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Iterative roots of upper semicontinuous multifunctions

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Abstract

The square iterative roots for strictly monotonic and upper semicontinuous functions with one set-valued point were fully described in (Li et al. in *Publ. Math. (Debr.)* 75:203–220, 2009). As a continuation, we study both strictly monotonic and nonmonotonic multifunctions. We present sufficient and necessary conditions under which those multifunctions have n th iterative roots. This equivalent condition and the construction method of n th iterative roots extend the previous results.

MSC: 39B12; 37E05

Keywords: iterative root; one-to-one; upper semicontinuous multifunction; set-valued point

1 Introduction

Given a mapping $F : X \rightarrow X$ and an integer $n \geq 2$, the iterative root problem is to find all self-mappings $f : X \rightarrow X$ such that their n th iterates satisfy the functional equation

$$f^n = F. \quad (1.1)$$

Babbage [2] investigated (1.1) for an identity mapping F as far back as the 1810s. After that, (1.1) has been studied in various aspects and settings since it is an important subject in the theory of functional equations; we refer to the survey papers [3–7], the monographs [8, 9], and the book [10]. For all we know, strictly increasing roots of strictly increasing and continuous functions were discussed by Bødewadt [11], and their strictly decreasing roots were presented by Haĭdukov [12]. In 1961, Kuczma [13] gave a complete description of strictly monotonic and continuous functions having roots. However, even simple nonmonotonic functions can have no iterative roots, for example, the hat functions $f(x) = \min\{\frac{x}{a}, \frac{1-x}{1-a}\}$ on the compact interval $[0, 1]$ for arbitrary $a \in (0, 1)$. In 1983, Zhang and Yang [14] investigated the roots of piecewise monotonic functions (abbreviated as PM functions). The main difficulties to find roots of PM functions lie in the continuously increasing number of nonmonotonic points under iteration (see [15]). Their method is based on the ‘characteristic interval’, which was developed in [7, 16]. In recent years, many important results on iterative roots of PM functions were presented in [17–19]. It is worth mentioning that those results are related to single-valued functions. In [20, 21] and [22], it is illustrated that the set of continuous functions having a root is a non-Borel subset of

$C([0, 1], \mathbb{R})$ and is small in $C([0, 1], [0, 1])$. That is to say, in the general case, no such roots exist, and the theory becomes extremely complicated if F is not bijective [23]. Therefore, it is a natural idea to extend the notion of iterative root.

In his survey paper [5], Targonski illustrated three ways to generalize iterative roots, extending or restricting the domain of the function or embedding the semigroup of self-mappings in a larger semigroup, and discussed the so-called phantom iterative root of continuous functions in [23]. Powierza and Jarczyk [24–26] gave set-valued functions as roots of single-valued functions. Maybe the best method to generalize iterative roots is replacing single-valued functions by set-valued functions for both F and f in (1.1) (see [4]). It seems that, up to now, there are only several results on set-valued iterative roots of multifunctions, even with a unique set-valued point. In 2007, Jarczyk and Zhang [27] considered the nonexistence of square iterative roots of multifunctions with exactly one set-value point and presented two sufficient conditions for the purely set-theoretical situation. Later, Li, Jarczyk, Jarczyk, and Zhang [1] gave new nonexistence results for purely set-theoretical case and fully described the square roots of strictly monotonic, upper semi-continuous (abbreviated as usc) multifunctions.

As a continuation of [1], in this paper, we study all strictly monotonic usc multifunctions having one set-valued point and partly nonmonotonic ones. We give sufficient and necessary conditions for the existence of n th iterative roots and their construction method, which extend the results on strictly monotonic usc multifunctions in [1]. In Section 2, we recall the basic definitions and present Lemmas 1-3. In Section 3, we give equivalent conditions for the existence of n th iterative roots and their expressions. Finally, in Section 4, we apply examples to illustrate our results.

2 Preliminaries

Given topological spaces X and Y , a multifunction $f : X \rightarrow 2^Y$ is called upper semicontinuous at a point $x_0 \in X$ if for every open set $V \subset Y$ with $f(x_0) \subset V$, there exists a neighborhood $U \subset X$ of x_0 such that $f(U) \subset V$. If f is upper semicontinuous at every point of $B \subset X$, then it is called upper semicontinuous on a set $B \subset X$. Let $F : X \rightarrow X$ and $G : Y \rightarrow Y$ be continuous functions. We say that F is topologically conjugate to G if there exists a homeomorphism $\varphi : X \rightarrow Y$ satisfying the equation $\varphi \circ F = G \circ \varphi$.

Let X be an interval $I := [a, b] \subset \mathbb{R}$. For a multifunction $f : I \rightarrow 2^I$, the image $f(A)$ for a set $A \subset I$ is defined by $f(A) := \bigcup_{x \in A} f(x)$. For a given positive integer n , we define the n th iterate of $f : I \rightarrow 2^I$ as the composition of n copies of f :

$$f^n(x) := \bigcup_{y \in f^{n-1}(x)} f(y),$$

where $f^0(x) := \{x\}$ for every $x \in I$. Let $\#A$ denote the cardinality of the set A . Then a point $x_0 \in (a, b)$ is said to be a set-valued point of f if the cardinality $\#f(x_0) \geq 2$. Denote by $\mathcal{S}_{x_0}(I, I)$ the set of all multifunctions $f : I \rightarrow 2^I$ that are continuous on I except for the single set-valued point x_0 . We say that f is one-to-one on I if $f(x_1) \neq f(x_2)$ for all different $x_1, x_2 \in I \setminus \{x_0\}$.

