}_i$, we have

$$(E|F|^p)^{\frac{1}{p}} \leq C_{p,q} (E|F|^2)^{\frac{1}{2}} \quad \text{for any } p \geq 2. \tag{11}$$

To prove the consistency of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$, we will use the following lemmas.

Lemma 1 (Kloeden and Neuenkirch [25]) *Let $\gamma > 0$ and $P_0 \in \mathbf{N}$. Moreover, let $\{Z_n\}_{n \in \mathbf{N}}$ be a sequence of random variables. Suppose that, for every $p \geq p_0$, there exists a constant $c_p > 0$ such that, for all $n \in \mathbf{N}$,*

$$(E|Z_n|^p)^{\frac{1}{p}} \leq c_p \cdot n^{-\gamma}.$$

Then, for all $\varepsilon > 0$, there exists a random variable η_ε such that, for any $n \in \mathbf{N}$,

$$|Z_n| \leq \eta_\varepsilon \cdot n^{-r+\varepsilon} \quad \text{a.s.}$$

Moreover, $E|\eta_\varepsilon|^p < \infty$ for all $p \geq 1$.

Lemma 2 *Assume that $-1 < a < 0$, $0 < b < a + 1$, $a + b > 0$, and $\Delta_n \rightarrow 0$ and $T_n \rightarrow \infty$ as $n \rightarrow \infty$. Then, for any $\alpha > 0$,*

$$e^{-2\theta T_n} S_n = \frac{\Delta_n}{e^{2\theta \Delta_n} - 1} \eta_{t_{n-1}}^2 + o(n^\alpha \Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n}) \quad \text{almost surely.} \tag{12}$$

Moreover, if $n \Delta_n^{1+\beta} \rightarrow \infty$ for some $\beta > 0$ as $n \rightarrow \infty$, then

$$e^{-2\theta T_n} S_n = \frac{\Delta_n}{e^{2\theta \Delta_n} - 1} \eta_{t_{n-1}}^2 + o(1) \quad \text{almost surely.} \tag{13}$$

In particular, as $n \rightarrow \infty$,

$$e^{-2\theta T_n} S_n \rightarrow \frac{\eta_\infty^2}{2\theta} \quad \text{almost surely,} \tag{14}$$

where $\eta_\infty := \int_0^\infty e^{-\theta s} dB_s^{a,b}$. Small $o(\bullet)$ notation is defined as an infinitesimal of higher order of \bullet .

Proof First, it is easy to get the solution of SDE (2):

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dB_s^{a,b}. \tag{15}$$

Denote

$$\eta_t = \int_0^t e^{-\theta s} dB_s^{a,b}, \quad t > 0.$$

Then

$$X_t = e^{\theta t} \eta_t. \tag{16}$$

Applying (16), we have

$$S_n = \Delta_n \sum_{i=1}^n X_{t_{i-1}}^2 = \Delta_n \sum_{i=1}^n e^{2\theta(i-1)\Delta_n} \eta_{t_{i-1}}^2.$$

Next, we need to deal with the term $e^{-2\theta T_n} S_n$:

$$\begin{aligned} e^{-2\theta T_n} S_n &= \Delta_n e^{-2\theta T_n} \sum_{i=1}^n e^{2\theta(i-1)\Delta_n} \eta_{t_{i-1}}^2 \\ &= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^n e^{2\theta(i-n)\Delta_n} \frac{e^{2\theta\Delta_n} - 1}{e^{2\theta\Delta_n}} \eta_{t_{i-1}}^2 \\ &= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^n e^{2\theta(i-n)\Delta_n} (1 - e^{-2\theta\Delta_n}) \eta_{t_{i-1}}^2 \\ &= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \sum_{i=1}^n [e^{2\theta(i-n)\Delta_n} - e^{2\theta(i-1-n)\Delta_n}] \eta_{t_{i-1}}^2 \\ &= \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \left[\eta_{t_{n-1}}^2 - \sum_{i=1}^{n-1} (\eta_{t_i}^2 - \eta_{t_{i-1}}^2) e^{2\theta(i-n)\Delta_n} \right]. \end{aligned}$$

Hence,

$$e^{-2\theta T_n} S_n - \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \eta_{t_{n-1}}^2 = -\frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \left[\sum_{i=1}^{n-1} (\eta_{t_i}^2 - \eta_{t_{i-1}}^2) e^{2\theta(i-n)\Delta_n} \right].$$

