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Some properties of algebraic difference equations of first order

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Abstract

We prove that if $g(z)$ is a finite-order transcendental meromorphic solution of

$$(\Delta_c g(z))^2 = A(z)g(z)g(z+c) + B(z),$$

where $A(z)$ and $B(z)$ are polynomials such that $\deg A(z) > 0$, then

$$1 \leq \rho(g) = \max\left\{\lambda(g), \lambda\left(\frac{1}{g}\right)\right\}.$$

MSC: 30D35; 39B12

Keywords: meromorphic functions; difference equations; value distribution; finite order

1 Introduction

Steinmetz [1] and Bank and Kaufman [2] proved that the equation

$$(g')^n = R(z, g)$$

can be reduced into a list of six simple differential equations by a suitable Möbius transformation with polynomial coefficients, which include

$$(g')^2 = p(z)(g - q(z))^2(g - \zeta)(g - \eta), \tag{1.1}$$

where ζ, η are constant, and $p(z), q(z)$ are rational functions. Let $q(z) \in \mathbb{C}$. Then equation (1.1) can be transformed into

$$(g')^2 = P(z)(g^2 - 1).$$

Ishizaki and Korhonen [3] investigated meromorphic solutions of

$$(\triangle g(z))^2 = P(z)(g(z)g(z+1) - Q(z)). \tag{1.2}$$

They proved that equation (1.2) possesses a continuous limit to the equation

$$\left(g'\right)^2 = P(z)\left(g^2 - 1\right),$$

which extends to solutions in certain cases.

We assume that the reader is familiar with the basic notions of Nevanlinna theory (see, e.g., [4, 5]). Of late, several scholars [3, 6–14] studied the properties of finite-order meromorphic solutions of algebraic difference equations and obtained many interesting results.

For the special case of (1.2), Whittaker [15] has shown that the equation

$$g(z+1) = q(z)g(z),$$

where $q(z)$ is a meromorphic function of finite order $\rho(q)$, has a meromorphic solution g such that $\rho(q) \leq \rho(g) \leq \rho(q) + 1$. Here $\rho(g)$ denotes the order of growth of the meromorphic function $g(z)$.

Chen [7] has extended this above result and proved that the Pielou logistic equation

$$g(z+1) = \frac{R(z)g(z)}{P(z) + Q(z)g(z)},$$

where $R(z)$, $P(z)$, and $Q(z)$ are polynomials with $P(z)R(z)Q(z) \not\equiv 0$, has a finite-order transcendental meromorphic solution g such that $1 \leq \rho(g)$.

Replacing $g(z+1)$ with $\triangle g(z)$, Ishizaki [10] was concerned with the growth and value distributions of transcendental meromorphic solutions of the algebraic difference equation

$$\left(\triangle g(z)\right)^2 = P(z)g(z).$$

In 2014, Liu [12] considered the Nevanlinna growth of an equation related to (1.2). It is interesting to consider some properties of (1.2), and our results will be stated in Section 2.

2 Main results

Theorem 2.1 *Let $c \in \mathbb{C} \setminus \{0\}$, and let $A(z)$ and $B(z)$ be polynomials such that* $\deg A(z) > 0$. *If $g(z)$ is a finite-order transcendental meromorphic solution of*

$$\left(\triangle_c g(z)\right)^2 = A(z)g(z)g(z+c) + B(z), \tag{2.1}$$

then

$$1 \leq \rho(g) = \max\left\{\lambda(g), \lambda\left(\frac{1}{g}\right)\right\}.$$

Remark It is a curious problem to construct a transcendental meromorphic solution of (2.1) for the case $\deg A > 0$.

Theorem 2.2 *Let $c \in \mathbb{C} \setminus \{0\}$, and let $E(z) = \frac{D(z)}{F(z)}$ be an irreducible rational function, where $D(z)$ and $F(z)$ are polynomials with* $\deg D(z) = d$ *and* $\deg F(z) = f$. *If the equation*

$$\left(\triangle_c g(z)\right)^2 = g(z)g(z+c) + E(z) \tag{2.2}$$

has a rational solution

$$g(z) = \frac{H(z)}{K(z)} = \frac{l_h z^h + \cdots + l_0}{m_k z^k + \cdots + m_0},$$

where $l_h\ (\neq 0), \ldots, l_0, m_k\ (\neq 0), \ldots, m_0$ *are constants,* $\deg H(z) = h$, *and* $\deg K(z) = k$.

(i) *If* $d \geq f$ *and* $d - f$ *is zero or an even number, then*

$$h - k = \frac{d-f}{2}.$$

(ii) *If* $d < f$, *then* $h - k = \frac{d-f}{2}$.

Further, Example 2.3 shows that there exist rational solutions satisfying Theorem 2.2(i), and Example 2.4 shows that there exist rational solutions satisfying Theorem 2.2(ii).