Definition 1 ([1]) Let $\gamma : I \rightarrow I$ be a strictly increasing continuous function. Given a fixed point $\xi \in I$ of γ , we put

$$A_\xi = \{x \in \text{cl} I : \gamma(x) = x \leq \xi\}, \quad B_\xi = \{x \in \text{cl} I : \gamma(x) = x \geq \xi\}.$$

Then ξ is called a regular fixed point of γ if

- (a) $\min A_\xi = \inf I$ if and only if $\max B_\xi = \sup I$;
- (b) there exists a strictly decreasing function a mapping A_ξ onto B_ξ ;
- (c) for every component (a, b) of the set $\{x \in \text{cl} I : \gamma(x) \neq x\}$, where $a \in A_\xi \cup \{\inf I\}$, $b \in A_\xi$, and $a < b$, the graphs of $\gamma|_{(a,b)}$ and $\gamma|_{(\alpha(a), \alpha(b))}$ lie on the opposite sides of the diagonal.

Definition 2 ([1]) A multifunction $f : I \rightarrow 2^I$ is called strictly increasing (strictly decreasing) if $\sup f(x_1) < \inf f(x_2)$ ($\inf f(x_1) < \sup f(x_2)$) whenever $x_1, x_2 \in I$ and $x_1 < x_2$. Multifunctions that are either strictly increasing or strictly decreasing are called strictly monotonic.

It is clear that every strictly monotonic multifunction is one-to-one. Conversely, a one-to-one multifunction is not necessarily strictly monotonic. In this paper, we investigate the n th iterative roots of usc multifunctions $F \in \mathcal{S}_{x_0}(I, I)$ of the form

$$F(x) = \begin{cases} F|_{I_-}, & x \in I_- := [a, x_0), \\ [c, d] \text{ } ([d, c]), & x = x_0, \\ F|_{I_+}, & x \in I_+ := (x_0, b], \end{cases} \quad (2.1)$$

where $[c, d] \text{ } ([d, c]) \subset I$, and $F|_{I_-}$ and $F|_{I_+}$ are strictly monotonic and satisfy

$$\lim_{x \rightarrow x_0^-} F(x) = c, \quad \lim_{x \rightarrow x_0^+} F(x) = d.$$

The next lemmas describe the fundamental properties of the n th iterative roots of (2.1).

Lemma 1 If $F \in \mathcal{S}_{x_0}(I, I)$ is one-to-one, then every n th iterative root f of F belongs to $\mathcal{S}_{x_0}(I, I)$ and is also one-to-one.

Proof We first prove that f is one-to-one. Suppose on the contrary that there exist two different points $u, v \in I$ such that $f(u) = f(v)$. Then $f^n(u) = f^n(v)$, and thus $F(u) = F(v)$. Since F is one-to-one, only the case $u = v$ is possible, contrary to the assumption.

Now we show that $f \in \mathcal{S}_{x_0}(I, I)$. For any $y \in I \setminus \{x_0\}$, suppose on the contrary that $\#f(y) \geq 2$. Then we have

$$\#F(y) = \#f^n(y) \geq 2,$$

which contradicts $F \in \mathcal{S}_{x_0}(I, I)$. Thus, $f(y)$ is a singleton for all $y \in I \setminus \{x_0\}$. We claim that x_0 is a unique set-valued point of f . Otherwise, let $f(x_0) = \{p_0\}$. Then two cases are possible: either $p_0 = x_0$ or $p_0 \neq x_0$. From the former we have

$$F(x_0) = f^n(x_0) = x_0,$$

which contradicts $\#F(x_0) \geq 2$. For the case $p_0 \neq x_0, f \in \mathcal{S}_{x_0}(I, I)$ implies that $\#f(p_0) = 1$ since f is single-valued at $I \setminus \{x_0\}$. Without loss of generality, assume that $f(p_0) = \{p_1\}$. Then there again exist two cases: either $p_1 = x_0$ or $p_1 \neq x_0$. If $p_1 = x_0$, a contradiction comes from

$$F(x_0) = f^{n-2}(p_1) = f^n(x_0) = \{x_0\}.$$

If $p_1 \neq x_0$, we have $\#f(p_1) = 1$ since f is single-valued except for x_0 . Repeating this progress, we inductively obtain

$$\#f(p_{n-2}) = 1 \quad \text{for } p_{n-2} \neq x_0,$$

and, consequently, we have

$$\#F(x_0) = \#f^n(x_0) = \#f(p_{n-2}) = 1,$$

which contradicts $F \in \mathcal{S}_{x_0}(I, I)$. Thus, we prove $f \in \mathcal{S}_{x_0}(I, I)$. This completes the proof. \square

Lemma 2 *If $F \in \mathcal{S}_{x_0}(I, I)$ is one-to-one, then F has no n th iterative root $f \in \mathcal{S}_{x_0}(I, I)$ taking the value $\{x_0\}$.*