Because $e^{2\theta\Delta_n} - 1 = 2\theta\Delta_n + o(\Delta_n)$, we have

$$e^{-2\theta T_n} S_n - \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \eta_{t_{n-1}}^2 = -\frac{1}{2\theta + \frac{o(\Delta_n)}{\Delta_n}} F_n, \tag{17}$$

where $F_n := \sum_{i=1}^{n-1} (\eta_{t_i}^2 - \eta_{t_{i-1}}^2) e^{2\theta(i-n)\Delta_n}$. From the equality

$$\sqrt{\Delta_n} e^{\theta T_n} F_n = \sqrt{\Delta_n} \sum_{i=1}^{n-1} e^{\theta i\Delta_n} e^{\theta(i-n)\Delta_n} (\eta_{t_i}^2 - \eta_{t_{i-1}}^2),$$

applying the Minkowski inequality, we have

$$(E|\sqrt{\Delta_n} e^{\theta T_n} F_n|^2)^{\frac{1}{2}} \leq \sqrt{\Delta_n} \sum_{i=1}^{n-1} e^{\theta i\Delta_n} e^{\theta(i-n)\Delta_n} [E(\eta_{t_i}^2 - \eta_{t_{i-1}}^2)^2]^{\frac{1}{2}}. \tag{18}$$

By the Cauchy-Schwarz inequality we can rewrite (18) as

$$\begin{aligned} &(E|\sqrt{\Delta_n} e^{\theta T_n} F_n|^2)^{\frac{1}{2}} \\ &\leq 2\sqrt{\Delta_n} (E\eta_{\infty}^4)^{\frac{1}{4}} \sum_{i=1}^{n-1} e^{\theta i\Delta_n} e^{\theta(i-n)\Delta_n} [E(\eta_{t_i} - \eta_{t_{i-1}})^4]^{\frac{1}{4}} \\ &= 2\sqrt{3}\sqrt{\Delta_n} (E\eta_{\infty}^2)^{\frac{1}{2}} \sum_{i=1}^{n-1} e^{\theta i\Delta_n} e^{\theta(i-n)\Delta_n} [E(\eta_{t_i} - \eta_{t_{i-1}})^2]^{\frac{1}{2}} \\ &= 2\sqrt{3}\sqrt{\Delta_n} \left[\frac{b}{2^a \theta^{1+a+b}} \Gamma(1+a)\Gamma(b) \right]^{\frac{1}{2}} \sum_{i=1}^{n-1} e^{\theta i\Delta_n} e^{\theta(i-n)\Delta_n} [E(\eta_{t_i} - \eta_{t_{i-1}})^2]^{\frac{1}{2}}. \end{aligned}$$

According to (10), we can calculate

$$\begin{aligned} & [E(\eta_{t_i} - \eta_{t_{i-1}})^2] \\ &= E\left(\int_{t_{i-1}}^{t_i} e^{-\theta s} dB_s^{a,b}\right)^2 \\ &= b \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} e^{-\theta s} e^{-\theta r} (s \wedge r)^a (s \vee r - s \wedge r)^{b-1} ds dr. \end{aligned}$$

Letting $s = (u + i - 1)\Delta_n$ and $r = (v + i - 1)\Delta_n$ ($i = 1, 2, 3, \dots, n$), we have

$$\begin{aligned} & [E(e^{\theta i \Delta_n}(\eta_{t_i} - \eta_{t_{i-1}}))^2] \\ &= b \Delta_n^{a+b+1} e^{2\theta \Delta_n} \int_0^1 \int_0^1 e^{-\theta u \Delta_n} e^{-\theta v \Delta_n} ((u + i - 1) \wedge (v + i - 1))^a \\ &\quad \times ((u \vee v) - (u \wedge v))^{b-1} du dv \\ &\leq b \Delta_n^{a+b+1} e^{2\theta \Delta_n} \int_0^1 \int_0^1 ((u + i - 1) \wedge (v + i - 1))^a \\ &\quad \times ((u \vee v) - (u \wedge v))^{b-1} du dv \\ &= 2b \Delta_n^{a+b+1} e^{2\theta \Delta_n} \int_0^1 \int_v^1 (v + i - 1)^a (u - v)^{b-1} du dv \\ &\leq 2b \Delta_n^{a+b+1} e^{2\theta \Delta_n} \int_0^1 \int_v^1 v^a (u - v)^{b-1} du dv \\ &= 2 \Delta_n^{a+b+1} e^{2\theta \Delta_n} \int_0^1 v^a (1 - v)^b dv \\ &= 2 \Delta_n^{a+b+1} e^{2\theta \Delta_n} B(a + 1, b + 1) \\ &:= M \Delta_n^{a+b+1} e^{2\theta \Delta_n}, \end{aligned} \tag{19}$$

where $M = 2B(a + 1, b + 1)$. Hence,

$$\begin{aligned} & (E|\sqrt{\Delta_n} e^{\theta T_n} F_n|^2)^{\frac{1}{2}} \\ &\leq 2\sqrt{3M} (E\eta_\infty^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b}{2}+1} e^{\theta \Delta_n} \sum_{i=1}^{n-1} e^{\theta(i-n)\Delta_n} \\ &= 2\sqrt{3M} (E\eta_\infty^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b}{2}+1} e^{\theta(1-n)\Delta_n} \sum_{i=1}^{n-1} e^{i\theta \Delta_n} \\ &= 2\sqrt{3M} (E\eta_\infty^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b}{2}+1} \frac{1 - e^{\theta(1-n)\Delta_n}}{1 - e^{-\theta \Delta_n}} \\ &\leq 2\sqrt{3M} (E\eta_\infty^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b}{2}+1} \frac{1}{1 - e^{-\theta \Delta_n}} \\ &\leq 2\sqrt{3M} (E\eta_\infty^2)^{\frac{1}{2}} \frac{\Delta_n}{1 - e^{-\theta \Delta_n}} \Delta_n^{\frac{a+b}{2}} \\ &\leq c(a, b, \theta) \Delta_n^{\frac{a+b}{2}}, \end{aligned} \tag{20}$$