Example 2.3 The equation

$$\bigl(g(z+c) - g(z)\bigr)^2 = g(z+c)g(z) + c^2 - z^2 - (4+c)z - 2c - 4$$

has a rational solution $g(z) = z + 2$, where $d = 2, f = 0$, and $h - k = 1 = \frac{d-f}{2}$.

Example 2.4 The equation

$$\bigl(g(z+c) - g(z)\bigr)^2 = g(z+c)g(z) + \frac{c^2 - z(z+c)}{z^2(z+c)^2}$$

has a rational solution $g(z) = \frac{1}{z}$, where $d = 2, f = 4$, and $h - k = -1 = \frac{d-f}{2}$.

3 Proof of Theorem 2.1

Lemma 3.1 ([11]) *Let* $w(z)$ *be a transcendental meromorphic solution of finite order of the difference equation*

$$P(z, w) = 0,$$

where $P(z, w)$ *is a difference polynomial in* $w(z)$ *and its shift. If* $P(z, a) \not\equiv 0$ *for a slowly moving target function* a, *that is,* $T(r, a) = S(r, w)$, *then*

$$m\left(r, \frac{1}{w - a}\right) = S(r, w).$$

The following result obtained by Chiang and Feng [16] and Halburd and Korhonen [9, 17] independently. We state here the form stated in [16, Theorem 8.2(b)].

Lemma 3.2 ([16]) *Let* c_1, c_2 *be two arbitrary complex numbers, and let* $w(z)$ *be a meromorphic function of finite order* ρ. *Let* $\varepsilon > 0$ *be given. Then there exists a subset* $E \subset (1, \infty)$ *of finite logarithmic measure such that, for all* $|z| = r \notin E \cup [0, 1]$, *we have*

$$\exp\bigl(-r^{(\rho-1+\varepsilon)}\bigr) \leq \left|\frac{w(z+c_1)}{w(z+c_2)}\right| \leq \exp\bigl(r^{(\rho-1+\varepsilon)}\bigr).$$

Firstly, we prove that $\rho(g) = \rho \geq 1$. We consider the following two cases separately.

Case 1.1. If $g(z)$ has infinitely many poles, we can pick a pole z_0 of $g(z)$ such that $g(z_0) = \infty^{\pi}$, where $\pi \geq 1$, then we deduce by (2.1) that $g(z_0 + c) = \infty^{\pi_1}$, where $\pi_1 \geq m$. Substituting $z_0 + c$ for z into (2.1), we have

$$\big(g(z+2c) - g(z+c)\big)^2 = A(z+c)g(z+2c)g(z+c) + B(z+c). \tag{3.1}$$

Then (3.1) implies that $z_0 + 2c$ is a pole of g of multiplicity $\pi_2 \geq \pi_1 \geq \pi$.

Since $g(z)$ has infinitely many poles, following the previous steps, we pick a pole z_0 of $g(z)$ such that

$$g(z_0 + nc) = f(\xi_n) = \infty^{\pi_n},$$

where $\pi_n \geq \pi$ for all $n \in \mathbb{N}^0$. Hence, we can choose a sequence $\{\xi_n = z_0 + nc, n \in \mathbb{N}^0\}$ of poles of $g(z)$, the multiplicity of which is $\pi_n \geq \pi$, so we obtain $\lambda(\frac{1}{g}) \geq 1$, and therefore $\rho(g) \geq \lambda(\frac{1}{g}) \geq 1$.

Case 1.2. If $g(z)$ is a transcendental meromorphic function with finitely many poles, then we can rewrite $g(z)$ as

$$g(z) = \frac{g_1(z)}{P(z)}, \tag{3.2}$$

where $g_1(z)$ is a transcendental entire function, and $P(z)$ is a polynomial. Substituting (3.2) into (2.1), we have

$$\left(\frac{g_1(z+c)}{P(z+c)} - \frac{g_1(z)}{P(z)}\right)^2 = A(z)\frac{g_1(z+c)}{P(z+c)}\frac{g_1(z)}{P(z)} + B(z). \tag{3.3}$$

By computing (3.3) we have

$$\frac{P(z)}{P(z+c)}\frac{g_1(z+c)}{g_1(z)} + \frac{P(z+c)}{P(z)}\frac{g_1(z)}{g_1(z+c)} = 2 + A(z) + \frac{B(z)P(z)P(z+c)}{g_1(z)g_1(z+c)}. \tag{3.4}$$