Proof Assume that f is an n th iterative root of F . We claim that $f \in \mathcal{S}_{x_0}(I, I)$ by Lemma 1. Suppose on the contrary that $f(y_0) = \{x_0\}$ for a point $y_0 \in I \setminus \{x_0\}$. Then

$$F(y_0) = f^n(y_0) = f^{n-1}(x_0), \quad (2.2)$$

whence

$$\#F(y_0) = \#f^{n-1}(x_0) \geq 2,$$

implying $y_0 = x_0$, a contradiction. This completes the proof. \square

Lemma 3 *Suppose that $F \in \mathcal{S}_{x_0}(I, I)$ is one-to-one.*

- (i) *If F is strictly increasing, then F has no strictly decreasing n th iterative root for odd n .*
- (ii) *If F is strictly decreasing, then F has no strictly increasing n th iterative root for even n .*

This proof is trivial and omitted.

3 Main results

In this section, we give several sufficient and necessary conditions and expressions of n th iterative roots of (2.1). Theorem 1 and Theorem 3 characterize the strictly monotonic usc multifunctions, and nonmonotonic cases are investigated in Theorem 2 and Theorem 4. For convenience, let $f_1 := f|_{I_-}, f_2 := f|_{I_+}$.

Theorem 1 *Suppose that the usc multifunction (2.1) has $F(x_0) = [c, d]$. If $F|_{I_-}$ and $F|_{I_+}$ are strictly increasing, then F has n th iterative roots if and only if F satisfies $F(b) < x_0$ or $F(a) > x_0$ or $c \leq x_0 \leq d$.*

(i) If $F(b) < x_0$, then F has a strictly increasing n th iterative root

$$f(x) = \begin{cases} f_1(x), & x \in I_-, \\ f_1^{-(n-1)} \circ F(x), & x \in I_+ \cup \{x_0\}, \end{cases} \quad (3.1)$$

where $f_1 : I_- \rightarrow I_-$ is a strictly increasing function satisfying $f_1^n = F|_{I_-}$. Moreover, if n is even and $F|_{I_-}$ has a regular fixed point, then F also has a strictly decreasing n th iterative root of the form (3.1), in which f_1 is a strictly decreasing function satisfying $f_1^n = F|_{I_-}$.

(ii) If $F(a) > x_0$, then F has a strictly increasing n th iterative root

$$f(x) = \begin{cases} f_2^{-(n-1)} \circ F(x), & x \in I_- \cup \{x_0\}, \\ f_2(x), & x \in I_+, \end{cases} \quad (3.2)$$

where $f_2 : I_+ \rightarrow I_+$ is a strictly increasing function satisfying $f_2^n = F|_{I_+}$. Moreover, if n is even and $F|_{I_+}$ has a regular fixed point, then F also has a strictly decreasing n th iterative root of the form (3.2), in which f_2 is a strictly decreasing function satisfying $f_2^n = F|_{I_+}$.

(iii) If $c \leq x_0 \leq d$, then F has a strictly increasing n th iterative root

$$f(x) = \begin{cases} f_1(x), & x \in I_-, \\ [f_1^{-(n-1)}(c), f_2^{-(n-1)}(d)], & x = x_0, \\ f_2(x), & x \in I_+, \end{cases} \quad (3.3)$$

where f_1 and f_2 are defined as in (3.1) and (3.2), respectively. Moreover, if n is even and $F|_{I_+}$ is topologically conjugate to $F|_{I_-}$ by a strictly decreasing function $f_1 : I_- \rightarrow I_+$, that is, $F|_{I_+} = f_1 \circ F|_{I_-} \circ f_1^{-1}$, then F also has a strictly decreasing n th iterative root

$$f(x) = \begin{cases} f_1(x), & x \in I_-, \\ [f_1^{-1} \circ g|_{I_+} \circ (F|_{I_+})^{-1}(d), (g|_{I_+})^{-1} \circ f_1 \circ g|_{I_-} \circ (F|_{I_-})^{-1}(c)], & x = x_0, \\ g|_{I_-} \circ f_1^{-1}(x), & x \in I_+, \end{cases} \quad (3.4)$$

in which g is a strictly increasing $\frac{n}{2}$ th iterative root of F .

Proof Necessity. If F has n th iterative roots, then F does not take the value $\{x_0\}$ by Lemma 2. Therefore, only the following cases are possible:

Case 1. $F(I) \subset I_-$, i.e., $F(b) < x_0$, or

Case 2. $F(I) \subset I_+$, i.e., $F(a) > x_0$, or

Case 3. $F(I_-) \subset I_-$ and $F(I_+) \subset I_+$, i.e., $c \leq x_0 \leq d$.

Sufficiency. Consider case (i). Observing that $F|_{I_-} : I_- \rightarrow I_-$ is strictly increasing, it follows from Theorem 11.2.2 in [9] that there exists a strictly increasing function $f_1 : I_- \rightarrow I_-$ such that

$$f_1^n = F|_{I_-}. \quad (3.5)$$

For every $x \in I_+ \cup \{x_0\}$, using that $F(x) \subset I_-$ and f_1 is strictly increasing, from

$$F(x) = f^{n-1}(f(x)) = f_1^{n-1}(f(x)) \quad (3.6)$$

we get

$$f(x) = f_1^{-(n-1)} \circ F(x), \quad x \in I_+ \cup \{x_0\}. \quad (3.7)$$

Furthermore, by the upper semicontinuity property at the set-value point x_0 we have

$$\begin{aligned} c = F(x_0^-) &= f_1^{n-1} \circ f(x_0^-), \\ d = F(x_0^+) &= f_1^{n-1} \circ f(x_0^+). \end{aligned} \quad (3.8)$$

Since f_1^{n-1} is strictly increasing, it follows from (3.8) that

$$f(x_0^-) < f(x_0^+). \quad (3.9)$$

Hence, formulas (3.5), (3.7), and (3.9) give a strictly increasing n th iterative root (3.1) of F .