where $c(a, b, \theta)$ is a positive constant depending on a, b, θ , and its value may be different in different cases. Therefore, for any $\alpha > 0$,

$$\left[E \left| n^{-\alpha} \Delta_n^{\frac{1-(a+b)}{2}} e^{\theta T_n} F_n \right|^2 \right]^{\frac{1}{2}} \leq c(a, b, \theta) n^{-\alpha}.$$

According to (11) and Lemma 1, there exists a random variable X_α such that

$$\left| n^{-\alpha} \Delta_n^{\frac{1-(a+b)}{2}} e^{\theta T_n} F_n \right| \leq |X_\alpha| n^{-\frac{\alpha}{2}} \quad \text{a.s.} \tag{21}$$

for all $n \in N$. Moreover, $[E|X_\alpha|^p] < \infty$ for all $p \geq 1$. Therefore equality (12) is satisfied.

For the convergence (13), we assume that $n\Delta_n^{1+\beta} \rightarrow \infty$ for some $\beta > 0$ as $n \rightarrow \infty$. Then

$$(n\Delta_n^{1+\beta})^{\frac{2\alpha+1-(a+b)}{2\beta}} \rightarrow \infty.$$

Note that $T_n^{\alpha + \frac{2\alpha+1-(a+b)}{2\beta}} e^{-\theta T_n} \rightarrow 0$ as $n \rightarrow \infty$ and

$$n^\alpha \Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n} = \frac{T_n^{\alpha + \frac{2\alpha+1-(a+b)}{2\beta}} e^{-\theta T_n}}{(n\Delta_n^{1+\beta})^{\frac{2\alpha+1-(a+b)}{2\beta}}}.$$

Hence, using (12), we get the convergence (13).

For the convergence (14), note that $\frac{\Delta_n}{e^{2\theta\Delta_n-1}} \rightarrow 2\theta$ as $n \rightarrow \infty$, and using $\eta_t \rightarrow \eta_\infty := \int_0^\infty e^{-\theta s} dB_s^{a,b}$ as $t \rightarrow \infty$, we can easily obtain it by (13). □

3 Establishment and strong consistency of the estimators

In this section, we construct two estimators $\hat{\theta}_n$ and $\check{\theta}_n$ and prove the strong consistency.

Using $t_i = i\Delta_n$ ($i = 1, 2, \dots, n$) and (16), we have

$$X_{t_{i-1}} = e^{\theta t_{i-1}} \eta_{t_{i-1}} \tag{22}$$

and

$$X_{t_i} = e^{\theta t_i} \eta_{t_i}. \tag{23}$$

Applying (22) and (23), we can write the estimator $\hat{\theta}_n$ in (7) as

$$\begin{aligned} \hat{\theta}_n &= \frac{\sum_{i=1}^n e^{\theta t_i} \eta_{t_i} X_{t_{i-1}} - \sum_{i=1}^n X_{t_{i-1}}^2}{S_n} \\ &= \frac{\sum_{i=1}^n e^{\theta t_i} (\eta_{t_i} - \eta_{t_{i-1}}) X_{t_{i-1}} + (\sum_{i=1}^n e^{\theta t_i} \eta_{t_{i-1}} X_{t_{i-1}} - \sum_{i=1}^n X_{t_{i-1}}^2)}{S_n} \\ &= \frac{\sum_{i=1}^n e^{\theta t_i} (\eta_{t_i} - \eta_{t_{i-1}}) X_{t_{i-1}} + (e^{\theta \Delta_n} - 1) \sum_{i=1}^n X_{t_{i-1}}^2}{S_n} \\ &= \frac{G_n}{S_n} + \frac{e^{\theta \Delta_n} - 1}{\Delta_n}, \end{aligned} \tag{24}$$

where $G_n := \sum_{i=1}^n e^{\theta t_i} (\eta_{t_i} - \eta_{t_{i-1}}) X_{t_{i-1}}$.

Substituting (16) into (8), we can write the other estimator $\check{\theta}_n$ of θ as

$$\check{\theta}_n = \frac{e^{2\theta T_n} \eta_{T_n}^2}{2S_n} = \frac{\eta_{T_n}^2}{2e^{-2\theta T_n} S_n}. \tag{25}$$

Theorem 1 *Let $-1 < a < 0, 0 < b < a + 1, a + b > 0$. Assume that $\theta > 0$ and that $\Delta_n \rightarrow 0$ and $n\Delta_n^{1+\beta} \rightarrow \infty$ for some $\beta > 0$ as $n \rightarrow \infty$. Then*

$$\hat{\theta}_n \rightarrow \theta \quad a.s. \tag{26}$$

and

$$\check{\theta}_n \rightarrow \theta \quad a.s. \tag{27}$$

Proof To prove (26), we need to show that $\frac{G_n}{S_n} \rightarrow 0$ a.s. as $n \rightarrow \infty$.

