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Existence and multiplicity of homoclinic solutions for difference systems involving classical (ϕ_1, ϕ_2)-Laplacian and a parameter

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Abstract

In this paper, we investigate the existence and multiplicity of homoclinic solutions for a class of nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian and a parameter:

$$\begin{cases} \Delta(\rho_1(n-1)\phi_1(\Delta u_1(n-1))) - \rho_3(n)\phi_3(u_1(n)) \\ \quad + \lambda \nabla_{u_1} F(n, u_1(n), u_2(n)) = f_1(n), \\ \Delta(\rho_2(n-1)\phi_2(\Delta u_2(n-1))) - \rho_4(n)\phi_4(u_2(n)) \\ \quad + \lambda \nabla_{u_2} F(n, u_1(n), u_2(n)) = f_2(n). \end{cases}$$

When F is not periodic in n and has (p,q)-sublinear growth or (p,q)-linear growth, by using the least action principle, we obtain that a system with classical (ϕ_1, ϕ_2)-Laplacian has at least one homoclinic solution and, by using Clark's theorem, we see that a system with $f_1 = f_2 \equiv 0$ has at least m distinct pairs of homoclinic solutions.

Keywords: difference systems; classical (ϕ_1, ϕ_2)-Laplacian; homoclinic solutions; variational method

1 Introduction

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ be the integers, and N be a fixed positive integer. $(\cdot,\cdot)$ stands for the usual product in $\mathbb{R}^N$, $|\cdot|$ is the induced norm, and $\mathbb{Z}[1,N] = \{1, 2, \dots, N\}$. $(\cdot)^\tau$ stands for the transpose of a vector. In this paper, we investigate the existence and multiplicity of homoclinic solutions for the following nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian:

$$\begin{cases} \Delta(\rho_1(n-1)\phi_1(\Delta u_1(n-1))) - \rho_3(n)\phi_3(u_1(n)) \\ \quad + \lambda \nabla_{u_1} F(n, u_1(n), u_2(n)) = f_1(n), \\ \Delta(\rho_2(n-1)\phi_2(\Delta u_2(n-1))) - \rho_4(n)\phi_4(u_2(n)) \\ \quad + \lambda \nabla_{u_2} F(n, u_1(n), u_2(n)) = f_2(n), \end{cases} \tag{1.1}$$

where $\lambda > 0$, Δ is the forward difference operator, $n \in \mathbb{Z}$, $u_m(n) \in \mathbb{R}^N$, $f_m : \mathbb{Z} \to \mathbb{R}^N$ with $f_m = (f_{m1}, \dots, f_{mN})^\tau$, $m = 1, 2$, and $\rho_i : \mathbb{Z} \to \mathbb{R}^+$ and ϕ_i, $i = 1, 2, 3, 4$ satisfy the following conditions:

(ρ) $0 < \inf_{n\in\mathbb{Z}} \rho_i \leq \sup_{n\in\mathbb{Z}} \rho_i < +\infty$, $i = 1,2,3,4$;

($\mathcal{A}_0$) ϕ_i is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ such that $\phi_i(0) = 0$ and $\phi_i = \nabla\Phi_i$, with $\Phi_i \in C^1(\mathbb{R}^N, [0,+\infty))$ strictly convex and $\Phi_i(0) = 0$, $i = 1,2,3,4$.

Remark 1.1 Assumption ($\mathcal{A}_0$) is given in [1], which is used to characterize the classical homeomorphism. If, furthermore, $\Phi_i : \mathbb{R}^N \to \mathbb{R}$ is coercive (*i.e.*, $\Phi_i(x) \to +\infty$ as $|x| \to \infty$), then there exists $\delta_i > 0$ such that

$$\Phi_i(x) \geq \delta_i\bigl(|x| - 1\bigr), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\delta_i = \min_{|x|=1} \Phi_m(x)$, $i = 1,2,3,4$ (see [1]).

As usual, we say that a solution $u(n) = (u_1(n), u_2(n))$ of system (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$, then $u(n)$ is called a nontrivial homoclinic solution.

It is well known that the existence and multiplicity of homoclinic orbits for difference systems have been extensively studied in many recent papers via critical point theory (for example, see [2–12]). In [5], by using a linking theorem from [13], the author obtained that a second-order self-adjoint discrete Hamiltonian system has infinitely many nontrivial homoclinic solutions, when potential function W is indefinite sign and subquadratic. In [6], by using a variant of the mountain pass theorem from [14], the authors obtained that a class of p-Laplacian difference systems has at least one nontrivial homoclinic solution when the potential function possesses asymptotically p-linear properties at infinity. In [7], Tang and Lin investigated the following second-order self-adjoint discrete difference system:

$$\Delta\bigl[p(n)\Delta u(n-1)\bigr] - L(n)u(n) + \nabla W\bigl(n, u(n)\bigr) = 0, \tag{1.3}$$

where $p(n)$ and $L(n)$ are $N \times N$ real symmetric positive definite matrices for all $n \in \mathbb{Z}$. By using the least action principle, they obtained that system (1.3) has at least one homoclinic solution and, by using the Clark theorem, they obtained that system (1.3) has infinitely many homoclinic solutions. To be precise, they obtained the following theorems.

Theorem A *Assume that $p(n)$ is an $N \times N$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$. Assume L and W satisfy the following conditions*:

(L) *$L(n)$ is an $N \times N$ real symmetric positive definite matrix for all $n \in \mathbb{Z}$ and there exists a constant $\beta > 0$ such that*

$$\bigl(L(n)x, x\bigr) \geq \beta|x|^2, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

(W1) *For every $n \in \mathbb{Z}$, W is continuously differentiable in x and there exist two constants $1 < \gamma_1 < \gamma_2 < 2$ and two functions $a_1, a_2 \in l^{2/(2-\gamma_1)}(\mathbb{Z}, [0,+\infty))$ such that*

$$\bigl|W(n,x)\bigr| \leq a_1(n)|x|^{\gamma_1}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N, |x| \leq 1$$

and

$$\left|W(n,x)\right| \le a_2(n)|x|^{\gamma_2}, \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N, |x| \ge 1.$$

(W2) *There exist two functions* $b \in l^{2/(2-\gamma_1)}$ *and* $\varphi \in C([0,+\infty),[0,+\infty))$ *such that*

$$\left|\nabla W(n,x)\right| \le b(n)\varphi\left(|x|\right), \quad \forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N,$$

where $\varphi(s) = O(s^{\gamma_1-1})$ *as* $s \to 0^+$.

(W3) *There exist* $n_0 \in \mathbb{Z}$ *and two constants* $\eta > 0$ *and* $\gamma_3 \in (1,2)$ *such that*

$$W(n_0,x) \ge \eta|x|^{\gamma_3}, \quad \forall x \in \mathbb{R}^N, |x| \le 1.$$

Then system (1.3) *possesses at least one non-trivial homoclinic solution.*

Theorem B *Assume that* $p(n)$ *is an* $N \times N$ *real symmetric positive definite matrix for all* $n \in \mathbb{Z}$. *Assume* L *and* W *satisfy* (L), (W1), (W2), *and the following conditions*:

(W3)′ *There exist two constants* $\eta > 0$ *and* $\gamma_3 \in (1,2)$ *and a set* $J \subset \mathbb{Z}$ *with* $m > 0$ *elements such that*

$$W(n,x) \ge \eta|x|^{\gamma_3}, \quad \forall (n,x) \in J \times \mathbb{R}^N, |x| \le 1.$$

(W4) $W(n,-x) = W(n,x)$, $\forall (n,x) \in \mathbb{Z} \times \mathbb{R}^N$.

Then system (1.3) *possesses at least m distinct pairs of non-trivial homoclinic solutions.*

Recently, in [1] and [15], Mawhin investigated the following second-order nonlinear difference systems with ϕ-Laplacian:

$$\Delta\phi\left(\Delta u(n-1)\right) = \nabla_u F\left(n,u(n)\right) + h(n) \quad (n \in \mathbb{Z}), \tag{1.4}$$

where ϕ is a homeomorphism from $X \subset \mathbb{R}^N$ onto $Y \subset \mathbb{R}^N$, with three possible cases:

(1) classical homeomorphism if $X = Y = \mathbb{R}^N$;
(2) bounded homeomorphism if $X = \mathbb{R}^N$, $Y = B_a$ ($a < +\infty$);
(3) singular homeomorphism if $X = B_a$, $Y = \mathbb{R}^N$,

where B_a is a ball with its center at origin and radius a. Inspired by [1, 15], and [10], Zhang and Wang in [8] studied the existence of homoclinic solutions for the following nonlinear difference systems with classical (ϕ_1,ϕ_2)-Laplacian:

$$\begin{cases} \Delta\phi_1(\Delta u_1(n-1)) + \nabla_{u_1} V(n,u_1(n),u_2(n)) = f_1(n), \\ \Delta\phi_2(\Delta u_2(n-1)) + \nabla_{u_2} V(n,u_1(n),u_2(n)) = f_2(n), \end{cases} \tag{1.5}$$

where $n \in \mathbb{Z}$, $u_m(n) \in \mathbb{R}^N$, $m = 1,2$, and ϕ_m, $m = 1,2$ satisfy assumption $(\mathcal{A}_0)$ and $V(n,x_1,x_2) = -K(n,x_1,x_2) + W(n,x_1,x_2)$, where $K, W : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $K(n,x_1,x_2)$ and $W(n,x_1,x_2)$ are T-periodic in n, K has p-sublinear growth, W has p-superlinear growth, and $f_m : \mathbb{Z} \to \mathbb{R}^N$, $m = 1,2$ satisfy some reasonable growth conditions. By using a linking theorem due to [16], they obtained some existence results of homoclinic solutions for system (1.5).

In this paper, motivated by [1, 6–8, 15], the purpose is to obtain some results like Theorem A and Theorem B for system (1.1). To be precise, by using the least action principle and Clark's theorem, we obtain some existence and multiplicity results of homoclinic solutions for system (1.1) when $F(n,x_1,x_2)$ is not periodic in n and possesses (p,q)-sublinear growth or (p,q)-linear growth. Our results are different from those in [8]. Moreover, since system (1.1) has a parameter λ and perturbation terms f_m ($m = 1,2$), some new cases cannot be covered by [7] even if system (1.1) reduces to the second-order difference system. For example, by virtue of perturbation terms f_m ($m = 1,2$), (I) $F(n_0,x_1,x_2)$ can be negative in a small interval of $(|x_1|,|x_2|)$, which is impossible in (W3) (see Theorem 1.1 below), (II) the restriction of f_m ($m = 1,2$) only aims at two components of f_m ($m = 1,2$), that is, f_{1i_0} and f_{2j_0}, which gives the idea that the other components of f_m ($m = 1,2$) can be arbitrary even if $f_{1i_0} + f_{2j_0} = 0$, which is also impossible according to Theorem A (see Theorem 1.2 below), and (III) we consider the case in which F has (p,q)-linear growth, which was not considered in [7] (see Theorem 1.3 below).

Let

$$\underline{\rho_i} = \inf_{n\in Z} \rho_i(n), \qquad \overline{\rho_i} = \sup_{n\in Z} \rho_i(n), \quad i = 1,2,3,4.$$

Next, we present our main results.

Theorem 1.1 *Suppose that* (ρ), $(\mathcal{A}_0)$, *and the following conditions hold*:

$(\mathcal{A}_1)$ *There exist positive constants* b_i, d_i, $i = 1,3$, b_j, d_j, $j = 2,4$, *and* $p > 1$, $q > 1$ *such that*

$$b_i|x|^p \le \Phi_i(x) \le d_i|x|^p, \quad i = 1,3,$$
$$b_j|y|^q \le \Phi_j(y) \le d_j|y|^q, \quad j = 2,4, \forall x,y \in \mathbb{R}^N.$$

$(\mathcal{A}_2)$ *There exist positive constants* k_m, $m = 1,2$, c_i, $i = 1,3$, c_j, $j = 2,4$ *such that*

$$\bigl|\phi_i(x)\bigr| \le k_m|x|^{p-1}, \quad m = 1,2$$

and

$$\bigl(\phi_i(x) - \phi_i(y), x - y\bigr) \ge c_i|x - y|^p, \quad i = 1,3, \forall x,y \in \mathbb{R}^N, \mathit{if}\ p > 2,$$
$$\bigl(\phi_j(x) - \phi_j(y), x - y\bigr) \ge c_j|x - y|^q, \quad j = 2,4, \forall x,y \in \mathbb{R}^N, \mathit{if}\ q > 2,$$
$$\bigl(\phi_i(x) - \phi_i(y), x - y\bigr) \ge c_i|x - y|^2\bigl(|x| + |y|\bigr)^{p-2}, \quad i = 1,3, \forall x,y \in \mathbb{R}^N, \mathit{if}\ 1 < p \le 2,$$
$$\bigl(\phi_j(x) - \phi_j(y), x - y\bigr) \ge c_j|x - y|^2\bigl(|x| + |y|\bigr)^{q-2}, \quad j = 2,4, \forall x,y \in \mathbb{R}^N, \mathit{if}\ 1 < q \le 2.$$

(F_1) $F(n,0,0) = 0$ *for all* $n \in \mathbb{Z}$ *and there exist* $\gamma_1 \in (1,p)$, $\gamma_2 \in (1,q)$, *and functions* $a_1 \in l^{p/(p-\gamma_1)}(\mathbb{Z},[0,+\infty))$, $a_2 \in l^{q/(q-\gamma_2)}(\mathbb{Z},[0,+\infty))$, $b_1 \in l^{\frac{p}{p-1}}(\mathbb{Z},[0,+\infty))$, *and* $b_2 \in l^{\frac{q}{q-1}}(\mathbb{Z},[0,+\infty))$ *such that*

$$\bigl|\nabla_{x_1}F(n,x_1,x_2)\bigr| \le a_1(n)|x_1|^{\gamma_1-1} + b_1(n),$$

$$\left|\nabla_{x_2}F(n,x_1,x_2)\right| \le a_2(n)|x_2|^{\gamma_2-1} + b_2(n),$$

for all $(n,x_1,x_2) \in \mathbb{Z}\times\mathbb{R}^N\times\mathbb{R}^N$.