Assuming that n is even, the left end-point a is not a fixed point of $F|_-$ since it has a regular fixed point, and there exists a strictly decreasing n th iterative root $f_1 : I_- \rightarrow I_-$ of $F|_-$ (see [9, p.427]). Moreover, since f_1 is strictly decreasing, using (3.8), we have

$$f(x_0^-) > f(x_0^+). \quad (3.10)$$

Thus, f_1 and (3.7) and (3.10) give a strictly decreasing n th iterative root (3.1) of F .

The proof of case (ii) is immediately obtained by the transformation

$$g(x) := b - a - f(x) \quad \text{for } x \in I. \quad (3.11)$$

Case (iii). The condition $c \leq x_0 \leq d$ implies that $F(I_-) \subset I_-$ and $F(I_+) \subset I_+$. Since $F|_{I_-}$ and $F|_{I_+}$ are strictly increasing, their strictly increasing n th iterative roots f_1 and f_2 exist and are defined as in (3.1) and (3.2), respectively. Observing that f is one-to-one by Lemma 1 and f_1 and f_2 are strictly increasing, we have

$$\begin{aligned} c = F(x_0^-) &= f^n(x_0^-) = f^{n-1}(f(x_0^-)) = f_1^{n-1}(f(x_0^-)), \\ d = F(x_0^+) &= f^n(x_0^+) = f^{n-1}(f(x_0^+)) = f_2^{n-1}(f(x_0^+)), \end{aligned} \quad (3.12)$$

which yields (3.9). Consequently, f_1 and f_2 together with (3.12) and (3.9) give a strictly increasing n th iterative root (3.3) of F .

If n is even and $F|_{I_+}$ is topologically conjugate to $F|_{I_-}$ by a strictly decreasing function $f_1 : I_- \rightarrow I_+$, then we will prove that F also has a strictly decreasing n th iterative root f of F . In fact, this problem reduces itself to a solution of the system

$$\begin{cases} f^2 = g, \\ g^{\frac{n}{2}} = F. \end{cases} \quad (3.13)$$

As in the previous argument, F possesses a strictly increasing $\frac{n}{2}$ th iterative root g , and we confine ourselves to the first equation $f^2 = g$ to find a strictly decreasing solution f of F . The given condition $F|_{I_+} = f_1 \circ F|_{I_-} \circ f_1^{-1}$ implies that the strictly increasing functions $g|_{I_+}$ and $g|_{I_-}$ satisfy the conjugacy equation

$$g|_{I_+} = f_1 \circ g|_{I_-} \circ f_1^{-1}, \quad (3.14)$$

and the equality $g^{\frac{n}{2}} = F$ shows that

$$g|_{I_+} = f_1 \circ f_2, \quad g|_{I_-} = f_2 \circ f_1. \quad (3.15)$$

The second equality in (3.15) yields that

$$f_2 = g|_{I_-} \circ f_1^{-1}, \quad (3.16)$$

implying that $f_2 : I_+ \rightarrow I_-$ also is strictly decreasing. Moreover,

$$\begin{aligned} c &= F(x_0^-) \\ &= g^{\frac{n}{2}}(x_0^-) \\ &= (g|_{I_-})^{\frac{n}{2}-1} \circ f_2 \circ f_1(x_0^-) \\ &= F|_{I_-} \circ (g|_{I_-})^{-1} \circ f_2 \circ f_1(x_0^-), \\ d &= F(x_0^+) \\ &= g^{\frac{n}{2}}(x_0^+) \\ &= (g|_{I_+})^{\frac{n}{2}-1} \circ f_1 \circ f_2(x_0^+) \\ &= F|_{I_+} \circ (g|_{I_+})^{-1} \circ f_1 \circ f_2(x_0^+). \end{aligned} \quad (3.17)$$

Since F and g are strictly increasing, $f_1 : I_- \rightarrow I_+$ and $f_2 : I_+ \rightarrow I_-$ are strictly decreasing, and f is one-to-one, it follows from (3.17) that (3.10) holds. Thus, the given f_1 , together with (3.10), (3.16), and (3.17), leads to (3.4). Refer to Figure 1. This completes the proof. \square

Theorem 2 Suppose that the usc multifunction (2.1) with $F(x_0) = [d, c]$ is one-to-one. If $F|_{I_-}$ and $F|_{I_+}$ are strictly increasing, then F has n th iterative roots if and only if F satisfies $F(b) < F(a) < c \leq x_0$ or $F(a) > F(b) > d \geq x_0$ or $F(b) < x_0 < F(a)$.