According to (14), it suffices to show that

$$e^{-2\theta T_n} G_n \rightarrow 0 \quad a.s., n \rightarrow \infty. \tag{28}$$

By the Minkowski inequality and (20) we have

$$\begin{aligned} & (E|e^{-2\theta T_n} G_n|^2)^{\frac{1}{2}} \\ &= e^{-2\theta T_n} \left\{ E \left[\sum_{i=1}^n e^{\theta t_i} (\eta_{t_i} - \eta_{t_{i-1}}) X_{t_{i-1}} \right]^2 \right\}^{\frac{1}{2}} \\ &\leq e^{-2\theta T_n} \sum_{i=1}^n e^{\theta i \Delta_n} (EX_{t_{i-1}}^2)^{\frac{1}{2}} [E(\eta_{t_i} - \eta_{t_{i-1}})^2]^{\frac{1}{2}} \\ &\leq e^{-2\theta T_n} \sqrt{M} \Delta_n^{\frac{a+b+1}{2}} e^{\theta \Delta_n} \sum_{i=1}^n (EX_{t_{i-1}}^2)^{\frac{1}{2}} \nu \\ &\leq \sqrt{M} e^{\theta \Delta_n} (E\eta_{\infty}^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b+1}{2}} e^{-2\theta T_n} \sum_{i=1}^n e^{\theta(i-1)\Delta_n} \\ &= \sqrt{M} e^{\theta \Delta_n} (E\eta_{\infty}^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b+1}{2}} e^{-2\theta T_n} \frac{1 - e^{\theta T_n}}{1 - e^{\theta \Delta_n}} \\ &\leq \sqrt{M} e^{\theta \Delta_n} (E\eta_{\infty}^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b+1}{2}} e^{-\theta T_n} \frac{1 - e^{-\theta T_n}}{e^{\theta \Delta_n} - 1} \\ &\leq \sqrt{M} e^{\theta \Delta_n} (E\eta_{\infty}^2)^{\frac{1}{2}} \Delta_n^{\frac{a+b+1}{2}} e^{-\theta T_n} \frac{1}{e^{\theta \Delta_n} - 1} \\ &\leq c(a, b, \theta) \Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n}. \end{aligned} \tag{29}$$

Noting that $\Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n} = o(n^{-\alpha}), \alpha > 0$, as $n \rightarrow \infty$, we have that, for any $\gamma > 0$,

$$\Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n} = n^{-\alpha-\gamma},$$

and thus (29) can be written as follows:

$$(E|e^{-2\theta T_n} G_n|^2)^{\frac{1}{2}} \leq c(a, b, \theta)n^{-\alpha-\gamma}.$$

By (11) and Lemma 1 there exists a random variable X_α such that

$$|e^{-2\theta T_n} G_n| \leq |X_\alpha|n^{-\alpha} \quad \text{a.s.}$$

for all $n \in N$. Moreover, $[E|X_\alpha|^p] < \infty$ for all $p \geq 1$. Hence, the convergence (28) is satisfied. Observe that $\frac{e^{\theta \Delta_n} - 1}{\Delta_n} \rightarrow \theta$ as $n \rightarrow \infty$, and then the convergence (26) is proved.

Now, it remains to show that the convergence (27) is satisfied.

Since $\eta_t \rightarrow \eta_\infty := \int_0^\infty e^{-\theta s} dB_s^{a,b}$ a.s. as $t \rightarrow \infty$, the convergence (27) is easily proved from (14) and (25). This completes proof. \square

4 Rate consistency of the estimators

In this section, we show that $\sqrt{T_n}(\hat{\theta}_n - \theta)$ and $\sqrt{T_n}(\check{\theta}_n - \theta)$ are bounded in probability.

Theorem 2 *Let $-1 < a < 0$, $0 < b < a + 1$, $a + b > 0$. Assume that $\Delta_n \rightarrow 0$, $T_n \rightarrow \infty$, and $n\Delta_n^{1+\beta} \rightarrow \infty$ for some $\beta > 0$ as $n \rightarrow \infty$,*

(1) *for any $q \geq 0$,*

$$\Delta_n^q e^{\theta T_n}(\hat{\theta}_n - \theta) \text{ is not bounded in probability.} \tag{30}$$

(2) *if $n\Delta_n^3 \rightarrow 0$ as $n \rightarrow \infty$, then*

$$\sqrt{T_n}(\hat{\theta}_n - \theta) \text{ is bounded in probability.} \tag{31}$$

Proof (1) Firstly, we consider the case of $q = 1$.