(F_2) *There exist* $n_0 \in \mathbb{Z}$ *and constants* $\eta_j > 0$, $j = 1,2$, $\delta_0 \in (0,1)$, *and* $\gamma_3,\gamma_4 \in (1,+\infty)$ *such that*

$$F(n_0,x_1,x_2) \ge -\eta_1|x_1|^{\gamma_3} - \eta_2|x_2|^{\gamma_4}, \quad \forall (x_1,x_2) \in \mathbb{R}^N\times\mathbb{R}^N, |x_1| \le \delta_0, |x_2| \le \delta_0.$$

(f) $f_1 \in l^{\frac{p}{p-1}}(\mathbb{Z},\mathbb{R}^N)$, $f_2 \in l^{\frac{q}{q-1}}(\mathbb{Z},\mathbb{R}^N)$, *and there exist* $i_0,j_0 \in \mathbb{Z}[1,N]$ *such that*

$$f_{1i_0}(n_0) + f_{2j_0}(n_0) < 0.$$

Then system (1.1) *with* $\lambda > 0$ *possesses at least one nontrivial homoclinic solution.*

Remark 1.2 There exist examples satisfying (ρ). For example, let $\rho_i(n) = \frac{1}{n^2+1} + 1$, $i = 1,2,3,4$. Then $\overline{\rho_i} = 2$ and $\underline{\rho_i} = 1$, $i = 1,2,3,4$. Moreover, there exist examples satisfying $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, and $(\mathcal{A}_2)$. For example, as in [8]:

(I) Assume $N = 1$. Let $p = 3$, $q = 4$,

$$\phi_1(x_1) = \phi_3(x_1) = \begin{cases} 3|x_1|^2, & x_1 > 0, \\ 6|x_1|^2, & x_1 \le 0, \end{cases}$$

and

$$\phi_2(x_2) = \phi_4(x_2) = \begin{cases} 4|x_1|^2, & x_2 > 0, \\ 8|x_1|^2, & x_2 \le 0. \end{cases}$$

(II) Assume $N \ge 1$. Let

$$\phi_1(x_1) = \phi_3(x_1) = 3a_0|x_1|^2, \qquad \phi_2(x_2) = \phi_4(x_2) = 4b_0|x_2|^3,$$

for some $a_0, b_0 > 0$.

Remark 1.3 There exist examples satisfying Theorem 1.1. For example, we take $N > 1$, p, q, ρ_i, and ϕ_i, $i = 1,2,3,4$ as in Remark 1.2. Let

$$\begin{aligned} F(n,x_1,x_2) = \frac{1}{n^2+1}\Big(&|x_1|^{\frac{5}{2}} + |x_2|^{\frac{7}{2}} + |x_1|^{\frac{1}{2}}\ln\big(1+|x_1|^2\big) \\ &+ |x_2|^{\frac{5}{2}}\ln\big(1+|x_2|^2\big) - \ln\big(1+|x_1|^{\frac{5}{2}}\big) - \ln\big(1+|x_2|^{\frac{7}{2}}\big)\Big). \end{aligned}$$

Take $\gamma_1 = \frac{5}{2}$, $\gamma_2 = \frac{7}{2}$, $a_1(n) = a_2(n) = \frac{4}{n^2+1}$, $b_1(n) = b_2(n) = 0$, $\eta_1 = \eta_2 = 1$, and $n_0 = 1$. Then it is easy to verify that F satisfies (F_1) and (F_2). Let

$$f_1(n) = \left(\frac{1}{n^2+2}, \frac{1}{n^2+1}, \dots, \frac{1}{n^2+1}\right)^{\tau}, \qquad f_2(n) = \frac{1}{n^2+1}(-1,\dots,1)^{\tau}.$$

Take $i_0 = j_0 = 1$. Then it is easy to see that (f) holds.

Theorem 1.2 *Suppose that* (ρ), $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, (F_1), *and the following conditions hold*:

$(F_2)'$ *there exist* $n_0 \in \mathbb{Z}$ *and constants* $\eta_j > 0$, $j = 1, 2$, $\delta_0 \in (0, 1)$, $\gamma_3 \in (1, p)$, *and* $\gamma_4 \in (1, q)$ *such that*

$$F(n_0, x_1, x_2) \geq \eta_1 |x_1|^{\gamma_3} + \eta_2 |x_2|^{\gamma_4}, \quad \forall (x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N, |x_1| \leq \delta_0, |x_2| \leq \delta_0;$$

$(f)'$ $f_1 \in l^{\frac{p}{p-1}}(\mathbb{Z}, \mathbb{R}^N)$, $f_2 \in l^{\frac{q}{q-1}}(\mathbb{Z}, \mathbb{R}^N)$, *and there exist* $i_0, j_0 \in \mathbb{Z}[1, N]$ *such that*

$$f_{1i_0}(n_0) + f_{2j_0}(n_0) = 0.$$

Then system (1.1) *with* $\lambda > 0$ *possesses at least one nontrivial homoclinic solution.*

Remark 1.4 There exist examples satisfying Theorem 1.2. For example, we take $N > 1$, p, q, ρ_i, and ϕ_i, $i = 1, 2, 3, 4$ as in Remark 1.2. Let

$$F(n, x_1, x_2) = \frac{1}{n^2 + 1}\left(|x_1|^{\frac{5}{2}} + |x_2|^{\frac{7}{2}} + \ln\left(1 + |x_1|^{\frac{5}{2}}\right) + \ln\left(1 + |x_2|^{\frac{7}{2}}\right)\right).$$

Take $\gamma_1 = \gamma_3 = \frac{5}{2}$, $\gamma_2 = \gamma_4 = \frac{7}{2}$, $a_1(n) = a_2(n) = \frac{4}{n^2+1}$, $b_1(n) = b_2(n) = 0$, $\eta_1 = \eta_2 = 1$, and $n_0 = 1$. Then it is easy to verify that F satisfies (F_1) and $(F_2)'$. Let

$$f_1(n) = \frac{1}{n^2 + 1}(1, \ldots, 1)^\tau, \qquad f_2(n) = \frac{1}{n^2 + 1}(-1, \ldots, 1)^\tau.$$

Take $i_0 = j_0 = 1$. Then it is easy to see that $(f)'$ holds.

Theorem 1.3 *Suppose that* (ρ), $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, (f), (F_2), *and the following condition hold*:

$(F_1)'$ $F(n, 0, 0) = 0$ *and there exist functions* $a_1, a_2 \in l^\infty(\mathbb{Z}, [0, +\infty))$ *with* $a_i(n) \to 0$ *as* $n \to \infty$, $i = 1, 2$, $b_1 \in l^{\frac{p}{p-1}}(\mathbb{Z}, [0, +\infty))$, *and* $b_2 \in l^{\frac{q}{q-1}}(\mathbb{Z}, [0, +\infty))$ *such that*

$$\left|\nabla_{x_1} F(n, x_1, x_2)\right| \leq a_1(n)|x_1|^{p-1} + b_1(n),$$
$$\left|\nabla_{x_2} F(n, x_1, x_2)\right| \leq a_2(n)|x_2|^{q-1} + b_2(n),$$

for all $(n, x_1, x_2) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N$.

Then system (1.1) *with* $\lambda \in (0, \min\{\frac{p \min\{\underline{\rho_1} b_1, \underline{\rho_3} b_3\}}{\|a_1\|_\infty}, \frac{q \min\{\underline{\rho_2} b_2, \underline{\rho_4} b_4\}}{\|a_2\|_\infty}\})$ *possesses at least one nontrivial homoclinic solution.*

Remark 1.5 There exist examples satisfying Theorem 1.3. For example, we take $N > 1$, p, q, ρ_i, and ϕ_i, $i = 1, 2, 3, 4$, as in Remark 1.2. Let

$$F(n, x_1, x_2) = \frac{1}{n^2 + 1}\left(|x_1|^3 + |x_2|^4 + |x_2|^2 \ln\left(1 + |x_2|^2\right) - \ln\left(1 + |x_1|^3\right) - \ln\left(1 + |x_2|^4\right)\right).$$

Take $a_1(n) = a_2(n) = \frac{4}{n^2+1}$, $b_1(n) = b_2(n) = 0$, $\eta_1 = \eta_2 = 1$, $\gamma_3 = 3$, $\gamma_4 = 4$, and $n_0 = 1$. Then it is easy to verify that F satisfies $(F_1)'$ and (F_2). Let

$$f_1(n) = \left(\frac{1}{n^2 + 2}, \frac{1}{n^2 + 1}, \ldots, \frac{1}{n^2 + 1}\right)^\tau, \qquad f_2(n) = \frac{1}{n^2 + 1}(-1, \ldots, 1)^\tau.$$

Take $i_0 = j_0 = 1$. Then it is easy to see that (f) holds.

Theorem 1.4 *Suppose that* (ρ), $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, (F_1), *and the following conditions hold*:

$(F_2)'''$ *there exist constants* $\delta_0 \in (0,1)$, $\eta_j > 0$, $j = 1,2$, $\gamma_3, \gamma_4 \in (1, \min\{p,q\})$, *and a set* $J \subset Z$ *with* $m \in \mathbb{Z}[1,N]$ *elements such that*

$$F(n,x_1,x_2) \geq \eta_1|x_1|^{\gamma_3} + \eta_2|x_2|^{\gamma_4}, \quad \forall (n,x_1,x_2) \in J \times \mathbb{R}^N \times \mathbb{R}^N, |x_1| \leq \delta_0, |x_2| \leq \delta_0;$$

(F_3) $F(n,-x_1,-x_2) = F(n,x_1,x_2)$, $\forall (n,x_1,x_2) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N$;

$(f)''$ $f_1 = f_2 \equiv 0$.

Then, *for every* $\lambda > 0$, *system* (1.1) *possesses at least m distinct pairs of nontrivial homoclinic solutions*.

Remark 1.6 There exist examples satisfying Theorem 1.4. For example, we take $N > 4$, p, q, ρ_i, and ϕ_i, $i = 1,2,3,4$, as in Remark 1.2. Let

$$F(n,x_1,x_2) = \frac{1}{n^2+1}\left(|x_1|^{\frac{5}{2}} + |x_2|^{\frac{7}{2}} + \ln\left(1 + |x_1|^{\frac{5}{2}}\right) + \ln\left(1 + |x_2|^{\frac{7}{2}}\right)\right).$$

Take $\gamma_1 = \gamma_3 = \frac{5}{2}$, $\gamma_2 = \gamma_4 = \frac{7}{2}$, $a_1(n) = a_2(n) = \frac{4}{n^2+1}$, $b_1(n) = b_2(n) = 0$, $\eta_1 = \eta_2 = \frac{1}{18}$, and $J = \{1,2,3,4\}$. Then it is easy to verify that F satisfies (F_1) and $(F_2)'''$. Hence, Theorem 1.4 implies that system (1.1) possesses at least four distinct pairs of nontrivial homoclinic solutions for every $\lambda > 0$.

2 Preliminaries

Define

$$\begin{aligned} S &= \left\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^N, n \in \mathbb{Z}\right\}, \\ E_\kappa &= \left\{u \in S : \sum_{n\in\mathbb{Z}}\left[\left|\Delta u(n)\right|^\kappa + \left|u(n)\right|^\kappa\right] < +\infty\right\}, \end{aligned} \tag{2.1}$$

where $1 < \kappa < +\infty$ and for $v \in E_\kappa$ we define

$$\|v\|_\kappa = \left\{\sum_{n\in\mathbb{Z}}\left[\left|\Delta v(n)\right|^\kappa + \left|v(n)\right|^\kappa\right]\right\}^{1/\kappa}. \tag{2.2}$$