(i) If $F(b) < F(a) < c \leq x_0$, then F has a nonmonotonic n th iterative root defined by (3.1). Moreover, if n is even and $F|_{I_-}$ has a regular fixed point, then F also has a nonmonotonic

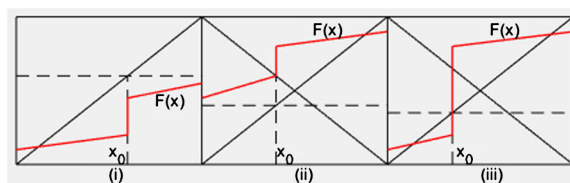


Figure 1 Three cases in Theorem 1.

n th iterative root of the form (3.1), in which f_1 is a strictly decreasing function satisfying $f_1^n = F|_{I_-}$.

(ii) If $F(a) > F(b) > d \geq x_0$, then F has a nonmonotonic n th iterative root defined by (3.2). Moreover, if n is even and $F|_{I_+}$ has a regular fixed point, then F has also a nonmonotonic n th iterative root of the form (3.2), in which f_2 is a strictly decreasing function satisfying $f_2^n = F|_{I_+}$.

(iii) If $F(b) < x_0 < F(a)$ and n is odd, then F has a nonmonotonic n th iterative root

$$f(x) = \begin{cases} F|_{I_-} \circ g^{-\frac{n-1}{2}}(x), & x \in I_-, \\ [(g|_{I_-})^{-\frac{n-1}{2}}(d), (g|_{I_+})^{-\frac{n-1}{2}}(c)], & x = x_0, \\ F|_{I_+} \circ g^{-\frac{n-1}{2}}(x), & x \in I_+, \end{cases} \quad (3.18)$$

where g is a strictly increasing n th iterative root of F^2 . If n is even, then F has no iterative roots.

Proof The necessity directly comes from Lemma 2. In what follows, our attention is paid to the sufficiency.

Sufficiency. For case (i), we first construct a strictly increasing function (3.5) and a multifunction (3.7) as in case (i) of Theorem 1. Since f_1 is strictly increasing, from (3.8) and $c > d$ we have (3.10). Thus, (3.5), (3.7), and (3.10) yield a nonmonotonic n th iterative root (3.1) of F .

Assuming that n is even and $F|_{I_-}$ has a regular fixed point, we can construct a strictly decreasing function f_1 and a multifunction (3.7) as in case (i) of Theorem 1. Since f_1 is strictly decreasing, (3.8) and $c > d$ lead to (3.9). Therefore, f_1 together with (3.7) and (3.9) gives a nonmonotonic n th iterative root (3.1) of F .

The proof of case (ii) is obtained from case (i) by the translation (3.11).

Case (iii). The assumption $F(b) < x_0 < F(a)$ shows that

$$F(I_-) \subset I_+, \quad F(I_+) \subset I_-. \quad (3.19)$$

If F has an n th iterative root f , then, whether n is odd or even, we assert that

$$f(I_-) \subset I_+, \quad f(I_+) \subset I_-. \quad (3.20)$$

In fact, suppose on the contrary that $f(I_-) \subset I_-$, whether $f(I_+) \subset I_-$ or $f(I_+) \subset I_+$. Then we always have $F(I_-) = f^n(I_-) \subset I_-$, contrary to the first formula in (3.19). This implies that $f(I_-) \subset I_+$. The proof for $f(I_+) \subset I_-$ is similar, and thus we have proved (3.20). It follows from (3.19) and (3.20) that

$$\begin{aligned} F|_{I_-} &= (f_1 \circ f_2)^{\frac{n-1}{2}} \circ f_1, \\ F|_{I_+} &= (f_2 \circ f_1)^{\frac{n-1}{2}} \circ f_2, \end{aligned} \quad (3.21)$$

whence

$$\begin{aligned} F^2|_{I_-} &= F|_{I_+} \circ F|_{I_-} = (f_2 \circ f_1)^n, \\ F^2|_{I_+} &= F|_{I_-} \circ F|_{I_+} = (f_1 \circ f_2)^n. \end{aligned} \quad (3.22)$$

By Theorem 1(iii), (3.22) shows that there exists a strictly increasing n th iterative root g of F^2 satisfying

$$g|_{I_-} = f_2 \circ f_1, \quad g|_{I_+} = f_1 \circ f_2. \quad (3.23)$$

Substituting (3.23) into (3.21), we obtain

$$\begin{aligned} f_1 &= (g|_{I_+})^{-\frac{n-1}{2}} \circ F|_{I_-}, \\ f_2 &= (g|_{I_-})^{-\frac{n-1}{2}} \circ F|_{I_+}, \end{aligned} \quad (3.24)$$

which implies that f_1 and f_2 are strictly increasing. Moreover, from (3.24) we have

$$\begin{aligned} f_1(x_0^-) &= (g|_{I_+})^{-\frac{n-1}{2}} \circ F|_{I_-}(x_0^-) = (g|_{I_+})^{-\frac{n-1}{2}}(c), \\ f_2(x_0^+) &= (g|_{I_-})^{-\frac{n-1}{2}} \circ F|_{I_+}(x_0^+) = (g|_{I_-})^{-\frac{n-1}{2}}(d), \end{aligned} \quad (3.25)$$

implying $f_1(x_0^-) > f_2(x_0^+)$, that is, (3.10) holds. Thus, (3.24), (3.25), and (3.10) show that F has a nonmonotonic n th iterative root (3.18).