According to (24), we have

$$\Delta_n e^{\theta T_n}(\hat{\theta}_n - \theta) = e^{\theta T_n}(e^{\theta \Delta_n} - 1 - \theta \Delta_n) + \frac{\Delta_n e^{-\theta T_n} G_n}{e^{-2\theta T_n} S_n}. \tag{32}$$

Because $e^{\theta \Delta_n} - 1 - \theta \Delta_n \sim \frac{\theta^2}{2} \Delta_n^2$ as $\Delta_n \rightarrow 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{\theta T_n}(e^{\theta \Delta_n} - 1 - \theta \Delta_n) &= \lim_{n \rightarrow \infty} \frac{\theta^2}{2} \Delta_n^2 e^{\theta T_n} \\ &= \lim_{n \rightarrow \infty} \frac{\theta^2}{2} (n\Delta_n^{1+\beta})^{\frac{2}{\beta}} \frac{e^{\theta T_n}}{T_n^{\frac{2}{\beta}}}. \end{aligned}$$

Because $n\Delta_n^{1+\beta} \rightarrow \infty$ as $n \rightarrow \infty$ for some $\beta > 0$ and $\frac{e^{\theta T_n}}{T_n^{\frac{2}{\beta}}} \rightarrow \infty$ as $n \rightarrow \infty$, we obtain that, as $n \rightarrow \infty$,

$$e^{\theta T_n}(e^{\theta \Delta_n} - 1 - \theta \Delta_n) \rightarrow \infty. \tag{33}$$

Applying (29), we have that, as $n \rightarrow \infty$,

$$E|\Delta_n e^{-\theta T_n} G_n| \leq c(a, b, \theta) \Delta_n^{\frac{a+b+1}{2}} \rightarrow 0. \tag{34}$$

Therefore we get the result (30) when $q = 1$ by combining (14), (32), (33), and (34). Next, we consider the case of $q > 1$.

According to (32), we have

$$\Delta_n^q e^{\theta T_n} (\hat{\theta}_n - \theta) = \Delta_n^{q-1} e^{\theta T_n} (e^{\theta \Delta_n} - 1 - \theta \Delta_n) + \Delta_n^{q-1} \frac{\Delta_n e^{-\theta T_n} G_n}{e^{-2\theta T_n} S_n}. \tag{35}$$

Because $e^{\theta \Delta_n} - 1 - \theta \Delta_n \sim \frac{\theta^2}{2} \Delta_n^2$ as $\Delta_n \rightarrow 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \Delta_n^{q-1} e^{\theta T_n} (e^{\theta \Delta_n} - 1 - \theta \Delta_n) &= \lim_{n \rightarrow \infty} \frac{\theta^2}{2} \Delta_n^{q+1} e^{\theta T_n} \\ &= \lim_{n \rightarrow \infty} \frac{\theta^2}{2} (n \Delta_n^{1+\beta})^{\frac{1+q}{\beta}} \frac{e^{\theta T_n}}{T_n^{\frac{1+q}{\beta}}}. \end{aligned}$$

Because $n \Delta_n^{1+\beta} \rightarrow \infty$ as $n \rightarrow \infty$ for some $\beta > 0$ and $\frac{e^{\theta T_n}}{T_n^{\frac{1+q}{\beta}}} \rightarrow \infty$ as $n \rightarrow \infty$, we obtain that, as $n \rightarrow \infty$,

$$\Delta_n^{q-1} e^{\theta T_n} (e^{\theta \Delta_n} - 1 - \theta \Delta_n) \rightarrow \infty. \tag{36}$$

Then by using (14), (34), (35), and (36) the result (30) is obtained when $q > 1$.

Finally, the case of $0 \leq q < 1$ is a direct result. Thus the proof of (30) is completed.

(2) According to (32), we have

$$\begin{aligned} \sqrt{T_n} (\hat{\theta}_n - \theta) &= \sqrt{n \Delta_n^3} \frac{e^{\theta \Delta_n} - 1 - \theta \Delta_n}{\Delta_n^2} + \frac{\sqrt{T_n} e^{-2\theta T_n} G_n}{e^{-2\theta T_n} S_n} \\ &= \sqrt{n \Delta_n^3} \frac{o(\Delta_n^2)}{\Delta_n^2} + \frac{\sqrt{T_n} e^{-2\theta T_n} G_n}{e^{-2\theta T_n} S_n}, \end{aligned} \tag{37}$$

where $o(\Delta_n^2)$ is an infinitesimal of higher order than Δ_n^2 as $n \rightarrow \infty$.

Because $\Delta_n \rightarrow 0$ and $n \Delta_n^3 \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\sqrt{n \Delta_n^3} \frac{o(\Delta_n^2)}{\Delta_n^2} \rightarrow 0. \tag{38}$$

Let us now show that, as $n \rightarrow \infty$,

$$\frac{\sqrt{T_n} e^{-2\theta T_n} G_n}{e^{-2\theta T_n} S_n} \rightarrow 0 \quad \text{in probability.}$$

Applying (14), it remains to prove $\sqrt{T_n} e^{-2\theta T_n} G_n \rightarrow 0$ as $n \rightarrow \infty$ in probability.