Let $E = E_p \times E_q$. For $u = (u_1,u_2) \in E$, we define

$$\|u\| = \|u_1\|_p + \|u_2\|_q. \tag{2.3}$$

Then E is a uniformly convex Banach space with this norm. As in [7], for $1 < \kappa < +\infty$, set

$$\begin{aligned} l^\kappa &:= l^\kappa\left(\mathbb{Z},\mathbb{R}^N\right) = \left\{u \in S : \sum_{n\in\mathbb{Z}}\left|u(n)\right|^\kappa < +\infty\right\}, \\ l^\infty &:= l^\infty\left(\mathbb{Z},\mathbb{R}^N\right) = \left\{u \in S : \sup_{n\in\mathbb{Z}}\left|u(n)\right| < +\infty\right\}, \end{aligned} \tag{2.4}$$

with the norms

$$\begin{aligned}
&\|u\|_{l^\kappa} = \left(\sum_{n\in\mathbb{Z}}\bigl|u(n)\bigr|^\kappa\right)^{1/\kappa}, \quad \forall u \in l^\kappa\bigl(\mathbb{Z},\mathbb{R}^N\bigr), \\
&\|u\|_\infty = \sup\bigl\{\bigl|u(n)\bigr| : n \in \mathbb{Z}\bigr\}, \quad \forall u \in l^\infty\bigl(\mathbb{Z},\mathbb{R}^N\bigr),
\end{aligned} \tag{2.5}$$

respectively. For $u \in E_\kappa$, it is easy to obtain

$$\|u\|_\infty \le \|u\|_{l^\kappa} \le \|u\|_\kappa. \tag{2.6}$$

Lemma 2.1 *Assume that* (ρ), $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, *and* (F_1) *hold. Then, for all* $\lambda > 0$, $f_1 \in l^{\frac{p}{p-1}}(\mathbb{Z},\mathbb{R}^N)$, *and* $f_2 \in l^{\frac{q}{q-1}}(\mathbb{Z},\mathbb{R}^N)$, *the functional* $\mathcal{J} : E \to \mathbb{R}$ *defined by*

$$\begin{aligned}
\mathcal{J}(u) = \sum_{n\in\mathbb{Z}}\bigl[&\rho_1(n)\Phi_1\bigl(\Delta u_1(n)\bigr) + \rho_2(n)\Phi_2\bigl(\Delta u_2(n)\bigr) + \rho_3(n)\Phi_3\bigl(u_1(n)\bigr) \\
&+ \rho_4(n)\Phi_4\bigl(u_2(n)\bigr) - \lambda F\bigl(n,u_1(n),u_2(n)\bigr) \\
&+ \bigl(f_1(n),u_1(n)\bigr) + \bigl(f_2(n),u_2(n)\bigr)\bigr], \quad \forall u \in E,
\end{aligned} \tag{2.7}$$

is well defined and of class $C^1(E,\mathbb{R})$ *and*

$$\begin{aligned}
\bigl\langle \mathcal{J}'(u),v\bigr\rangle &= \bigl\langle \mathcal{J}'(u_1,u_2),(v_1,v_2)\bigr\rangle \\
&= \sum_{n\in\mathbb{Z}}\bigl[\rho_1(n)\bigl(\phi_1\bigl(\Delta u_1(n)\bigr),\Delta v_1(n)\bigr) \\
&\quad + \rho_2(n)\bigl(\phi_2\bigl(\Delta u_2(n)\bigr),\Delta v_2(n)\bigr) \\
&\quad + \rho_3(n)\bigl(\phi_3\bigl(u_1(n)\bigr),v_1(n)\bigr) + \rho_4(n)\bigl(\phi_4\bigl(u_2(n)\bigr),v_2(n)\bigr) \\
&\quad - \lambda\bigl(\nabla_{u_1}F\bigl(n,u_1(n),u_2(n)\bigr),v_1(n)\bigr) \\
&\quad - \lambda\bigl(\nabla_{u_2}F\bigl(n,u_1(n),u_2(n)\bigr),v_2(n)\bigr) \\
&\quad + \bigl(f_1(n),v_1(n)\bigr) + \bigl(f_2(n),v_2(n)\bigr)\bigr], \quad \forall u,v \in E.
\end{aligned} \tag{2.8}$$

Furthermore, the critical points of $\mathcal{J}$ *in E are solutions of* (1.1) *with* $u(\pm\infty) = 0$.

Proof Firstly, we show that $\mathcal{J} : E \to \mathbb{R}$ is well defined. In fact,

$$\begin{aligned}
F(n,x_1,x_2) &= \int_0^1\bigl(\nabla_{x_1}F(n,sx_1,x_2),x_1\bigr)\,ds + F(n,0,x_2) \\
&= \int_0^1\bigl(\nabla_{x_1}F(n,sx_1,x_2),x_1\bigr)\,ds \\
&\quad + \int_0^1\bigl(\nabla_{x_2}F(n,0,tx_2),x_2\bigr)\,dt \\
&\quad + F(n,0,0).
\end{aligned} \tag{2.9}$$

Then, by (F_1), we have

$$\begin{aligned}
\left|F(n,x_1,x_2)\right| &\le \int_0^1 \left|\nabla_{x_1}F(n,sx_1,x_2)\right||x_1|\,ds + \int_0^1 \left|\nabla_{x_2}F(n,0,tx_2)\right||x_2|\,dt \\
&\le \int_0^1 \left(\left|a_1(n)\right||sx_1|^{\gamma_1-1} + b_1(n)\right)|x_1|\,ds + \int_0^1 \left(\left|a_2(n)\right||tx_2|^{\gamma_2-1}\right. \\
&\quad \left. + b_2(n)\right)|x_2|\,dt \\
&= \frac{|a_1(n)|}{\gamma_1}|x_1|^{\gamma_1} + \frac{|a_2(n)|}{\gamma_2}|x_2|^{\gamma_2} + b_1(n)|x_1| + b_2(n)|x_2|.
\end{aligned} \tag{2.10}$$

So, for $u = (u_1,u_2)^\tau \in E$, by (2.10), the Hölder inequality, and (2.6), we have

$$\begin{aligned}
\left|\sum_{n\in\mathbb{Z}} F\left(n,u_1(n),u_2(n)\right)\right| &\le \sum_{n\in\mathbb{Z}} \left|F\left(n,u_1(n),u_2(n)\right)\right| \\
&\le \sum_{n\in\mathbb{Z}} \left(\frac{|a_1(n)|}{\gamma_1}\left|u_1(n)\right|^{\gamma_1} + \frac{|a_2(n)|}{\gamma_2}\left|u_2(n)\right|^{\gamma_2}\right) \\
&\quad + \sum_{n\in\mathbb{Z}} \left(\left|b_1(n)\right|\left|u_1(n)\right| + \left|b_2(n)\right|\left|u_2(n)\right|\right) \\
&\le \frac{1}{\gamma_1}\left(\sum_{n\in\mathbb{Z}}\left|a_1(n)\right|^{\frac{p}{p-\gamma_1}}\right)^{\frac{p-\gamma_1}{p}} \left(\sum_{n\in\mathbb{Z}}\left|u_1(n)\right|^p\right)^{\frac{\gamma_1}{p}} \\
&\quad + \frac{1}{\gamma_2}\left(\sum_{n\in\mathbb{Z}}\left|a_2(n)\right|^{\frac{q}{q-\gamma_2}}\right)^{\frac{q-\gamma_2}{q}} \left(\sum_{n\in\mathbb{Z}}\left|u_2(n)\right|^q\right)^{\frac{\gamma_2}{q}} \\
&\quad + \left(\sum_{n\in\mathbb{Z}}\left|b_1(n)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum_{n\in\mathbb{Z}}\left|u_1(n)\right|^p\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{n\in\mathbb{Z}}\left|b_2(n)\right|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \left(\sum_{n\in\mathbb{Z}}\left|u_2(n)\right|^q\right)^{\frac{1}{q}} \\
&= \frac{1}{\gamma_1}\|a_1\|_{l^{p/(p-\gamma_1)}}\|u_1\|_{l^p}^{\gamma_1} + \frac{1}{\gamma_2}\|a_2\|_{l^{q/(q-\gamma_2)}}\|u_2\|_{l^q}^{\gamma_2} \\
&\quad + \|b_1\|_{l^{p/(p-1)}}\|u_1\|_{l^p} + \|b_2\|_{l^{q/(q-1)}}\|u_2\|_{l^q} \\
&\le \frac{1}{\gamma_1}\|a_1\|_{l^{p/(p-\gamma_1)}}\|u_1\|_p^{\gamma_1} + \frac{1}{\gamma_2}\|a_2\|_{l^{q/(q-\gamma_2)}}\|u_2\|_q^{\gamma_2} \\
&\quad + \|b_1\|_{l^{p/(p-1)}}\|u_1\|_p + \|b_2\|_{l^{q/(q-1)}}\|u_2\|_q.
\end{aligned} \tag{2.11}$$

It follows from (ρ), $(\mathcal{A}_1)$, (2.7), and (2.11) that

$$\begin{aligned}
\mathcal{J}(u) &\le \sum_{n\in\mathbb{Z}}\left[\overline{\rho_1}d_1\left|\Delta u_1(n)\right|^p + \overline{\rho_2}d_2\left|\Delta u_2(n)\right|^q + \overline{\rho_3}d_3\left|u_1(n)\right|^p + \overline{\rho_4}d_4\left|u_2(n)\right|^q\right] \\
&\quad + \frac{\lambda}{\gamma_1}\|a_1\|_{l^{p/(p-\gamma_1)}}\|u_1\|_p^{\gamma_1} + \frac{\lambda}{\gamma_2}\|a_2\|_{l^{q/(q-\gamma_2)}}\|u_2\|_q^{\gamma_2} \\
&\quad + \lambda\|b_1\|_{l^{p/(p-1)}}\|u_1\|_p + \lambda\|b_2\|_{l^{q/(q-1)}}\|u_2\|_q \\
&\quad + \|f_1\|_{l^{\frac{p}{p-1}}}\left(\sum_{n\in\mathbb{Z}}\left|u_1(n)\right|^p\right)^{1/p} + \|f_2\|_{l^{\frac{q}{q-1}}}\left(\sum_{n\in\mathbb{Z}}\left|u_2(n)\right|^q\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\le \max\{\overline{\rho_1}d_1, \overline{\rho_3}d_3\}\|u_1\|_p^p + \max\{\overline{\rho_2}d_2, \overline{\rho_4}d_4\}\|u_2\|_q^q \\
&\quad + \frac{\lambda}{\gamma_1}\|a_1\|_{l^{p/(p-\gamma_1)}}\|u_1\|_p^{\gamma_1} + \frac{\lambda}{\gamma_2}\|a_2\|_{l^{q/(q-\gamma_2)}}\|u_2\|_q^{\gamma_2} \\
&\quad + \lambda\|b_1\|_{l^{p/(p-1)}}\|u_1\|_p + \lambda\|b_2\|_{l^{q/(q-1)}}\|u_2\|_q \\
&\quad + \|f_1\|_{l^{\frac{p}{p-1}}}\|u_1\|_p + \|f_2\|_{l^{\frac{q}{q-1}}}\|u_2\|_q,
\end{aligned}
$$

which shows that J is well defined.

Next, we prove that $\mathcal{J} \in C^1(E,\mathbb{R})$. We denote $\mathcal{J}$ as follows:

$$
\mathcal{J}(u) = \mathcal{J}_1(u) - \lambda\mathcal{J}_2(u) + \mathcal{J}_3(u), \tag{2.12}
$$

where

$$
\begin{aligned}
\mathcal{J}_1(u) &:= \sum_{n\in\mathbb{Z}}\bigl[\rho_1(n)\Phi_1\bigl(\Delta u_1(n)\bigr) + \rho_2(n)\Phi_2\bigl(\Delta u_2(n)\bigr) \\
&\quad + \rho_3(n)\Phi_3\bigl(u_1(n)\bigr) + \rho_4(n)\Phi_4\bigl(u_2(n)\bigr)\bigr], \\
\mathcal{J}_2(u) &:= \sum_{n\in\mathbb{Z}} F\bigl(n, u_1(n), u_2(n)\bigr), \\
\mathcal{J}_3(u) &:= \sum_{n\in\mathbb{Z}}\bigl[\bigl(f_1(n), u_1(n)\bigr) + \bigl(f_2(n), u_2(n)\bigr)\bigr].
\end{aligned} \tag{2.13}
$$

First, by $(\mathcal{A}_0)$, it is easy to prove that $\mathcal{J}_1 \in C^1(E,\mathbb{R})$ and

$$
\begin{aligned}
\bigl\langle \mathcal{J}_1'(u), v\bigr\rangle &= \sum_{n\in\mathbb{Z}}\bigl[\rho_1(n)\bigl(\phi_1\bigl(\Delta u_1(n)\bigr), \Delta v_1(n)\bigr) \\
&\quad + \rho_2(n)\bigl(\phi_2\bigl(\Delta u_2(n)\bigr), \Delta v_2(n)\bigr) + \rho_3(n)\bigl(\phi_3\bigl(u_1(n)\bigr), v_1(n)\bigr) \\
&\quad + \rho_4(n)\bigl(\phi_4\bigl(u_2(n)\bigr), v_2(n)\bigr)\bigr], \quad \forall u, v \in E.
\end{aligned} \tag{2.14}
$$

Next, we prove that $\mathcal{J}_2 \in C^1(E,\mathbb{R})$ and

$$
\begin{aligned}
\bigl\langle \mathcal{J}_2'(u), v\bigr\rangle &= \sum_{n\in\mathbb{Z}}\bigl[\bigl(\nabla_{u_1}F\bigl(n, u_1(n), u_2(n)\bigr), v_1(n)\bigr) \\
&\quad + \bigl(\nabla_{u_2}F\bigl(n, u_1(n), u_2(n)\bigr), v_2(n)\bigr)\bigr].
\end{aligned} \tag{2.15}
$$

For any given $u = (u_1, u_2), v = (v_1, v_2) \in E$ and for any sequence $\{\theta_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ with $|\theta_n| < 1$ for $n \in \mathbb{Z}$ and any number $h \in (0,1)$, by (F_1) and the Hölder inequality, we have