If n is even, then suppose on the contrary that F has an n th iterative root f . Since g is a strictly increasing n th iterative root of F^2 , we have

$$\begin{aligned} F(I_-) &= (g|_{I_-})^{\frac{n}{2}}(I_-) = (f_2 \circ f_1)^{\frac{n}{2}} \subset I_-, \\ F(I_+) &= (g|_{I_+})^{\frac{n}{2}}(I_+) = (f_1 \circ f_2)^{\frac{n}{2}} \subset I_+, \end{aligned}$$

contrary to the fact that $F(I_-) \subset I_+$ and $F(I_+) \subset I_-$. Refer to Figure 2. This completes the proof. \square

Theorem 3 Suppose that the usc multifunction (2.1) has $F(x_0) = [d, c]$. If $F|_{I_-}$ and $F|_{I_+}$ are strictly decreasing, then F has n th iterative roots if and only if n is odd and F satisfies $F(a) < x_0$ or $F(b) > x_0$ or $d \leq x_0 \leq c$.

(i) If $F(a) < x_0$ and n is odd, then F has a strictly decreasing n th iterative root of the form (3.1), where f_1 is defined by

$$f_1(x) = \begin{cases} F|_{I_-} \circ \chi^{-\frac{n-1}{2}}(x), & a \leq x < x_1, \\ \chi^{\frac{n+1}{2}} \circ (F|_{I_-})^{-1}(x), & x_1 \leq x < x_0, \end{cases} \quad (3.26)$$

in which x_1 is the unique fixed point of $F|_{I_-}$, and χ is a strictly increasing n th iterative root of $(F|_{I_-})^2$.

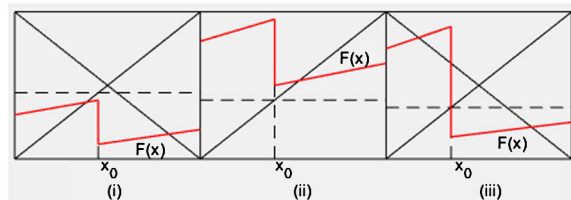


Figure 2 Three cases in Theorem 2.

(ii) If $F(b) > x_0$ and n is odd, then F has a strictly decreasing n th iterative root of the form (3.2), where f_2 is defined by

$$f_2(x) = \begin{cases} F|_{I_+} \circ \psi^{-\frac{n-1}{2}}(x), & x_0 < x \leq x_2, \\ \psi^{\frac{n+1}{2}} \circ (F|_{I_+})^{-1}(x), & x_2 < x \leq b, \end{cases} \quad (3.27)$$

in which x_2 is the unique fixed point of $F|_{I_+}$, and ψ is a strictly increasing n th iterative root of $(F|_{I_+})^2$.

(iii) If $d \leq x_0 \leq c$ and n is odd, then F has a strictly decreasing n th iterative root

$$f(x) = \begin{cases} F|_{I_-} \circ \phi^{-\frac{n-1}{2}}(x), & x \in I_-, \\ [(\phi|_{I_-})^{-\frac{n-1}{2}}(d), (\phi|_{I_+})^{-\frac{n-1}{2}}(c)], & x = x_0, \\ F|_{I_+} \circ \phi^{-\frac{n-1}{2}}(x), & x \in I_+, \end{cases} \quad (3.28)$$

where ϕ is a strictly increasing n th iterative root of F^2 .

Proof Necessity. Suppose on the contrary that n is even and F has an n th iterative root f . Then only the following cases are possible:

Case 1. $f(I) \subset I_-$, or

Case 2. $f(I) \subset I_+$, or

Case 3. $f(I_-) \subset I_-$, $f(I_+) \subset I_+$, or

Case 4. $f(I_-) \subset I_+$, $f(I_+) \subset I_-$.

If either $f(I) \subset I_-$ or $f(I) \subset I_+$, then it is clear that

$$F(I) = f^n(I) \subset I_+ \quad \text{or} \quad F(I) = f^n(I) \subset I_-,$$

contrary to the assumption on F . If $f(I_-) \subset I_-$ and $f(I_+) \subset I_+$, then

$$F(I_-) = f^n(I_-) \subset I_-, \quad F(I_+) = f^n(I_+) \subset I_+, \quad (3.29)$$

which also contradicts the assumption on F . If $f(I_-) \subset I_+$ and $f(I_+) \subset I_-$, then as n is even, we again get (3.29) and a contradiction. Thus, we have proved that n is odd. The remainder directly comes from Lemma 2.

Sufficiency. Case (i). Note that

$$F|_{I_-}([a, x_1]) \subset (x_1, x_0), \quad F|_{I_-}(x_1, x_0) \subset [a, x_1].$$

By Theorem 11.2.3 in [9] the strictly decreasing function $F|_{I_-}$ has a strictly decreasing n th iterative root f_1 of the form (3.26). From (3.6) we still have (3.7), and, moreover, (3.8) and $d < c$ yield (3.10). Hence, (3.26), (3.7), and (3.10) show that F has a strictly decreasing n th iterative root f of the form (3.1).

The proof of case (ii) is directly obtained from case (i) by the translation (3.11).