Using (37), we have

$$E|e^{-2\theta T_n} G_n| \leq (E|e^{-2\theta T_n} G_n|^2)^{\frac{1}{2}} \leq c(a, b, \theta) \Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n}.$$

Then

$$E|\sqrt{T_n} e^{-2\theta T_n} G_n| \leq c(a, b, \theta) \Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n} \sqrt{T_n}$$

$$\begin{aligned}
 &= c(a, b, \theta) \frac{T_n^{\frac{1}{2} + \frac{1-(a+b)}{2\beta}} e^{-\theta T_n}}{(n\Delta_n^{1+\beta})^{\frac{1-(a+b)}{2\beta}}} \\
 &\rightarrow 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{39}$$

The last convergence (39) follows from $n\Delta_n^{1+\beta} \rightarrow \infty$ as $n \rightarrow \infty$. Applying the Markov inequality, as $n \rightarrow \infty$, we have

$$\sqrt{T_n} e^{-2\theta T_n} G_n \rightarrow 0 \quad \text{in probability.}$$

Thus, combining (14), (38), and (39), we deduce the conclusion (2). □

Theorem 3 *Let $-1 < a < 0$, $0 < b < a + 1$, $a + b > 0$. Assume that $\Delta_n \rightarrow 0$, $T_n \rightarrow \infty$, and $n\Delta_n^{1+\beta} \rightarrow \infty$ for some $\beta > 0$ as $n \rightarrow \infty$,*

(1) *for any $q \geq 0$,*

$$\Delta_n^q e^{\theta T_n} (\check{\theta}_n - \theta) \text{ is not bounded in probability.} \tag{40}$$

(2) *If $n\Delta_n^3 \rightarrow 0$ as $n \rightarrow \infty$, then*

$$\sqrt{T_n} (\check{\theta}_n - \theta) \text{ is bounded in probability.} \tag{41}$$

Proof (1) First, we shall prove the case of $q = \frac{1}{2}$. By using (33) we calculate

$$\begin{aligned}
 &\sqrt{\Delta_n} e^{\theta T_n} (\check{\theta}_n - \theta) \\
 &= \sqrt{\Delta_n} e^{\theta T_n} \left(\frac{\eta_{T_n}^2}{2e^{-2\theta T_n} S_n} - \theta \right) \\
 &= \frac{\sqrt{\Delta_n} e^{\theta T_n}}{2e^{-2\theta T_n} S_n} (\eta_{T_n}^2 - 2\theta e^{-2\theta T_n} S_n) \\
 &= \frac{\sqrt{\Delta_n} e^{\theta T_n}}{2e^{-2\theta T_n} S_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \eta_{t_{n-1}}^2 \right. \\
 &\quad \left. - 2\theta \left(e^{-2\theta T_n} S_n - \frac{\Delta_n}{e^{2\theta \Delta_n} - 1} \eta_{t_{n-1}}^2 \right) \right] \\
 &= \frac{\sqrt{\Delta_n} e^{\theta T_n}}{2e^{-2\theta T_n} S_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \frac{2\theta}{2\theta + \frac{\rho(\Delta_n)}{\Delta_n}} F_n \right. \\
 &\quad \left. + \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \eta_{t_{n-1}}^2 \right],
 \end{aligned} \tag{42}$$

where (42) comes from (17).

Using the Minkowski and Cauchy inequalities, by (20) and (21) we have

$$\begin{aligned}
 &E \left| \sqrt{\Delta_n} e^{\theta T_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \frac{2\theta}{2\theta + \frac{\rho(\Delta_n)}{\Delta_n}} F_n \right] \right| \\
 &\leq E \left| \sqrt{\Delta_n} e^{\theta T_n} (\eta_{t_n}^2 - \eta_{t_{n-1}}^2) \right| + E \left| \frac{2\theta}{2\theta + \frac{\rho(\Delta_n)}{\Delta_n}} \sqrt{\Delta_n} e^{\theta T_n} F_n \right|
 \end{aligned}$$

$$\begin{aligned} &\leq 2E(\eta_\infty^2)^{\frac{1}{2}} \left\{ E[\sqrt{\Delta_n} e^{\theta T_n} (\eta_{t_n} - \eta_{t_{n-1}})]^2 \right\}^{\frac{1}{2}} + E \left| \frac{2\theta}{2\theta + \frac{\rho(\Delta_n)}{\Delta_n}} \sqrt{\Delta_n} e^{\theta T_n} F_n \right| \\ &\leq c(a, b, \theta) \Delta_n^{\frac{a+b+2}{2}} e^{\theta \Delta_n} + c(a, b, \theta) \Delta_n^{\frac{a+b}{2}}. \end{aligned}$$

The last inequality converges to 0 a.s. as $n \rightarrow \infty$.