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}\max_{h\in[0,1]}\bigl|\bigl(\nabla_{u_1}F\bigl(n, u_1(n) + \theta_n h v_1(n), u_2(n) + h v_2(n)\bigr), v_1(n)\bigr)\bigr| \\
&\quad + \sum_{n\in\mathbb{Z}}\max_{h\in[0,1]}\bigl|\bigl(\nabla_{u_2}F\bigl(n, u_1(n), u_2(n) + \theta_n h v_2(n)\bigr), v_2(n)\bigr)\bigr| \\
&\le \sum_{n\in\mathbb{Z}}\max_{h\in[0,1]}\bigl|\nabla_{u_1}F\bigl(n, u_1(n) + \theta_n h v_1(n), u_2(n) + h v_2(n)\bigr)\bigr|\bigl|v_1(n)\bigr| \\
&\quad + \sum_{n\in\mathbb{Z}}\max_{h\in[0,1]}\bigl|\nabla_{u_2}F\bigl(n, u_1(n), u_2(n) + \theta_n h v_2(n)\bigr)\bigr|\bigl|v_2(n)\bigr|
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{n\in\mathbb{Z}} \max_{h\in[0,1]} \bigl(|a_1(n)||u_1(n)+\theta_n h v_1(n)|^{\gamma_1-1}+b_1(n)\bigr)|v_1(n)| \\
&\quad + \sum_{n\in\mathbb{Z}} \max_{h\in[0,1]} \bigl(|a_2(n)||u_2(n)+\theta_n h v_2(n)|^{\gamma_2-1}+b_2(n)\bigr)|v_2(n)| \\
&\le 2^{\gamma_1-1}\sum_{n\in\mathbb{Z}}|a_1(n)|\bigl(|u_1(n)|^{\gamma_1-1}+|v_1(n)|^{\gamma_1-1}\bigr)|v_1(n)| \\
&\quad + 2^{\gamma_2-1}\sum_{n\in\mathbb{Z}}|a_2(n)|\bigl(|u_2(n)|^{\gamma_2-1}+|v_2(n)|^{\gamma_2-1}\bigr)|v_2(n)| \\
&\quad + \sum_{n\in\mathbb{Z}}|b_1(n)||v_1(n)| + \sum_{n\in\mathbb{Z}}|b_2(n)||v_2(n)| \\
&\le 2^{\gamma_1-1}\Bigl(\sum_{n\in\mathbb{Z}}|a_1(n)|^{\frac{p}{p-\gamma_1}}\Bigr)^{\frac{p-\gamma_1}{p}}\Bigl(\sum_{n\in\mathbb{Z}}|u_1(n)|^p\Bigr)^{\frac{\gamma_1-1}{p}}\Bigl(\sum_{n\in\mathbb{Z}}|v_1(n)|^p\Bigr)^{\frac{1}{p}} \\
&\quad + 2^{\gamma_1-1}\Bigl(\sum_{n\in\mathbb{Z}}|a_1(n)|^{\frac{p}{p-\gamma_1}}\Bigr)^{\frac{p-\gamma_1}{p}}\Bigl(\sum_{n\in\mathbb{Z}}|v_1(n)|^p\Bigr)^{\frac{\gamma_1}{p}} \\
&\quad + \Bigl(\sum_{n\in\mathbb{Z}}|b_1(n)|^{\frac{p}{p-1}}\Bigr)^{\frac{p-1}{p}}\Bigl(\sum_{n\in\mathbb{Z}}|v_1(n)|^p\Bigr)^{\frac{1}{p}} \\
&\quad + 2^{\gamma_2-1}\Bigl(\sum_{n\in\mathbb{Z}}|a_2(n)|^{\frac{q}{q-\gamma_2}}\Bigr)^{\frac{q-\gamma_2}{q}}\Bigl(\sum_{n\in\mathbb{Z}}|u_2(n)|^q\Bigr)^{\frac{\gamma_2-1}{q}}\Bigl(\sum_{n\in\mathbb{Z}}|v_2(n)|^q\Bigr)^{\frac{1}{q}} \\
&\quad + 2^{\gamma_2-1}\Bigl(\sum_{n\in\mathbb{Z}}|a_2(n)|^{\frac{q}{q-\gamma_2}}\Bigr)^{\frac{q-\gamma_2}{q}}\Bigl(\sum_{n\in\mathbb{Z}}|v_2(n)|^q\Bigr)^{\frac{\gamma_2}{q}} \\
&\quad + \Bigl(\sum_{n\in\mathbb{Z}}|b_2(n)|^{\frac{q}{q-1}}\Bigr)^{\frac{q-1}{q}}\Bigl(\sum_{n\in\mathbb{Z}}|v_2(n)|^q\Bigr)^{\frac{1}{q}} \\
&\le 2^{\gamma_1-1}\|a_1\|_{l^{p/(p-\gamma_1)}}\bigl(\|u_1\|_p^{\gamma_1-1}+\|v_1\|_p^{\gamma_1-1}\bigr)\|v_1\|_p \\
&\quad + 2^{\gamma_2-1}\|a_2\|_{l^{q/(q-\gamma_2)}}\bigl(\|u_2\|_q^{\gamma_2-1}+\|v_2\|_q^{\gamma_2-1}\bigr)\|v_2\|_q \\
&\quad + \|b_1\|_{l^{p/(p-1)}}\|v_1\|_p + \|b_2\|_{l^{q/(q-1)}}\|v_2\|_q \\
&< +\infty.
\end{aligned}
\tag{2.16}
$$

Then it follows from (2.13) and (2.16) that

$$
\begin{aligned}
\langle \mathcal{J}_2'(u), v\rangle &= \lim_{h\to 0^+}\frac{\mathcal{J}_2(u+hv)-\mathcal{J}_2(u)}{h} \\
&= \lim_{h\to 0^+}\frac{1}{h}\sum_{n\in\mathbb{Z}}\bigl[F\bigl(n,u_1(n)+hv_1(n),u_2(n)+hv_2(n)\bigr)-F\bigl(n,u_1(n),u_2(n)\bigr)\bigr] \\
&= \lim_{h\to 0^+}\sum_{n\in\mathbb{Z}}\bigl[\bigl(\nabla_{u_1}F\bigl(n,u_1(n)+\theta_n h v_1(n),u_2(n)+hv_2(n)\bigr),v_1(n)\bigr) \\
&\quad + \bigl(\nabla_{u_2}F\bigl(n,u_1(n),u_2(n)+\theta_n h v_2(n)\bigr),v_2(n)\bigr)\bigr] \\
&= \sum_{n\in\mathbb{Z}}\bigl[\bigl(\nabla_{u_1}F\bigl(n,u_1(n),u_2(n)\bigr),v_1(n)\bigr) \\
&\quad + \bigl(\nabla_{u_2}F\bigl(n,u_1(n),u_2(n)\bigr),v_2(n)\bigr)\bigr],
\end{aligned}
$$

which implies that (2.15) holds. Next, we prove $\mathcal{J}_2 \in C^1(E,\mathbb{R})$. For any sequence $\{u_k\} = \{(u_1^k, u_2^k)\}$ and any given $v \in E$, by the Hölder inequality and (2.6), we obtain

$$\begin{aligned}
&\bigl|\bigl\langle \mathcal{J}_2'(u_k) - \mathcal{J}_2'(u), v\bigr\rangle\bigr| \\
&\quad\le \Bigl|\sum_{n\in\mathbb{Z}} \bigl(\nabla_{u_1} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_1} F\bigl(n, u_1(n), u_2(n)\bigr), v_1(n)\bigr)\Bigr| \\
&\qquad + \Bigl|\sum_{n\in\mathbb{Z}} \bigl(\nabla_{u_2} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_2} F\bigl(n, u_1(n), u_2(n)\bigr), v_2(n)\bigr)\Bigr| \\
&\quad\le \sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_1} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_1} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|\bigl|v_1(n)\bigr| \\
&\qquad + \sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_2} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_2} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|\bigl|v_2(n)\bigr| \\
&\quad\le \|v_1\|_{l^p} \Bigl(\sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_1} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_1} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|^{\frac{p}{p-1}}\Bigr)^{\frac{p-1}{p}} \\
&\qquad + \|v_2\|_{l^q} \Bigl(\sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_2} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_2} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|^{\frac{q}{q-1}}\Bigr)^{\frac{q-1}{q}} \\
&\quad\le \|v_1\|_{p} \Bigl(\sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_1} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_1} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|^{\frac{p}{p-1}}\Bigr)^{\frac{p-1}{p}} \\
&\qquad + \|v_2\|_{q} \Bigl(\sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_2} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_2} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|^{\frac{q}{q-1}}\Bigr)^{\frac{q-1}{q}}.
\end{aligned} \tag{2.17}$$

Finally, we claim that

$$\sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_1} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_1} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|^{\frac{p}{p-1}} \to 0, \quad \text{as } k\to\infty, \tag{2.18}$$

and

$$\sum_{n\in\mathbb{Z}} \bigl|\nabla_{u_2} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_2} F\bigl(n, u_1(n), u_2(n)\bigr)\bigr|^{\frac{q}{q-1}} \to 0, \quad \text{as } k\to\infty, \tag{2.19}$$

if $u_k \to u$ in E. In fact, since $u_k \to u$, $\|u_1^k - u_1\|_p^p \to 0$ and $\|u_2^k - u_2\|_q^q \to 0$. Furthermore, by (2.6), we have $u_1^k \to u_1$ in l^p and $u_2^k \to u_2$ in l^q and

$$\lim_{k\to\infty} u_i^k(n) = u_i(n), \quad \forall n \in \mathbb{Z}, i = 1, 2. \tag{2.20}$$

Therefore, there exists a constant $C_0 > 0$ such that

$$\bigl\|u_1^k\bigr\|_{l^p} + \|u_1\|_{l^p} + \bigl\|u_2^k\bigr\|_{l^q} + \|u_2\|_{l^q} \le C_0.$$

By (F_1), we have

$$\begin{aligned}
&\left|\nabla_{u_1}F\left(n,u_1^k(n),u_2^k(n)\right)-\nabla_{u_1}F\left(n,u_1(n),u_2(n)\right)\right|^{\frac{p}{p-1}}\\
&\quad\le\left[\left|a_1(n)\right|\left(\left|u_1^k(n)\right|^{\gamma_1-1}+\left|u_1(n)\right|^{\gamma_1-1}\right)+2b_1(n)\right]^{\frac{p}{p-1}}\\
&\quad\le 2^{\frac{1}{p-1}}\left|a_1(n)\right|^{\frac{p}{p-1}}\left(\left|u_1^k(n)\right|^{\gamma_1-1}+\left|u_1(n)\right|^{\gamma_1-1}\right)^{\frac{p}{p-1}}+2^{\frac{p+1}{p-1}}\left(b_1(n)\right)^{\frac{p}{p-1}}\\
&\quad\le 2^{\frac{2}{p-1}}\left|a_1(n)\right|^{\frac{p}{p-1}}\left|u_1^k(n)\right|^{\frac{p(\gamma_1-1)}{p-1}}+2^{\frac{2}{p-1}}\left|a_1(n)\right|^{\frac{p}{p-1}}\left|u_1(n)\right|^{\frac{p(\gamma_1-1)}{p-1}}\\
&\qquad+2^{\frac{p+1}{p-1}}\left|b_1(n)\right|^{\frac{p}{p-1}}\\
&\quad:=g(n),\quad\forall k\in\mathbb{N},n\in\mathbb{Z}.
\end{aligned}\tag{2.21}$$

By (2.21) and the Hölder inequality, we obtain

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}g(n)&=2^{\frac{2}{p-1}}\sum_{n\in\mathbb{Z}}\left[\left|a_1(n)\right|^{\frac{p}{p-1}}\left|u_1^k(n)\right|^{\frac{p(\gamma_1-1)}{p-1}}+\left|a_1(n)\right|^{\frac{p}{p-1}}\left|u_1(n)\right|^{\frac{p(\gamma_1-1)}{p-1}}\right]\\
&\quad+2^{\frac{p+1}{p-1}}\sum_{n\in\mathbb{Z}}\left|b_1(n)\right|^{\frac{p}{p-1}}\\
&\le 2^{\frac{2}{p-1}}\|a_1\|_{l^{\frac{p}{p-\gamma_1}}}^{\frac{p}{p-1}}\left(\sum_{n\in\mathbb{Z}}\left|u_1^k(n)\right|^p\right)^{\frac{\gamma_1-1}{p-1}}\\
&\quad+2^{\frac{2}{p-1}}\|a_1\|_{l^{\frac{p}{p-\gamma_1}}}^{\frac{p}{p-1}}\left(\sum_{n\in\mathbb{Z}}\left|u_1(n)\right|^p\right)^{\frac{\gamma_1-1}{p-1}}\\
&\quad+2^{\frac{p+1}{p-1}}\sum_{n\in\mathbb{Z}}\left|b_1(n)\right|^{\frac{p}{p-1}}\\
&\le 2^{\frac{2}{p-1}}\|a_1\|_{l^{p/(p-\gamma_1)}}^{\frac{p}{p-1}}\left\|u_1^k\right\|_{l^p}^{\frac{p(\gamma_1-1)}{p-1}}+2^{\frac{2}{p-1}}\|a_1\|_{l^{p/(p-\gamma_1)}}^{\frac{p}{p-1}}\|u_1\|_{l^p}^{\frac{p(\gamma_1-1)}{p-1}}\\
&\quad+2^{\frac{p+1}{p-1}}\|b_1\|_{l^{\frac{p}{p-1}}}^{\frac{p}{p-1}}\\
&\le 2^{\frac{2}{p-1}}\|a_1\|_{l^{p/(p-\gamma_1)}}^{\frac{p}{p-1}}C_0^{\frac{p(\gamma_1-1)}{p-1}}+2^{\frac{2}{p-1}}\|a_1\|_{l^{p/(p-\gamma_1)}}^{\frac{p}{p-1}}C_0^{\frac{p(\gamma_1-1)}{p-1}}+2^{\frac{p+1}{p-1}}\|b_1\|_{l^{\frac{p}{p-1}}}^{\frac{p}{p-1}}\\
&<+\infty.
\end{aligned}\tag{2.22}$$

Since F is continuously differentiable in $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$, (2.20) implies that, for all $n\in\mathbb{Z}$,

$$\left|\nabla_{u_1}F\left(n,u_1^k(n),u_2^k(n)\right)-\nabla_{u_1}F\left(n,u_1(n),u_2(n)\right)\right|\to 0,\quad\text{as } k\to\infty.\tag{2.23}$$

Then it follows from (2.22) and (2.23) that

$$\sum_{n\in\mathbb{Z}}\left|\nabla_{u_1}F\left(n,u_1^k(n),u_2^k(n)\right)-\nabla_{u_1}F\left(n,u_1(n),u_2(n)\right)\right|^{\frac{p}{p-1}}\to 0,\quad\text{as } k\to\infty.\tag{2.24}$$

Hence, (2.18) holds. Similarly, we can obtain (2.19). Combining (2.18) and (2.19) with (2.17), we conclude that $\mathcal{J}_2\in C^1(E,\mathbb{R})$.