Case (iii). By the same argument as that of case (iii) in Theorem 2 we see that (3.21) still holds and there exists a strictly increasing n th iterative root ϕ of the strictly increasing

multifunction F^2 satisfying

$$\phi|_{I_-} = f_2 \circ f_1, \quad \phi|_{I_+} = f_1 \circ f_2. \quad (3.30)$$

Substituting (3.30) into (3.21), we obtain

$$\begin{aligned} f_1 &= (\phi|_{I_+})^{-\frac{n-1}{2}} \circ F|_{I_-}, \\ f_2 &= (\phi|_{I_-})^{-\frac{n-1}{2}} \circ F|_{I_+}, \end{aligned} \quad (3.31)$$

which implies both f_1 and f_2 are strictly decreasing. Moreover, since ϕ is strictly increasing, from (3.31) we have

$$\begin{aligned} f_1(x_0^-) &= (\phi|_{I_+})^{-\frac{n-1}{2}} \circ F|_{I_-}(x_0^-) = (\phi|_{I_+})^{-\frac{n-1}{2}}(c), \\ f_2(x_0^+) &= (\phi|_{I_-})^{-\frac{n-1}{2}} \circ F|_{I_+}(x_0^+) = (\phi|_{I_-})^{-\frac{n-1}{2}}(d), \end{aligned} \quad (3.32)$$

implying (3.10). Thus, formulas (3.31), (3.32), and (3.10) show that F has a strictly decreasing n th iterative root (3.28). Refer to Figure 3. This completes the proof. \square

Theorem 4 Suppose that the usc multifunction (2.1) with $F(x_0) = [c, d]$ is one-to-one, where $F|_{I_-}$ and $F|_{I_+}$ are strictly decreasing. Then F has n th iterative roots if and only if n is odd and F satisfies $F(a) < F(b) < d \leq x_0$ or $F(b) > F(a) > c \geq x_0$ or $F(a) < x_0 < F(b)$.

- (i) If $F(a) < F(b) < d \leq x_0$ and n is odd, then F has a nonmonotonic n th iterative root of the form (3.1), where f_1 is defined by (3.26).
- (ii) If $F(b) > F(a) > c \geq x_0$ and n is odd, then F has a nonmonotonic n th iterative root of the form (3.2), where f_2 is defined by (3.27).
- (iii) If $F(a) < x_0 < F(b)$ and n is odd, then F has a nonmonotonic n th iterative root of the form (3.3), where f_1 is defined by (3.26), and f_2 is defined by (3.27).

Proof The proof of necessity is similar to that of Theorem 3.

Sufficiency. Case (i). Using similar arguments as in the proof of case (i) of Theorem 3, we say that $F|_{I_-}$ has a strictly decreasing n th iterative root f_1 of the form (3.26). Moreover, (3.7) comes from (3.6), and (3.9) comes from (3.8) and $c < d$. Thus, (3.26), (3.7), and (3.9) prove that F has a nonmonotonic n th iterative root f of the form (3.1).

The proof of case (ii) is directly obtained from case (i) by the translation (3.11).

Case (iii). The assumption on F implies that $F(I_-) \subset I_-$ and $F(I_+) \subset I_+$. Since $F|_{I_-}$ and $F|_{I_+}$ are strictly decreasing, we can obtain their strictly decreasing n th iterative roots f_1 and f_2 defined by (3.26) and (3.27), respectively. Moreover, from (3.12) and $c < d$ we obtain (3.9).

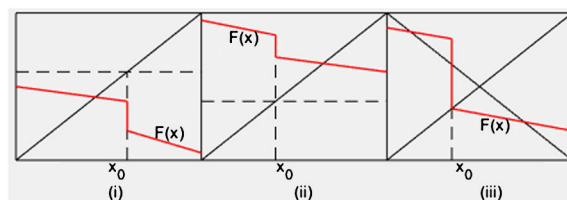


Figure 3 Three cases in Theorem 3.

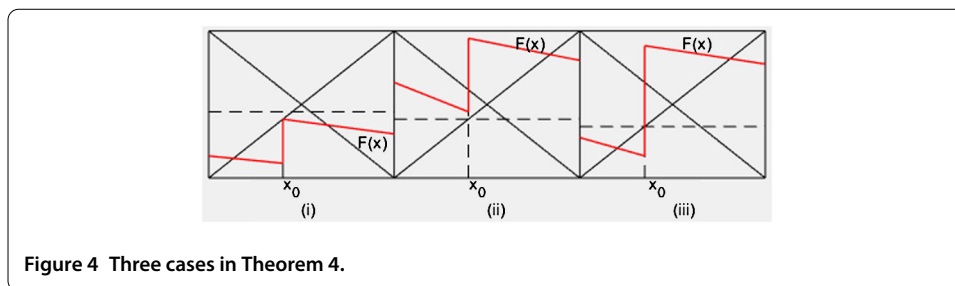


Figure 4 Three cases in Theorem 4.

Therefore, (3.26), (3.27), and (3.9) show that F has a nonmonotonic n th iterative root of the form (3.3). Refer to Figure 4. This completes the proof. \square

4 Examples

In this section, we give examples. For convenience, to demonstrate our main results and avoid complicated computations, we only present a third iterative root of the given usc multifunctions.