Thus by the Markov inequality and (14) we that, as $n \rightarrow \infty$,

$$\frac{\sqrt{\Delta_n} e^{\theta T_n}}{2e^{-2\theta T_n} S_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \frac{2\theta}{2\theta + \frac{\rho(\Delta_n)}{\Delta_n}} F_n \right] \rightarrow 0 \quad \text{in probability.} \tag{43}$$

Furthermore,

$$\begin{aligned} &\sqrt{\Delta_n} e^{\theta T_n} \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \\ &= \sqrt{\Delta_n} e^{\theta T_n} \frac{e^{2\theta \Delta_n} - 1 - 2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \\ &= \Delta_n^{\frac{3}{2}} e^{\theta T_n} \frac{e^{2\theta \Delta_n} - 1 - 2\theta \Delta_n}{\Delta_n^2} \frac{\Delta_n}{e^{2\theta \Delta_n} - 1} \\ &= (n \Delta_n^{1+\beta})^{\frac{3}{2\beta}} \frac{e^{\theta T_n}}{T_n^{\frac{3}{2\beta}}} \frac{e^{2\theta \Delta_n} - 1 - 2\theta \Delta_n}{\Delta_n^2} \frac{\Delta_n}{e^{2\theta \Delta_n} - 1}. \end{aligned}$$

Noting that $n \Delta_n^{1+\beta} \rightarrow \infty$ as $n \rightarrow \infty$ for some $\beta > 0$, $\frac{e^{\theta T_n}}{T_n^{\frac{3}{2\beta}}} \rightarrow \infty$ as $n \rightarrow \infty$, and $\frac{e^{2\theta \Delta_n} - 1 - 2\theta \Delta_n}{\Delta_n^2} \rightarrow 2\theta^2$ and $\frac{\Delta_n}{e^{2\theta \Delta_n} - 1} \rightarrow \frac{1}{2\theta}$ as $\Delta_n \rightarrow 0$, we obtain that, as $n \rightarrow \infty$,

$$\sqrt{\Delta_n} e^{\theta T_n} \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \rightarrow \infty. \tag{44}$$

Then we get (40) when $q = \frac{1}{2}$ by combining (14), (43), and (44).

Similarly, we can prove (40) when $q > \frac{1}{2}$ and $0 \leq q < \frac{1}{2}$.

Thus we have the conclusion (1) of Theorem 3.

(2) Now, we calculate

$$\begin{aligned} &\sqrt{T_n} (\check{\theta}_n - \theta) \\ &= \sqrt{T_n} \left(\frac{\eta_{T_n}^2}{2e^{-2\theta T_n} S_n} - \theta \right) \\ &= \frac{\sqrt{T_n}}{2e^{-2\theta T_n} S_n} (\eta_{T_n}^2 - 2\theta e^{-2\theta T_n} S_n) \\ &= \frac{\sqrt{T_n}}{2e^{-2\theta T_n} S_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \eta_{t_{n-1}}^2 \right. \\ &\quad \left. - 2\theta \left(e^{-2\theta T_n} S_n - \frac{\Delta_n}{e^{2\theta \Delta_n} - 1} \eta_{t_{n-1}}^2 \right) \right] \\ &= \frac{\sqrt{T_n}}{2e^{-2\theta T_n} S_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \frac{2\theta}{2\theta + \frac{\rho(\Delta_n)}{\Delta_n}} F_n \right. \\ &\quad \left. + \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \eta_{t_{n-1}}^2 \right], \end{aligned} \tag{45}$$

where equation (45) comes from (17).

Using the Minkowski and Cauchy inequalities, from (20) and (21) we have

$$\begin{aligned}
 & E \left| \sqrt{T_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \frac{2\theta}{2\theta + \frac{\sigma(\Delta_n)}{\Delta_n}} F_n \right] \right| \\
 & \leq E \left| \sqrt{T_n} (\eta_{t_n}^2 - \eta_{t_{n-1}}^2) \right| + E \left| \frac{2\theta}{2\theta + \frac{\sigma(\Delta_n)}{\Delta_n}} \sqrt{T_n} F_n \right| \\
 & \leq 2E(\eta_\infty^2)^{\frac{1}{2}} \{ E[\sqrt{T_n}(\eta_{t_n} - \eta_{t_{n-1}})]^2 \}^{\frac{1}{2}} + E \left| \frac{2\theta}{2\theta + \frac{\sigma(\Delta_n)}{\Delta_n}} \sqrt{T_n} F_n \right| \\
 & \leq c_1(a, b, \theta) \Delta_n^{\frac{a+b-1}{2}} e^{\theta \Delta_n} e^{-\theta T_n} \sqrt{n \Delta_n^3} + c_2(a, b, \theta) \Delta_n^{\frac{a+b-1}{2}} e^{-\theta T_n} \sqrt{T_n} \\
 & = c_1(a, b, \theta) \frac{T_n^{\frac{1}{2} + \frac{1-(a+b)}{2\beta}} e^{-\theta T_n}}{(n \Delta_n^{1+\beta})^{\frac{1-(a+b)}{2\beta}}} e^{\theta \Delta_n} \Delta_n + c_2(a, b, \theta) \frac{T_n^{\frac{1}{2} + \frac{1-(a+b)}{2\beta}} e^{-\theta T_n}}{(n \Delta_n^{1+\beta})^{\frac{1-(a+b)}{2\beta}}}.
 \end{aligned}$$