Finally, it is easy to check that $\mathcal{J}_3 \in C^1(E,\mathbb{R})$ and

$$\langle \mathcal{J}_3'(u), v\rangle := \sum_{n\in\mathbb{Z}} \bigl[\bigl(f_1(n), v_1(n)\bigr) + \bigl(f_2(n), v_2(n)\bigr)\bigr]. \tag{2.25}$$

Combining (2.14) and (2.15) with (2.25), we deduce that (2.8) holds. By $(\mathcal{A}_2)$ and the Hölder inequality, we obtain, for any given $u = (u_1, u_2), v = (v_1, v_2) \in E$,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \Delta\bigl(\rho_1(n-1)\phi_1\bigl(\Delta u_1(n-1)\bigr), v_1(n)\bigr) \\
&\quad\le \sum_{n\in\mathbb{Z}} \bigl[\bigl|\rho_1(n)\bigr|\bigl|\phi_1\bigl(\Delta u_1(n)\bigr)\bigr|\bigl|v_1(n+1)\bigr| + \bigl|\rho_1(n-1)\bigr|\bigl|\phi_1\bigl(\Delta u_1(n-1)\bigr)\bigr|\bigl|v_1(n)\bigr|\bigr] \\
&\quad\le \overline{\rho_1}\sum_{n\in\mathbb{Z}} k_1\bigl|\Delta u_1(n)\bigr|^{p-1}\bigl|v_1(n+1)\bigr| + \overline{\rho_1}\sum_{n\in\mathbb{Z}} k_1\bigl|\Delta u_1(n-1)\bigr|^{p-1}\bigl|v_1(n)\bigr| \\
&\quad\le \overline{\rho_1}k_1\biggl(\sum_{n\in\mathbb{Z}}\bigl|\Delta u_1(n)\bigr|^p\biggr)^{\frac{p-1}{p}}\biggl(\sum_{n\in\mathbb{Z}}\bigl|v_1(n+1)\bigr|^p\biggr)^{1/p} \\
&\qquad + \overline{\rho_1}k_1\biggl(\sum_{n\in\mathbb{Z}}\bigl|\Delta u_1(n-1)\bigr|^p\biggr)^{\frac{p-1}{p}}\biggl(\sum_{n\in\mathbb{Z}}\bigl|v_1(n)\bigr|^p\biggr)^{1/p},
\end{aligned}$$

which, together with the definition of E, implies that the series $\sum_{n\in\mathbb{Z}} \Delta(\rho_1(n-1)\phi_1(\Delta u_1(n-1)), v_1(n))$ is absolutely convergent and then it is easy to see that

$$\sum_{n\in\mathbb{Z}} \Delta\bigl(\rho_1(n-1)\phi_1\bigl(\Delta u_1(n-1)\bigr), v_1(n)\bigr) = 0.$$

Similarly, we have

$$\sum_{n\in\mathbb{Z}} \Delta\bigl(\rho_2(n-1)\phi_2\bigl(\Delta u_2(n-1)\bigr), v_2(n)\bigr) = 0.$$

Thus, for $u, v \in E$,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \bigl[\rho_1(n)\bigl(\phi_1\bigl(\Delta u_1(n)\bigr), \Delta v_1(n)\bigr) + \rho_2(n)\bigl(\phi_2\bigl(\Delta u_2(n)\bigr), \Delta v_2(n)\bigr) \\
&\qquad + \rho_3(n)\bigl(\phi_3\bigl(u_1(n)\bigr), v_1(n)\bigr) + \rho_4(n)\bigl(\phi_4\bigl(u_2(n)\bigr), v_2(n)\bigr) \\
&\qquad - \bigl(\nabla_{u_1}F\bigl(n, u_1(n), u_2(n)\bigr), v_1(n)\bigr) - \bigl(\nabla_{u_2}F\bigl(n, u_1(n), u_2(n)\bigr), v_2(n)\bigr) \\
&\qquad + \bigl(f_1(n), v_1(n)\bigr) + \bigl(f_2(n), v_2(n)\bigr)\bigr] \\
&\quad = \sum_{n\in\mathbb{Z}} \bigl[\Delta\bigl(\rho_1(n-1)\phi_1\bigl(\Delta u_1(n-1)\bigr), v_1(n)\bigr) \\
&\qquad - \bigl(\Delta\bigl(\rho_1(n-1)\phi_1\bigl(\Delta u_1(n-1)\bigr)\bigr), v_1(n)\bigr) \\
&\qquad + \Delta\bigl(\rho_2(n-1)\phi_2\bigl(\Delta u_2(n-1)\bigr), v_2(n)\bigr) \\
&\qquad - \bigl(\Delta\bigl(\rho_2(n-1)\phi_2\bigl(\Delta u_2(n-1)\bigr)\bigr), v_2(n)\bigr) \\
&\qquad + \rho_3(n)\bigl(\phi_3\bigl(u_1(n)\bigr), v_1(n)\bigr) + \rho_4(n)\bigl(\phi_4\bigl(u_2(n)\bigr), v_2(n)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \big(\nabla_{u_1} F\big(n, u_1(n), u_2(n)\big), v_1(n)\big) - \big(\nabla_{u_2} F\big(n, u_1(n), u_2(n)\big), v_2(n)\big) \\
&\quad + \big(f_1(n), v_1(n)\big) + \big(f_2(n), v_2(n)\big)\big] \\
&= \sum_{n\in\mathbb{Z}} \big[\big(-\Delta\big(\rho_1(n-1)\phi_1\big(\Delta u_1(n-1)\big)\big) + \rho_3(n)\phi_3\big(u_1(n)\big) \\
&\quad - \nabla_{u_1} F\big(n, u_1(n), u_2(n)\big), v_1(n)\big)\big] \\
&\quad + \sum_{n\in\mathbb{Z}} \big[\big(-\Delta\big(\rho_2(n-1)\phi_2\big(\Delta u_2(n-1)\big)\big) + \rho_4(n)\phi_4\big(u_2(n)\big) \\
&\quad - \nabla_{u_2} F\big(n, u_1(n), u_2(n)\big), v_2(n)\big)\big] \\
&\quad + \sum_{n\in\mathbb{Z}} \big[\big(f_1(n), v_1(n)\big) + \big(f_2(n), v_2(n)\big)\big].
\end{aligned}
$$

Using the above equation, it is easy to show that the critical points of $\mathcal{J}$ in E are weak solutions of (1.1) with $u(\pm\infty) = 0$. The proof is complete. □

Lemma 2.2 *Assume that* (ρ), $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, *and* $(F_1)'$ *hold. Then, for all* $\lambda > 0$, $f_1 \in l^{\frac{p}{p-1}}(\mathbb{Z}, \mathbb{R}^N)$, *and* $f_2 \in l^{\frac{q}{q-1}}(\mathbb{Z}, \mathbb{R}^N)$, *the functional* $\mathcal{J} : E \to \mathbb{R}$ *defined by* (2.7) *is well defined and of class* $C^1(E, \mathbb{R})$ *and* (2.8) *holds. Furthermore, the critical points of* $\mathcal{J}$ *in E are weak solutions of* (1.1) *with* $u(\pm\infty) = 0$.

Proof The proof is similar to Lemma 2.1. In the proof of Lemma 2.1, we only need to replace γ_1, γ_2, $\|a_1\|_{l^{p/(p-\gamma_1)}}$, and $\|a_2\|_{l^{q/(q-\gamma_2)}}$ with p, q, $\|a_1\|_{l^\infty}$, and $\|a_2\|_{l^\infty}$, respectively. We omit the details. □

Next, we introduce two lemmas which will be used to prove our main results.

Assume that E is a real Banach space. For $\varphi \in C^1(E, \mathbb{R})$, we say that φ satisfies the Palais-Smale (PS) condition if any sequence $\{u_m\} \subset E$ for which $\varphi(u_m)$ is bounded and $\varphi'(u_m) \to 0$ as $m \to \infty$ has a convergent subsequence.

Lemma 2.3 (see [17]) *Assume that E is a real Banach space and let* $\varphi \in C^1(E, \mathbb{R})$ *satisfy the PS condition. If* φ *is bounded from below, then* $c = \inf_E \varphi$ *is a critical value of* φ.

Lemma 2.4 (see [18]) *Assume that E is a real Banach space and* $\varphi \in C^1(E, \mathbb{R})$ *with* φ *even, bounded from below, and satisfying the PS condition. Suppose* $\varphi(0) = 0$. *Then there exists a set* $K \subset E$ *such that K is homeomorphic to* S^{j-1} ($j-1$ *dimension unit sphere*) *by an odd map and* $\sup_K \varphi < 0$. *Then* φ *has at least j distinct pairs of critical points.*

3 Proofs

Proof of Theorem 1.1 By Lemma 2.1, we have $\mathcal{J} \in C^1(E, \mathbb{R})$. It follows from (ρ), $(\mathcal{A}_1)$, and (2.11) that

$$
\begin{aligned}
\mathcal{J}(u) &= \sum_{n\in\mathbb{Z}} \rho_1(n)\Phi_1\big(\Delta u_1(n)\big) + \sum_{n\in\mathbb{Z}} \rho_2(n)\Phi_2\big(\Delta u_2(n)\big) \\
&\quad + \sum_{n\in\mathbb{Z}} \rho_3(n)\Phi_3\big(u_1(n)\big) + \sum_{n\in\mathbb{Z}} \rho_4(n)\Phi_4\big(u_2(n)\big) \\
&\quad - \lambda \sum_{n\in\mathbb{Z}} F\big(n, u_1(n), u_2(n)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n\in\mathbb{Z}}\bigl(f_1(n),u_1(n)\bigr)+\sum_{n\in\mathbb{Z}}\bigl(f_2(n),u_2(n)\bigr)\\
&\geq \underline{\rho_1}\sum_{n\in\mathbb{Z}}b_1\bigl|\Delta u_1(n)\bigr|^p+\underline{\rho_2}\sum_{n\in\mathbb{Z}}b_2\bigl|\Delta u_2(n)\bigr|^q\\
&+\underline{\rho_3}\sum_{n\in\mathbb{Z}}b_3\bigl|u_1(n)\bigr|^p+\underline{\rho_4}\sum_{n\in\mathbb{Z}}b_4\bigl|u_2(n)\bigr|^q\\
&-\lambda\sum_{n\in\mathbb{Z}}F\bigl(n,u_1(n),u_2(n)\bigr)\\
&-\|f_1\|_{l^{\frac{p}{p-1}}}\Bigl(\sum_{n\in\mathbb{Z}}\bigl|u_1(n)\bigr|^p\Bigr)^{1/p}-\|f_2\|_{l^{\frac{q}{q-1}}}\Bigl(\sum_{n\in\mathbb{Z}}\bigl|u_2(n)\bigr|^q\Bigr)^{1/q}\\
&\geq \min\{\underline{\rho_1}b_1,\underline{\rho_3}b_3\}\|u_1\|_p^p+\min\{\underline{\rho_2}b_2,\underline{\rho_4}b_4\}\|u_2\|_q^q\\
&-\frac{\lambda}{\gamma_1}\|a_1\|_{l^{p/(p-\gamma_1)}}\|u_1\|_p^{\gamma_1}-\frac{\lambda}{\gamma_2}\|a_2\|_{l^{q/(q-\gamma_2)}}\|u_2\|_q^{\gamma_2}\\
&-\lambda\|b_1\|_{l^{p/(p-1)}}\|u_1\|_p-\lambda\|b_2\|_{l^{q/(q-1)}}\|u_2\|_q\\
&-\|f_1\|_{l^{p/(p-1)}}\|u_1\|_p-\|f_2\|_{l^{q/(q-1)}}\|u_2\|_q.
\end{aligned}
\tag{3.1}
$$

Note that $1 < \gamma_1 < p$, $1 < \gamma_2 < q$. Then (3.1) and (ρ) show that $\mathcal{J}(u) \to +\infty$ as $\|u\| \to +\infty$, which implies that $\mathcal{J}$ is bounded from below.

Next, we show that $\mathcal{J}$ satisfies the PS condition. Suppose that $\{u_k = (u_1^k, u_2^k)\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{\mathcal{J}(u_k)\}_{k\in\mathbb{N}}$ is bounded and $\mathcal{J}'(u_k) \to 0$ as $k \to +\infty$. Then, by (3.1), there exists a constant $M_0 > 0$ such that

$$\|u_k\| = \bigl\|u_1^k\bigr\|_p + \bigl\|u_2^k\bigr\|_q \leq M_0, \quad k \in \mathbb{N}.$$

By (2.6), we have

$$\bigl\|u_1^k\bigr\|_\infty \leq \bigl\|u_1^k\bigr\|_p \leq M_0, \qquad \bigl\|u_2^k\bigr\|_\infty \leq \bigl\|u_2^k\bigr\|_q \leq M_0. \tag{3.2}$$

Hence, there exists a subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u_0$ for some $u_0 = (u_1^0, u_2^0)$ in E. Like the argument of Proposition 1.2 in [17], it is easy to verify that

$$\lim_{k\to+\infty} u_k(n) = u_0(n), \quad \forall n \in \mathbb{Z}. \tag{3.3}$$

Hence, by (3.2), (3.3), and the lower semi-continuity of norm, we have

$$\bigl\|u_1^0\bigr\|_\infty \leq M_0, \qquad \bigl\|u_2^0\bigr\|_\infty \leq M_0. \tag{3.4}$$

Note that $a_1 \in l^{p/(p-\gamma_1)}(\mathbb{Z}, [0,+\infty))$ and $b_1 \in l^{\frac{p}{p-1}}(\mathbb{Z}, [0,+\infty))$. Then, for any given $\varepsilon > 0$, there exists an integer $M_1 > 0$ such that

$$\Bigl(\sum_{|n|>M_1}\bigl|a_1(n)\bigr|^{\frac{p}{p-\gamma_1}}\Bigr)^{\frac{p-\gamma_1}{p}} < \varepsilon, \qquad \Bigl(\sum_{|n|>M_1}\bigl|b_1(n)\bigr|^{\frac{p}{p-1}}\Bigr)^{\frac{p-1}{p}} < \varepsilon. \tag{3.5}$$