Example 1 Consider the usc multifunction $\phi_1 : [0, 1] \rightarrow 2^{[0,1]}$ defined by

$$\phi_1(x) = \begin{cases} \frac{27}{125}x + \frac{49}{250}, & x \in [0, \frac{1}{3}), \\ [\frac{67}{250}, \frac{1761}{2000}], & x = \frac{1}{3}, \\ \frac{27}{8000}x + \frac{1407}{1600}, & x \in (\frac{1}{3}, 1]. \end{cases}$$

Clearly, both $\phi_1|_{[0,1]_-}$ and $\phi_1|_{[0,1]_+}$ are strictly increasing. Since $\phi_1(\frac{1}{3}^-) < \frac{1}{3} < \phi_1(\frac{1}{3}^+)$, in view of case (iii) of Theorem 1, we say that ϕ_1 has a strictly increasing n th iterative root (3.3). In fact, ϕ_1 has a third iterative root as follows:

$$\varphi_1(x) = \begin{cases} \frac{3}{5}x + \frac{1}{10}, & x \in [0, \frac{1}{3}), \\ [\frac{3}{10}, \frac{4}{5}], & x = \frac{1}{3}, \\ \frac{3}{20}x + \frac{3}{4}, & x \in (\frac{1}{3}, 1]. \end{cases}$$

Example 2 Consider the usc multifunction $\phi_2 : [0, 1] \rightarrow 2^{[0,1]}$ defined by

$$\phi_2(x) = \begin{cases} \frac{8}{125}x + \frac{87}{125}, & x \in [0, \frac{1}{2}), \\ [\frac{43}{125}, \frac{91}{125}], & x = \frac{1}{2}, \\ \frac{8}{125}x + \frac{6}{25}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

We see that $\phi_2|_{[0,1]_-}$ and $\phi_2|_{[0,1]_+}$ are strictly increasing. As $\phi_2(1) < \frac{1}{2} < \phi_2(0)$, using case (iii) of Theorem 2, ϕ_2 has a nonmonotonic n th iterative root (3.18). Here, we give a third iterative root as follows:

$$\varphi_2(x) = \begin{cases} \frac{2}{5}x + \frac{3}{5}, & x \in [0, \frac{1}{2}), \\ [\frac{1}{5}, \frac{4}{5}], & x = \frac{1}{2}, \\ \frac{2}{5}x, & x \in (\frac{1}{2}, 1]. \end{cases}$$

Example 3 Consider the usc multifunction $\phi_3 : [0, 1] \rightarrow 2^{[0,1]}$ defined by

$$\phi_3(x) = \begin{cases} -\frac{3}{320}x + \frac{467}{960}, & x \in [0, \frac{1}{3}), \\ [\frac{29}{60}, \frac{185}{384}], & x = \frac{1}{3}, \\ -\frac{1}{512}x + \frac{247}{512}, & x \in (\frac{1}{3}, 1]. \end{cases}$$

Note that $\phi_3|_{[0,1]_-}$ and $\phi_3|_{[0,1]_+}$ are strictly decreasing and $\phi_3(1) > \frac{1}{3}$. From case (ii) of Theorem 3 we see that ϕ_3 has a strictly decreasing n th iterative root of the form (3.2). We present a third iterative root by simple calculation:

$$\varphi_3(x) = \begin{cases} -\frac{3}{5}x + \frac{4}{5}, & x \in [0, \frac{1}{3}), \\ [\frac{1}{2}, \frac{3}{5}], & x = \frac{1}{3}, \\ -\frac{1}{8}x + \frac{13}{24}, & x \in (\frac{1}{3}, 1]. \end{cases}$$

Example 4 Consider the usc multifunction $\phi_4 : [0, 1] \rightarrow 2^{[0,1]}$ defined by

$$\phi_4(x) = \begin{cases} -\frac{27}{1000}x + \frac{79}{125}, & x \in [0, \frac{2}{3}), \\ [\frac{167}{500}, \frac{307}{500}], & x = \frac{2}{3}, \\ -\frac{9}{500}x + \frac{173}{500}, & x \in (\frac{2}{3}, 1]. \end{cases}$$

Note that $\phi_4|_{[0,1]_-}$ and $\phi_4|_{[0,1]_+}$ are strictly decreasing and $\phi_4(0) < \phi_4(1) < \phi_4(\frac{2}{3}^+) < \frac{2}{3}$. In view of case (i) of Theorem 4, ϕ_4 has a nonmonotonic n th iterative root of the form (3.1). Here, a nonmonotone third iterative root of ϕ_4 is given as

$$\varphi_4(x) = \begin{cases} -\frac{3}{10}x + \frac{2}{5}, & x \in [0, \frac{2}{3}), \\ [\frac{1}{5}, \frac{3}{5}], & x = \frac{2}{3}, \\ -\frac{1}{5}x + \frac{11}{15}, & x \in (\frac{2}{3}, 1]. \end{cases}$$

Acknowledgements

The authors would like to thank the referees for their valuable comments and suggestions. This work is supported by Shandong Provincial Natural Science Foundation (ZR2017MA019, ZR2014AL003) and Scientific Research Fund of Binzhou University (BZXYL1703).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors completed the paper together. They also read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 January 2017 Accepted: 20 October 2017 Published online: 25 November 2017

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