The last term converges to 0 almost surely since $n \Delta_n^{1+\beta} \rightarrow \infty$ as $n \rightarrow \infty$. Thus by the Markov inequality and (14) we have that, as $n \rightarrow \infty$,

$$\frac{\sqrt{T_n}}{2e^{-2\theta T_n} S_n} \left[(\eta_{t_n}^2 - \eta_{t_{n-1}}^2) + \frac{2\theta}{2\theta + \frac{\sigma(\Delta_n)}{\Delta_n}} F_n \right] \rightarrow 0 \quad \text{in probability.} \tag{46}$$

Next, we consider the convergence of $\sqrt{T_n}(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1})$:

$$\begin{aligned}
 & \sqrt{T_n} \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \\
 & = \frac{\sqrt{n \Delta_n^3} e^{2\theta \Delta_n} - 1 - 2\theta \Delta_n}{\Delta_n e^{2\theta \Delta_n} - 1} \\
 & = \sqrt{n \Delta_n^3} \frac{e^{2\theta \Delta_n} - 1 - 2\theta \Delta_n}{\Delta_n^2} \frac{\Delta_n}{e^{2\theta \Delta_n} - 1}.
 \end{aligned}$$

Because $n \Delta_n^3 \rightarrow 0$, $\frac{e^{2\theta \Delta_n} - 1 - 2\theta \Delta_n}{\Delta_n^2} \rightarrow 2\theta^2$, and $\frac{\Delta_n}{e^{2\theta \Delta_n} - 1} \rightarrow \frac{1}{2\theta}$ as $n \rightarrow \infty$, we obtain that, as $n \rightarrow \infty$,

$$\sqrt{T_n} \left(1 - \frac{2\theta \Delta_n}{e^{2\theta \Delta_n} - 1} \right) \rightarrow 0. \tag{47}$$

Therefore, combining (14), (46), and (47), we have the conclusion (2) of Theorem 3. \square

5 Numerical simulations

In this section, we study the efficiency of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ of θ based on the simulated path of X_t for different values of a, b, θ . We simulate 200 sample paths of X_t on the time interval $[0, 1]$ with the equidistant time $\Delta_n = 0.0001$. At the end, we get a series of data sets about the two estimators $\hat{\theta}_n$ and $\check{\theta}_n$ by (5) and (6). The simulation of $\hat{\theta}_n$ and $\check{\theta}_n$ is given in Table 1.

According to Table 1, we can easily see that the standard deviations of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ are small. Also, we see that the means and medians of the constructed parameter estimators are close to the true parameter values. Therefore, the numerical simulations confirm the theoretical research.

Table 1 The means, medians, and standard deviations of estimators

	$a = -0.2, b = 0.4$	$a = -0.2, b = 0.5$	$a = -0.2, b = 0.6$
Panel 1. Low parameter value $\theta = 0.8000$			
Mean ($\hat{\theta}_n$)	0.6109	0.7290	0.7544
Median ($\hat{\theta}_n$)	0.8029	0.8185	0.8247
Std. dev. ($\hat{\theta}_n$)	0.5560	0.3874	0.3370
Mean ($\check{\theta}_n$)	0.7741	0.7967	0.7921
Median ($\check{\theta}_n$)	0.8407	0.8374	0.8339
Std. dev. ($\check{\theta}_n$)	0.2917	0.2568	0.3061
Panel 2. Medium parameter value $\theta = 1.6931$			
Mean ($\hat{\theta}_n$)	1.5894	1.6275	1.6262
Median ($\hat{\theta}_n$)	1.6865	1.6936	1.6919
Std. dev. ($\hat{\theta}_n$)	0.4326	0.3594	0.3242
Mean ($\check{\theta}_n$)	1.6430	1.6523	1.6424
Median ($\check{\theta}_n$)	1.6931	1.6956	1.6940
Std. dev. ($\check{\theta}_n$)	0.2529	0.2696	0.2739
Panel 3. High parameter value $\theta = 3.7097$			
Mean ($\hat{\theta}_n$)	3.7004	3.7087	3.7093
Median ($\hat{\theta}_n$)	3.7098	3.7097	3.7097
Std. dev. ($\hat{\theta}_n$)	0.1967	0.0235	0.0182
Mean ($\check{\theta}_n$)	3.7087	3.7103	3.7107
Median ($\check{\theta}_n$)	3.7110	3.7110	3.7111
Std. dev. ($\check{\theta}_n$)	0.1395	0.0299	0.0182

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

Author details

¹Institute of Mathematics and Finance, Chuzhou University, Fengle Road, Chuzhou, China. ²Department of Mathematics, Anhui Normal University, Wuhu, 241000, China.

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