It follows from (3.2)-(3.4) and (F_1) that

$$\sum_{n=-M_1}^{M_1} \big|\nabla_{u_1} F\big(n, u_1^k(n), u_2^k(n)\big) - \nabla_{u_1} F\big(n, u_1^0(n), u_2^0(n)\big)\big|\big|u_1^k(n) - u_1^0(n)\big| \to 0, \quad \text{as } k \to \infty. \tag{3.6}$$

On the other hand, it follows from (3.2), (3.4), (3.5), (F_1), and Young's inequality that

$$\begin{aligned}
&\sum_{|n|>M_1} \big|\nabla_{u_1} F\big(n, u_1^k(n), u_2^k(n)\big) - \nabla_{u_1} F\big(n, u_1^0(n), u_2^0(n)\big)\big|\big|u_1^k(n) - u_1^0(n)\big| \\
&\quad \le \sum_{|n|>M_1} \big[\big|a_1(n)\big|\big(\big|u_1^k(n)\big|^{\gamma_1 - 1} + \big|u_1^0(n)\big|^{\gamma_1 - 1}\big) + 2b_1(n)\big]\big(\big|u_1^k(n)\big| + \big|u_1^0(n)\big|\big) \\
&\quad \le 3 \sum_{|n|>M_1} \big|a_1(n)\big|\big(\big|u_1^k(n)\big|^{\gamma_1} + \big|u_1^0(n)\big|^{\gamma_1}\big) \\
&\qquad + 2 \sum_{|n|>M_1} b_1(n)\big(\big|u_1^k(n)\big| + \big|u_1^0(n)\big|\big) \\
&\quad \le 3\Big(\sum_{|n|>M_1} \big|a_1(n)\big|^{\frac{p}{p-\gamma_1}}\Big)^{\frac{p-\gamma_1}{p}} \big(\big\|u_1^k\big\|_{l^p}^{\gamma_1} + \big\|u_1^0\big\|_{l^p}^{\gamma_1}\big) \\
&\qquad + 2\Big(\sum_{|n|>M_1} \big|b_1(n)\big|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \big(\big\|u_1^k\big\|_{l^p} + \big\|u_1^0\big\|_{l^p}\big) \\
&\quad \le 3\Big(\sum_{|n|>M_1} \big|a_1(n)\big|^{\frac{p}{p-\gamma_1}}\Big)^{\frac{p-\gamma_1}{p}} \big(\big\|u_1^k\big\|_{p}^{\gamma_1} + \big\|u_1^0\big\|_{p}^{\gamma_1}\big) \\
&\qquad + 2\Big(\sum_{|n|>M_1} \big|b_1(n)\big|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \big(\big\|u_1^k\big\|_{p} + \big\|u_1^0\big\|_{p}\big) \\
&\quad \le 3\varepsilon\big(M_0^{\gamma_1} + \big\|u_1^0\big\|_p^{\gamma_1}\big) + 2\varepsilon\big(M_0 + \big\|u_1^0\big\|_p\big), \quad k \in N.
\end{aligned} \tag{3.7}$$

Then the arbitrariness of ε, together with (3.6), implies that

$$\sum_{n\in\mathbb{Z}} \big(\nabla_{u_1} F\big(n, u_1^k(n), u_2^k(n)\big) - \nabla_{u_1} F\big(n, u_1^0(n), u_2^0(n)\big), u_1^k(n) - u_1^0(n)\big) \to 0, \quad \text{as } k \to +\infty. \tag{3.8}$$

Similarly, we have

$$\sum_{n\in\mathbb{Z}} \big(\nabla_{u_2} F\big(n, u_1^k(n), u_2^k(n)\big) - \nabla_{u_2} F\big(n, u_1^0(n), u_2^0(n)\big), u_2^k(n) - u_2^0(n)\big) \to 0, \quad \text{as } k \to +\infty. \tag{3.9}$$

By $(\mathcal{A}_2)$, we have

$$\big(\phi_i(x) - \phi_i(y), x - y\big) \ge 0, \quad \forall x, y \in R^N, i = 1, 2, 3, 4.$$

Then

$$\begin{aligned}
&\langle \mathcal{J}'(u_k) - \mathcal{J}'(u_0), u_k - u_0 \rangle \\
&\quad = \langle \mathcal{J}'(u_1^k, u_2^k) - \mathcal{J}'(u_1^0, u_2^0), (u_1^k - u_1^0, u_2^k - u_2^0) \rangle \\
&\quad \geq \underline{\rho_1} \sum_{n\in\mathbb{Z}} \bigl(\phi_1\bigl(\Delta u_1^k(n)\bigr) - \phi_1\bigl(\Delta u_1^0(n)\bigr), \Delta u_1^k(n) - \Delta u_1^0(n)\bigr) \\
&\qquad + \underline{\rho_2} \sum_{n\in\mathbb{Z}} \bigl(\phi_2\bigl(\Delta u_2^k(n)\bigr) - \phi_2\bigl(\Delta u_2^0(n)\bigr), \Delta u_2^k(n) - \Delta u_2^0(n)\bigr) \\
&\qquad + \underline{\rho_3} \sum_{n\in\mathbb{Z}} \bigl(\phi_3\bigl(u_1^k(n)\bigr) - \phi_3\bigl(u_1^0(n)\bigr), u_1^k(n) - u_1^0(n)\bigr) \\
&\qquad + \underline{\rho_4} \sum_{n\in\mathbb{Z}} \bigl(\phi_4\bigl(u_2^k(n)\bigr) - \phi_4\bigl(u_2^0(n)\bigr), u_2^k(n) - u_2^0(n)\bigr) \\
&\qquad - \lambda \sum_{n\in\mathbb{Z}} \bigl[\bigl(\nabla_{u_1} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_1} F\bigl(n, u_1^0(n), u_2^0(n)\bigr), u_1^k(n) - u_1^0(n)\bigr) \\
&\qquad + \bigl(\nabla_{u_2} F\bigl(n, u_1^k(n), u_2^k(n)\bigr) - \nabla_{u_2} F\bigl(n, u_1^0(n), u_2^0(n)\bigr), u_2^k(n) - u_2^0(n)\bigr)\bigr].
\end{aligned} \tag{3.10}$$

Moreover, since $\mathcal{J}'(u_k) \to 0$ and $u_k \rightharpoonup u_0$ as $k \to \infty$, we have

$$\langle \mathcal{J}'(u_k) - \mathcal{J}'(u_0), u_k - u_0 \rangle \to 0, \quad \text{as } k \to \infty. \tag{3.11}$$

Since $(\phi_i(x) - \phi_i(y), x - y) \geq 0$ for all $x, y \in R^N$, $\lambda > 0$, (3.10) and (3.11), together with (3.8) and (3.9), imply that

$$\sum_{n\in\mathbb{Z}} \bigl(\phi_1\bigl(\Delta u_1^k(n)\bigr) - \phi_1\bigl(\Delta u_1^0(n)\bigr), \Delta u_1^k(n) - \Delta u_1^0(n)\bigr) \to 0, \quad \text{as } k \to +\infty, \tag{3.12}$$

$$\sum_{n\in\mathbb{Z}} \bigl(\phi_2\bigl(\Delta u_2^k(n)\bigr) - \phi_2\bigl(\Delta u_2^0(n)\bigr), \Delta u_2^k(n) - \Delta u_2^0(n)\bigr) \to 0, \quad \text{as } k \to +\infty, \tag{3.13}$$

$$\sum_{n\in\mathbb{Z}} \bigl(\phi_3\bigl(u_1^k(n)\bigr) - \phi_3\bigl(u_1^0(n)\bigr), u_1^k(n) - u_1^0(n)\bigr) \to 0, \quad \text{as } k \to +\infty, \tag{3.14}$$

$$\sum_{n\in\mathbb{Z}} \bigl(\phi_4\bigl(u_2^k(n)\bigr) - \phi_4\bigl(u_2^0(n)\bigr), u_2^k(n) - u_2^0(n)\bigr) \to 0, \quad \text{as } k \to +\infty. \tag{3.15}$$

If $1 < p \leq 2$, then it follows from $(\mathcal{A}_2)$ and the Hölder inequality that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \bigl|\Delta u_1^k(n) - \Delta u_1^0(n)\bigr|^p \\
&\quad = \sum_{n\in\mathbb{Z}} \bigl|\Delta u_1^k(n) - \Delta u_1^0(n)\bigr|^{\frac{2p}{2}} \\
&\quad \leq \frac{1}{c_1^{\frac{p}{2}}} \sum_{n\in\mathbb{Z}} \bigl(\phi_1\bigl(\Delta u_1^k(n)\bigr) - \phi_1\bigl(\Delta u_1^0(n)\bigr), \Delta u_1^k(n) - \Delta u_1^0(n)\bigr)^{\frac{p}{2}} \\
&\qquad \cdot \bigl(\bigl|\Delta u_1^k(n)\bigr| + \bigl|\Delta u_1^0(n)\bigr|\bigr)^{\frac{p(2-p)}{2}} \\
&\quad \leq \frac{1}{c_1^{\frac{p}{2}}} \biggl(\sum_{n\in\mathbb{Z}} \bigl(\phi_1\bigl(\Delta u_1^k(n)\bigr) - \phi_1\bigl(\Delta u_1^0(n)\bigr), \Delta u_1^k(n) - \Delta u_1^0(n)\bigr)\biggr)^{\frac{p}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\left(\sum_{n\in\mathbb{Z}}\left(\left|\Delta u_1^k(n)\right|+\left|\Delta u_1^0(n)\right|\right)^p\right)^{\frac{2-p}{2}}\\
&\le \frac{2^{\frac{p(2-p)}{2}}}{c_1^{\frac{p}{2}}}\left(\sum_{n\in\mathbb{Z}}\left(\phi_1\left(\Delta u_1^k(n)\right)-\phi_1\left(\Delta u_1^0(n)\right),\Delta u_1^k(n)-\Delta u_1^0(n)\right)\right)^{\frac{p}{2}}\\
&\cdot\left(\sum_{n\in\mathbb{Z}}\left(\left|\Delta u_1^k(n)\right|^p+\left|\Delta u_1^0(n)\right|^p\right)\right)^{\frac{2-p}{2}}\\
&\le \frac{2^{\frac{p(2-p)}{2}}}{c_1^{\frac{p}{2}}}\left(\sum_{n\in\mathbb{Z}}\left(\phi_1\left(\Delta u_1^k(n)\right)-\phi_1\left(\Delta u_1^0(n)\right),\Delta u_1^k(n)-\Delta u_1^0(n)\right)\right)^{\frac{p}{2}}\\
&\cdot\left(\left\|u_1^k\right\|_p^p+\left\|u_1^0\right\|_p^p\right)^{\frac{2-p}{2}}.
\end{aligned}
\tag{3.16}
$$

Similarly, we have

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}\left|u_1^k(n)-u_1^0(n)\right|^p\\
&\le \frac{2^{\frac{p(2-p)}{2}}}{c_3^{\frac{p}{2}}}\left(\sum_{n\in\mathbb{Z}}\left(\phi_3\left(u_1^k(n)\right)-\phi_3\left(u_1^0(n)\right),u_1^k(n)-u_1^0(n)\right)\right)^{\frac{p}{2}}\\
&\cdot\left(\left\|u_1^k\right\|_p^p+\left\|u_1^0\right\|_p^p\right)^{\frac{2-p}{2}}.
\end{aligned}
\tag{3.17}
$$

If $p > 2$, then it follows from $(\mathcal{A}_2)$ and the Hölder inequality that

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}\left|\Delta u_1^k(n)-\Delta u_1^0(n)\right|^p\\
&\le \frac{1}{c_1}\sum_{n\in\mathbb{Z}}\left(\phi_1\left(\Delta u_1^k(n)\right)-\phi_1\left(\Delta u_1^0(n)\right),\Delta u_1^k(n)-\Delta u_1^0(n)\right),
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}\left|u_1^k(n)-u_1^0(n)\right|^p\\
&\le \frac{1}{c_3}\sum_{n\in\mathbb{Z}}\left(\phi_1\left(u_1^k(n)\right)-\phi_1\left(u_1^0(n)\right),u_1^k(n)-u_1^0(n)\right).
\end{aligned}
\tag{3.19}
$$

By (3.12)-(3.19), it is easy to see that $u_1^k \to u_1^0$ in E_p for any $p > 1$. Similarly, we can obtain $u_2^k \to u_2^0$ in E_q for any $q > 1$. So, $u_k \to u_0$ in E, that is, $\mathcal{J}$ satisfies the PS condition.

Let $\varphi = \mathcal{J}$. By Lemma 2.3, $c = \inf_E \mathcal{J}(u)$ is a critical value of $\mathcal{J}$, that is, there exists a critical point $u^* \in E$ such that $\mathcal{J}(u^*) = c$.

Finally, we show that $u^* \neq 0$. Let $u_*(n_0) = (u_{1*}(n_0), u_{2*}(n_0))$ where $u_{1*}(n_0) = (0,\dots,1,\dots,0)^\tau \in \mathbb{R}^N$ with 1 is the i_0th component of the vector, $u_{2*}(n_0) = (0,\dots,1,\dots 0)^\tau \in \mathbb{R}^N$ with 1 is the j_0th component of the vector, and $u_*(n) = 0$ for $n \neq n_0$, where i_0, j_0 are defined in assumption (f). Then, by (F_2) and (2.7), we have

$$
\begin{aligned}
\mathcal{J}(su_*) = \sum_{n\in\mathbb{Z}}\Big[&\rho_1(n)\Phi_1\left(\Delta su_{1*}(n)\right)+\rho_2(n)\Phi_2\left(\Delta su_{2*}(n)\right)\\
&+\rho_3(n)\Phi_3\left(su_{1*}(n)\right)+\rho_4(n)\Phi_4\left(su_{2*}(n)\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- \lambda \sum_{n\in\mathbb{Z}} F\big(n, su_{1*}(n), su_{2*}(n)\big) + \sum_{n\in\mathbb{Z}} \big(f_1(n), u_{1*}(n)\big) + \sum_{n\in\mathbb{Z}} \big(f_2(n), u_{2*}(n)\big) \\
\le{}& \overline{\rho_1} s^p d_1 \sum_{n\in\mathbb{Z}} \big|\Delta u_{1*}(n)\big|^p + \overline{\rho_2} s^q d_2 \sum_{n\in\mathbb{Z}} \big|\Delta u_{2*}(n)\big|^q + \overline{\rho_3} s^p d_3 \sum_{n\in\mathbb{Z}} \big|u_{1*}(n)\big|^p \\
&+ \overline{\rho_4} s^q d_4 \sum_{n\in\mathbb{Z}} \big|u_{2*}(n)\big|^q - \lambda F\big(n_0, su_{1*}(n_0), su_{2*}(n_0)\big) \\
&+ \big(f_1(n_0), su_{1*}(n_0)\big) + \big(f_2(n_0), su_{2*}(n_0)\big) \\
\le{}& \overline{\rho_1} s^p d_1 \big(\big|\Delta u_{1*}(n_0)\big|^p + \big|\Delta u_{1*}(n_0-1)\big|^p\big) \\
&+ \overline{\rho_2} s^q d_2 \big(\big|\Delta u_{2*}(n_0)\big|^q + \big|\Delta u_{2*}(n_0-1)\big|^q\big) \\
&+ \overline{\rho_3} s^p d_3 \big|u_{1*}(n_0)\big|^p + \overline{\rho_4} s^q d_4 \big|u_{2*}(n_0)\big|^q \\
&+ \lambda\eta_1 s^{\gamma_3} \big|u_{1*}(n_0)\big|^{\gamma_3} + \lambda\eta_2 s^{\gamma_4} \big|u_{2*}(n_0)\big|^{\gamma_4} + sf_{1i_0}(n_0) + sf_{2j_0}(n_0) \\
={}& (2\overline{\rho_1} d_1 + \overline{\rho_3} d_3) s^p + (2\overline{\rho_2} d_2 + \overline{\rho_4} d_4) s^q + \lambda\eta_1 s^{\gamma_3} + \lambda\eta_2 s^{\gamma_4} \\
&+ s\big(f_{1i_0}(n_0) + f_{2j_0}(n_0)\big),
\end{aligned} \tag{3.20}
$$

for all $0 < s < \delta_0$. Since $p, q, \gamma_3, \gamma_4 \in (1, +\infty)$, it follows from (f) that $\mathcal{J}(su_*) < 0$ for $s > 0$ small enough. Hence, $\mathcal{J}(u^*) = c = \inf_E \mathcal{J}(u) < 0$, which implies that $u^* \in E$ is a nontrivial critical point of $\mathcal{J}$ and so $u^* = u^*(n)$ is a nontrivial homoclinic solution of system (1.1). The proof is complete. □

Proof of Theorem 1.2 By the proof of Theorem 1.1, we know that there exists a critical point $u^* \in E$ such that $\mathcal{J}(u^*) = c$. Next, we prove that $u^* \neq 0$ when $(F_2)'$ and $(f)'$ hold. We define the same u_* as Theorem 1.1. Then, by $\lambda > 0$, $(F_2)'$, and $(f)'$, we have

$$
\begin{aligned}
\mathcal{J}(su_*) ={}& \sum_{n\in\mathbb{Z}} \big[\rho_1(n)\Phi_1\big(\Delta su_{1*}(n)\big) + \rho_2(n)\Phi_2\big(\Delta su_{2*}(n)\big) \\
&+ \rho_3(n)\Phi_3\big(su_{1*}(n)\big) + \rho_4(n)\Phi_4\big(su_{2*}(n)\big)\big] \\
&- \lambda \sum_{n\in\mathbb{Z}} F\big(n, su_{1*}(n), su_{2*}(n)\big) + \sum_{n\in\mathbb{Z}} \big(f_1(n), u_{1*}(n)\big) + \sum_{n\in\mathbb{Z}} \big(f_2(n), u_{2*}(n)\big) \\
\le{}& \overline{\rho_1} s^p d_1 \sum_{n\in\mathbb{Z}} \big|\Delta u_{1*}(n)\big|^p + \overline{\rho_2} s^q d_2 \sum_{n\in\mathbb{Z}} \big|\Delta u_{2*}(n)\big|^q + \overline{\rho_3} s^p d_3 \sum_{n\in\mathbb{Z}} \big|u_{1*}(n)\big|^p \\
&+ \overline{\rho_4} s^q d_4 \sum_{n\in\mathbb{Z}} \big|u_{2*}(n)\big|^q - \lambda F\big(n_0, su_{1*}(n_0), su_{2*}(n_0)\big) \\
&+ \big(f_1(n_0), su_{1*}(n_0)\big) + \big(f_2(n_0), su_{2*}(n_0)\big) \\
\le{}& \overline{\rho_1} s^p d_1 \big(\big|\Delta u_{1*}(n_0)\big|^p + \big|\Delta u_{1*}(n_0-1)\big|^p\big) \\
&+ \overline{\rho_2} s^q d_2 \big(\big|\Delta u_{2*}(n_0)\big|^q + \big|\Delta u_{2*}(n_0-1)\big|^q\big) + \overline{\rho_3} s^p d_3 \big|u_{1*}(n_0)\big|^p \\
&+ \overline{\rho_4} s^q d_4 \big|u_{2*}(n_0)\big|^q - \lambda\eta_1 s^{\gamma_3} \big|u_{1*}(n_0)\big|^{\gamma_3} \\
&- \lambda\eta_2 s^{\gamma_4} \big|u_{2*}(n_0)\big|^{\gamma_4} + sf_{1i_0}(n_0) + sf_{2j_0}(n_0) \\
={}& (2\overline{\rho_1} d_1 + \overline{\rho_3} d_3) s^p + (2\overline{\rho_2} d_2 + \overline{\rho_4} d_4) s^q - \lambda\eta_1 s^{\gamma_3} - \lambda\eta_2 s^{\gamma_4},
\end{aligned} \tag{3.21}
$$

for all $0 < s < \delta_0$. Since $1 < \gamma_3 < p$ and $1 < \gamma_4 < q$, $\mathcal{J}(su_*) < 0$ for $s > 0$ small enough. Hence, $\mathcal{J}(u^*) = c = \inf_E \mathcal{J}(u) < 0$, which implies that $u^* \in E$ is a nontrivial critical point of $\mathcal{J}$

and so $u^* = u^*(n)$ is a nontrivial homoclinic solution of system (1.1). The proof is complete. □

Proof of Theorem 1.3 By Lemma 2.2, $\mathcal{J} \in C^1(E,\mathbb{R})$. Similar to (3.1), it follows from (ρ), $(\mathcal{A}_1)$, $(F_1)'$, and (2.11), by replacing γ_1, γ_2, $\|a_1\|_{l^{p/(p-\gamma_1)}}$, and $\|a_2\|_{l^{q/(q-\gamma_2)}}$ with p, q, $\|a_1\|_{l^\infty}$, and $\|a_2\|_{l^\infty}$, respectively, that

$$\begin{aligned}\mathcal{J}(u) \geq{}& \min\{\underline{\rho_1}b_1, \underline{\rho_3}b_3\}\|u_1\|_p^p + \min\{\underline{\rho_2}b_2, \underline{\rho_4}b_4\}\|u_2\|_q^q \\ &- \frac{\lambda}{p}\|a_1\|_\infty\|u_1\|_p^p - \frac{\lambda}{q}\|a_2\|_\infty\|u_2\|_q^q \\ &- \lambda\|b_1\|_{l^{p/(p-1)}}\|u_1\|_p - \lambda\|b_2\|_{l^{q/(q-1)}}\|u_2\|_q \\ &- \|f_1\|_{l^{\frac{p}{p-1}}}\|u_1\|_p - \|f_2\|_{l^{\frac{q}{q-1}}}\|u_2\|_q. \end{aligned} \tag{3.22}$$

Note that $\lambda < \min\{\frac{p\min\{\underline{\rho_1}b_1,\underline{\rho_3}b_3\}}{\|a_1\|_\infty}, \frac{q\min\{\underline{\rho_2}b_2,\underline{\rho_4}b_4\}}{\|a_2\|_\infty}\}$. Then (3.22) shows that $\mathcal{J}(u) \to +\infty$ as $\|u\| \to +\infty$, which implies that $\mathcal{J}$ is bounded from below.

Next, we show that $\mathcal{J}$ satisfies the PS condition. Suppose that $\{u_k = (u_1^k, u_2^k)\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{\mathcal{J}(u_k)\}_{k\in\mathbb{N}}$ is bounded and $\mathcal{J}'(u_k) \to 0$ as $k \to +\infty$. Similar to the proof of Theorem 1.1, by (3.22), there exists a constant $M_0 > 0$ such that (3.2)-(3.4) hold. Note that $a_1(n) \to 0$ as $n \to \infty$ and $b_1 \in l^{\frac{p}{p-1}}(\mathbb{Z}, [0,+\infty))$. Then, for any given $\varepsilon > 0$, there exists an integer $M_1 > 0$ such that

$$\sup_{|n|>M_1}|a_1(n)| < \varepsilon, \qquad \Big(\sum_{|n|>M_1}|b_1(n)|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} < \varepsilon. \tag{3.23}$$

It follows from (3.2)-(3.4) and $(F_1)'$ that (3.6) holds. On the other hand, it follows from (3.2), (3.4), (3.23), $(F_1)'$, and Young's inequality that

$$\begin{aligned}&\sum_{|n|>M_1}\big|\nabla_{u_1}F\big(n,u_1^k(n),u_2^k(n)\big) - \nabla_{u_1}F\big(n,u_1^0(n),u_2^0(n)\big)\big|\big|u_1^k(n) - u_1^0(n)\big| \\ &\quad\leq \sum_{|n|>M_1}\big[|a_1(n)|\big(|u_1^k(n)|^{p-1} + |u_1^0(n)|^{p-1}\big) + 2b_1(n)\big]\big(|u_1^k(n)| + |u_1^0(n)|\big) \\ &\quad\leq 3\sum_{|n|>M_1}|a_1(n)|\big(|u_1^k(n)|^p + |u_1^0(n)|^p\big) + 2\sum_{|n|>M_1}b_1(n)\big(|u_1^k(n)| + |u_1^0(n)|\big) \\ &\quad\leq 3\sup_{|n|>M_1}|a_1(n)|\big(\|u_1^k\|_{l^p}^p + \|u_1^0\|_{l^p}^p\big) \\ &\qquad + 2\Big(\sum_{|n|>M_1}|b_1(n)|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\big(\|u_1^k\|_{l^p} + \|u_1^0\|_{l^p}\big) \\ &\quad\leq 3\sup_{|n|>M_1}|a_1(n)|\big(\|u_1^k\|_p^p + \|u_1^0\|_p^p\big) \\ &\qquad + 2\Big(\sum_{|n|>M_1}|b_1(n)|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\big(\|u_1^k\|_p + \|u_1^0\|_p\big) \\ &\quad\leq 3\varepsilon\big(M_0^p + \|u_1^0\|_p^p\big) + 2\varepsilon\big(M_0 + \|u_1^0\|_p\big), \quad \forall k \in N.\end{aligned}$$

Then arbitrariness of ε, together with (3.6), implies that

$$\sum_{n\in\mathbb{Z}}\bigl(\nabla_{u_1}F\bigl(n,u_1^k(n),u_2^k(n)\bigr)\\ -\nabla_{u_1}F\bigl(n,u_1^0(n),u_2^0(n)\bigr),u_1^k(n)-u_1^0(n)\bigr)\to 0,\quad \text{as } k\to+\infty.$$

Similarly, we have

$$\sum_{n\in\mathbb{Z}}\bigl(\nabla_{u_2}F\bigl(n,u_1^k(n),u_2^k(n)\bigr)\\ -\nabla_{u_2}F\bigl(n,u_1^0(n),u_2^0(n)\bigr),u_2^k(n)-u_2^0(n)\bigr)\to 0,\quad \text{as } k\to+\infty.$$

Following the argument of Theorem 1.1, we can obtain $u_k\to u_0$ in E, that is, $\mathcal{J}$ satisfies the PS condition.

Let $\varphi=\mathcal{J}$. By Lemma 2.3, $c=\inf_E\mathcal{J}(u)$ is a critical value of $\mathcal{J}$, that is, there exists a critical point $u^*\in E$ such that $\mathcal{J}(u^*)=c$.

Finally, with the same argument as Theorem 1.1, we know that $u^*\neq 0$. The proof is complete. □

Proof of Theorem 1.4 In view of Lemma 2.1 and the proof of Theorem 1.1, $\mathcal{J}\in C^1(E,\mathbb{R})$ is bounded from below and satisfies the PS condition. It follows from $(\mathcal{A}_0)$, (F_1), (F_3), and $(f)''$ that $\mathcal{J}$ is even and $\mathcal{J}(0)=0$. In order to apply Lemma 2.4, let $\varphi=\mathcal{J}$. We prove now that there is a set $K\subset E$ such that K is homeomorphic to S^{m-1} by an odd map and $\sup_K\mathcal{J}<0$. The proof is motivated by [7] and [19]. Let

$$J=\{n_1,n_2,\dots,n_m\},$$

where $n_1<n_2<\cdots<n_m$. Note that $m\le N$. Define

$$u_j^i(n)=\begin{cases}(0,\dots,0,\underset{\substack{\downarrow\\ i}}{1},0,\dots,0)^\tau\in\mathbb{R}^N, & n=n_i,\\ 0, & n\neq n_i,\end{cases}\qquad i=1,2,\dots,m, j=1,2,$$

$$u^i(n)=\bigl(u_1^i(n),u_2^i(n)\bigr)^\tau,\quad i=1,2,\dots,m,$$

and

$$E_m=\operatorname{span}\bigl\{u^1,u^2,\dots,u^m\bigr\},\qquad K_m=\bigl\{u\in E_m:\|u\|_{(2)}=\delta_0\bigr\},\tag{3.24}$$

where $\|u\|_{(2)}$ is defined by $\|u\|_{(2)}=\|u_1\|_{l^2}+\|u_2\|_{l^2}$. For any $u\in E_m$, there exist $\lambda_i\in\mathbb{R}$, $i=1,2,\dots,m$, such that

$$u=\sum_{i=1}^m\lambda_iu^i\quad\text{and}\quad\bigl(u_1(n),u_2(n)\bigr)=\sum_{i=1}^m\lambda_i\bigl(u_1^i(n),u_2^i(n)\bigr),\quad\text{for } n\in\mathbb{Z}.\tag{3.25}$$

Then

$$\begin{aligned}\|u_1\|_{l^{\gamma_3}} &= \left(\sum_{n\in\mathbb{Z}}\bigl|u_1(n)\bigr|^{\gamma_3}\right)^{1/\gamma_3} = \left(\sum_{i=1}^{m}|\lambda_i|^{\gamma_3}\bigl|u_1^i(n_i)\bigr|^{\gamma_3}\right)^{1/\gamma_3},\\ \|u_2\|_{l^{\gamma_4}} &= \left(\sum_{n\in\mathbb{Z}}\bigl|u_2(n)\bigr|^{\gamma_4}\right)^{1/\gamma_4} = \left(\sum_{i=1}^{m}|\lambda_i|^{\gamma_4}\bigl|u_2^i(n_i)\bigr|^{\gamma_4}\right)^{1/\gamma_4}.\end{aligned}\tag{3.26}$$

Note that $|u_1^i(n_i)|^2 = |u_2^i(n_i)|^2 = 1$, $i = 1,2,\ldots,m$. Hence

$$\begin{aligned}\|u\|_{(2)}^2 &= \bigl(\|u_1\|_{l^2} + \|u_2\|_{l^2}\bigr)^2\\ &= \|u_1\|_{l^2}^2 + 2\|u_1\|_{l^2}\|u_2\|_{l^2} + \|u_2\|_{l^2}^2\\ &= \sum_{n\in\mathbb{Z}}\bigl|u_1(n)\bigr|^2 + 2\left(\sum_{n\in\mathbb{Z}}\bigl|u_1(n)\bigr|^2\right)^{1/2}\left(\sum_{n\in\mathbb{Z}}\bigl|u_2(n)\bigr|^2\right)^{1/2} + \sum_{n\in\mathbb{Z}}\bigl|u_2(n)\bigr|^2\\ &= \sum_{n\in\mathbb{Z}}\left(\sum_{i=1}^{m}\lambda_i u_1^i(n), \sum_{i=1}^{m}\lambda_i u_1^i(n)\right)\\ &\quad + \sum_{n\in\mathbb{Z}}\left(\sum_{i=1}^{m}\lambda_i u_2^i(n), \sum_{i=1}^{m}\lambda_i u_2^i(n)\right)\\ &\quad + 2\left(\sum_{n\in\mathbb{Z}}\left(\sum_{i=1}^{m}\lambda_i u_1^i(n), \sum_{i=1}^{m}\lambda_i u_1^i(n)\right)\right)^{1/2}\\ &\quad \cdot\left(\sum_{n\in\mathbb{Z}}\left(\sum_{i=1}^{m}\lambda_i u_2^i(n), \sum_{i=1}^{m}\lambda_i u_2^i(n)\right)\right)^{1/2}\\ &= \sum_{i=1}^{m}\lambda_i^2\bigl|u_1^i(n_i)\bigr|^2 + \sum_{i=1}^{m}\lambda_i^2\bigl|u_2^i(n_i)\bigr|^2\\ &\quad + 2\left(\sum_{i=1}^{m}\lambda_i^2\bigl|u_1^i(n_i)\bigr|^2\right)^{1/2}\left(\sum_{i=1}^{m}\lambda_i^2\bigl|u_2^i(n_i)\bigr|^2\right)^{1/2}\\ &= 4\sum_{i=1}^{m}\lambda_i^2.\end{aligned}\tag{3.27}$$

Since all the norms of a finite dimensional normed space are equivalent, there are constants $R_i > 0, i = 1,2,3,4$, such that

$$\begin{aligned}&\|u_1\|_p \le R_1\|u_1\|_{l^2}, \qquad \|u_2\|_q \le R_2\|u_2\|_{l^2},\\ &R_3\|u_1\|_{l^2} \le \|u_1\|_{l^{\gamma_3}}, \qquad R_4\|u_2\|_{l^2} \le \|u_2\|_{l^{\gamma_4}}, \quad \text{for } u_1,u_2 \in E_m.\end{aligned}\tag{3.28}$$

Note that $\delta_0 \in (0,1)$. Then, for all $u \in K_m$, we have

$$\begin{aligned}&\min\bigl\{\lambda\eta_1(sR_3)^{\gamma_3}, \lambda\eta_2(sR_4)^{\gamma_4}\bigr\}\bigl(\|u_1\|_{l^2} + \|u_2\|_{l^2}\bigr)^{\max\{\gamma_3,\gamma_4\}}\\ &\quad\le 2^{\max\{\gamma_3,\gamma_4\}}\min\bigl\{\lambda\eta_1(sR_3)^{\gamma_3}, \lambda\eta_2(sR_4)^{\gamma_4}\bigr\}\bigl(\|u_1\|_{l^2}^{\gamma_3} + \|u_2\|_{l^2}^{\gamma_4}\bigr)\\ &\quad\le 2^{\max\{\gamma_3,\gamma_4\}}\bigl[\lambda\eta_1(sR_3)^{\gamma_3}\|u_1\|_{l^2}^{\gamma_3} + \lambda\eta_2(sR_4)^{\gamma_4}\|u_2\|_{l^2}^{\gamma_4}\bigr].\end{aligned}\tag{3.29}$$

Note that $F(n,0,0) = 0$ for all $n \in \mathbb{Z}$ and $\lambda > 0$. Then, by $(\mathcal{A}_1)$, $(F_2)'''$, $(f)''$, (2.7), (3.24), (3.26), (3.28), and (3.29), we have

$$\begin{aligned}
\mathcal{J}(su) &= \sum_{n\in\mathbb{Z}} \rho_1(n)\Phi_1\big(\Delta su_1(n)\big) + \sum_{n\in\mathbb{Z}} \rho_2(n)\Phi_2\big(\Delta su_2(n)\big) \\
&\quad + \sum_{n\in\mathbb{Z}} \rho_3(n)\Phi_3\big(su_1(n)\big) + \sum_{n\in\mathbb{Z}} \rho_4(n)\Phi_4\big(su_2(n)\big) \\
&\quad - \lambda \sum_{n\in\mathbb{Z}} F\big(n, su_1(n), su_2(n)\big) \\
&\le \overline{\rho_1} d_1 s^p \sum_{n\in\mathbb{Z}} \big|\Delta u_1(n)\big|^p + \overline{\rho_2} d_2 s^q \sum_{n\in\mathbb{Z}} \big|\Delta u_2(n)\big|^q \\
&\quad + \overline{\rho_3} d_3 s^p \sum_{n\in\mathbb{Z}} \big|u_1(n)\big|^p \\
&\quad + \overline{\rho_4} d_4 s^q \sum_{n\in\mathbb{Z}} \big|u_2(n)\big|^q - \lambda \sum_{i=1}^{m} F\big(n_i, s\lambda_i u_1^i(n_i), s\lambda_i u_2^i(n_i)\big) \\
&\le \max\{\overline{\rho_1} d_1, \overline{\rho_3} d_3\} s^p \|u_1\|_p^p + \max\{\overline{\rho_2} d_2, \overline{\rho_4} d_4\} s^q \|u_2\|_q^q \\
&\quad - \lambda \sum_{i=1}^{m} \big[\eta_1 \big|\lambda_i s u_1^i(n_i)\big|^{\gamma_3} + \eta_2 \big|\lambda_i s u_2^i(n_i)\big|^{\gamma_4}\big] \\
&= \max\{\overline{\rho_1} d_1, \overline{\rho_3} d_3\} s^p \|u_1\|_p^p + \max\{\overline{\rho_2} d_2, \overline{\rho_4} d_4\} s^q \|u_2\|_q^q \\
&\quad - \lambda \eta_1 s^{\gamma_3} \sum_{i=1}^{m} |\lambda_i|^{\gamma_3} \big|u_1^i(n_i)\big|^{\gamma_3} - \lambda \eta_2 s^{\gamma_4} \sum_{i=1}^{m} |\lambda_i|^{\gamma_4} \big|u_2^i(n_i)\big|^{\gamma_4} \\
&= \max\{\overline{\rho_1} d_1, \overline{\rho_3} d_3\} s^p \|u_1\|_p^p + \max\{\overline{\rho_2} d_2, \overline{\rho_4} d_4\} s^q \|u_2\|_q^q \\
&\quad - \lambda \eta_1 s^{\gamma_3} \|u_1\|_{l^{\gamma_3}}^{\gamma_3} - \lambda \eta_2 s^{\gamma_4} \|u_2\|_{l^{\gamma_4}}^{\gamma_4} \\
&\le \max\{\overline{\rho_1} d_1, \overline{\rho_3} d_3\} (sR_1)^p \|u_1\|_{l^2}^p + \max\{\overline{\rho_2} d_2, \overline{\rho_4} d_4\} (sR_2)^q \|u_2\|_{l^2}^q \\
&\quad - \lambda \eta_1 (sR_3)^{\gamma_3} \|u_1\|_{l^2}^{\gamma_3} - \lambda \eta_2 (sR_4)^{\gamma_4} \|u_2\|_{l^2}^{\gamma_4} \\
&\le \max\{\overline{\rho_1} d_1, \overline{\rho_3} d_3\} (sR_1)^p \|u_1\|_{l^2}^p + \max\{\overline{\rho_2} d_2, \overline{\rho_4} d_4\} (sR_2)^q \|u_2\|_{l^2}^q \\
&\quad - \frac{1}{2^{\max\{\gamma_3,\gamma_4\}}} \min\big\{\lambda \eta_1 (sR_3)^{\gamma_3}, \lambda \eta_2 (sR_4)^{\gamma_4}\big\} \\
&\quad \cdot \big(\|u_1\|_{l^2} + \|u_2\|_{l^2}\big)^{\max\{\gamma_3,\gamma_4\}} \\
&\le \max\{\overline{\rho_1} d_1, \overline{\rho_3} d_3\} (sR_1)^p \delta_0^p + \max\{\overline{\rho_2} d_2, \overline{\rho_4} d_4\} (sR_2)^q \delta_0^q \\
&\quad - \frac{\lambda}{2^{\max\{\gamma_3,\gamma_4\}}} \min\big\{\eta_1 (sR_3)^{\gamma_3}, \eta_2 (sR_4)^{\gamma_4}\big\} \delta_0^{\max\{\gamma_3,\gamma_4\}} \\
&\le \max\big\{\overline{\rho_1} d_1 R_1^p \delta_0^p, \overline{\rho_3} d_3 R_1^p \delta_0^p, \overline{\rho_2} d_2 R_2^q \delta_0^q, \overline{\rho_4} d_4 R_2^q \delta_0^q\big\} s^{\min\{p,q\}} \\
&\quad - \frac{\lambda}{2^{\max\{\gamma_3,\gamma_4\}}} \min\big\{\eta_1 (sR_3)^{\gamma_3}, \eta_2 (sR_4)^{\gamma_4}\big\} \delta_0^{\max\{\gamma_3,\gamma_4\}},
\end{aligned} \tag{3.30}$$

for all $u = (u_1, u_2)^\tau \in K_m$ and $0 < s < \min\{1, \delta_0(\sum_{i=1}^m |\lambda_i|)^{-1}\}$. Note that $\gamma_3, \gamma_4 \in (1, \min\{p,q\})$. Then (3.30) implies that, for any given $\lambda > 0$, there exist sufficiently small $s_{0\lambda} \in (0,1)$ and $\varepsilon > 0$ such that

$$\mathcal{J}(s_{0\lambda} u) < -\varepsilon, \quad \forall u \in K_m. \tag{3.31}$$

Let

$$K_m^{s_{0\lambda}} = \{s_{0\lambda} u : u \in K_m\}$$

and

$$S^{m-1} = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_m)^\tau \in R^m : \sum_{i=1}^{m} \lambda_i^2 = 1 \right\}. \tag{3.32}$$

Then

$$K_m^{s_{0\lambda}} = \left\{ \sum_{i=1}^{m} \lambda_i u^i : \sum_{i=1}^{m} \lambda_i^2 = \frac{s_{0\lambda}^2 \delta_0^2}{4} \right\}. \tag{3.33}$$

Define the map $\psi : K_m^{s_{0\lambda}} \to S^{m-1}$ by

$$\psi(u) = \frac{4}{s_{0\lambda}^2 \delta_0^2} (\lambda_1, \lambda_2, \ldots, \lambda_m)^\tau, \quad \forall u \in K_m^{s_{0\lambda}}. \tag{3.34}$$

It is easy to verify that $\psi : K_m^{s_{0\lambda}} \to S^{m-1}$ is an odd homeomorphic map. On the other hand, by (3.31), we have

$$\mathcal{J}(u) < -\varepsilon, \quad \text{for } u \in K_m^{s_{0\lambda}}, \tag{3.35}$$

and so $\sup_{K_m^{s_{0\lambda}}} \mathcal{J} \le -\varepsilon < 0$. Therefore, by Lemma 2.4, $\mathcal{J}$ has at least m distinct pairs of critical points, so system (1.1) possesses at least m distinct pairs of nontrivial homoclinic solutions. The proof is complete. □

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