We prove that $\rho(g) = \rho(g_1) = \rho \geq 1$. Suppose, on the contrary to the assertion, that $\rho(g) = \rho(g_1) = \rho < 1$. For any given ε $(0 < \varepsilon < \frac{1-\rho(g_1)}{2})$, by Lemma 3.2 we obtain

$$\begin{aligned} \left|\frac{g_1(z+c)}{g_1(z)}\right| &\leq \exp\big(r^{\rho(g_1)-1+\varepsilon}\big) = \exp\big(o(1)\big), \\ \left|\frac{g_1(z)}{g_1(z+c)}\right| &\leq \exp\big(r^{\rho(g_1)-1+\varepsilon}\big) = \exp\big(o(1)\big) \end{aligned} \tag{3.5}$$

outside a finite logarithmic measure E. As z_k satisfies $|g_1(z_k)| = M(r_k, g_1)$, $|z_k| = r_k \notin E$, $r_k \to \infty$, we deduce by (3.4) and (3.5) that

$$\begin{aligned} |A(z_k)| &= \left| \frac{P(z_k)}{P(z_k+c)} \frac{g_1(z_k+c)}{g_1(z_k)} + \frac{P(z_k+c)}{P(z_k)} \frac{g_1(z_k)}{g_1(z_k+c)} - \frac{B(z_k)P(z_k)P(z_k+c)}{g_1(z_k)g_1(z_k+c)} - 2 \right| \\ &\le \left| \frac{P(z_k)}{P(z_k+c)} \frac{g_1(z_k+c)}{g_1(z_k)} \right| + \left| \frac{P(z_k+c)}{P(z_k)} \frac{g_1(z_k)}{g_1(z_k+c)} \right| \\ &\quad + \left| \frac{B(z_k)P(z_k)P(z_k+c)}{M(r_k,g_1)^2} \frac{g_1(z_k)}{g_1(z_k+c)} \right| + 2 \\ &\le M, \end{aligned}$$

where M is some finite constant, a contradiction, since $\deg A(z) > 0$. Hence we have $\rho(g) \ge 1$.

Next, we prove that $\max\{\lambda(g), \lambda(\frac{1}{g})\} = \rho(g)$. If $B(z) \not\equiv 0$, then we set

$$P(z,g) = \big(g(z+c) - g(z)\big)^2 - A(z)g(z+c)g(z) - B(z).$$

Since $P(z,0) = -B(z) \not\equiv 0$, by Lemma 3.1 we deduce that

$$N\left(r, \frac{1}{g}\right) = T(r,g) + S(r,f).$$

Hence $\lambda(g) = \rho(g)$.

If $B(z) \equiv 0$, then (2.1) can be reduced to

$$\big(g(z+c) - g(z)\big)^2 = A(z)g(z+c)g(z).$$

Next, we prove that $\max\{\lambda(g), \lambda(\frac{1}{g})\} = \rho(g)$. Suppose, on the contrary to the assertion, that $\max\{\lambda(g), \lambda(\frac{1}{g})\} = \alpha < \rho(g)$. We next divide the proof into the following two cases.

Case 1. Suppose that $\rho(g) = 1$. Then we obtain

$$g(z) = m(z) \exp^{qz+p}, \tag{3.6}$$

where $q \ne 0$ and p are constants, and $m(z)$ is a meromorphic function such that $\rho(m) = \alpha < 1$. Substituting (3.6) into (2.1), we obtain

$$\big(m(z+c) \exp^{q(z+c)+p} - m(z) \exp^{qz+p}\big)^2 = A(z)m(z+c) \exp^{q(z+c)+p} m(z) \exp^{qz+p}. \tag{3.7}$$

By computing (3.7) we obtain

$$\begin{aligned} &m^2(z+c) \exp^{2qc+2p} \exp^{2qz} + m^2(z) \exp^{2p} \exp^{2qz} \\ &\quad = \big(A(z)+2\big) m(z)m(z+c) \exp^{qc+2p} \exp^{2qz}, \end{aligned} \tag{3.8}$$

that is,

$$\big(A(z)+2\big) \exp^{qc+2p} \exp^{2qz} = \frac{m(z+c)}{m(z)} \exp^{2qc+2p} \exp^{2qz} + \frac{m(z)}{m(z+c)} \exp^{2p} \exp^{2qz}. \tag{3.9}$$

By Lemma 3.2 we obtain

$$\begin{aligned} \left|\frac{m(z+c)}{m(z)}\right| &\leq \exp\bigl(r^{\rho(m)-1+\varepsilon}\bigr) = \exp\bigl(o(1)\bigr), \\ \left|\frac{m(z)}{m(z+c)}\right| &\leq \exp\bigl(r^{\rho(m)-1+\varepsilon}\bigr) = \exp\bigl(o(1)\bigr) \end{aligned} \tag{3.10}$$

outside a finite logarithmic measure. By (3.9) and (3.10), as $|z| \to \infty$, we obtain

$$\begin{aligned} \bigl|\bigl(A(z)+2\bigr)\exp^{qc+2p}\bigr| &= \left|\frac{m(z+c)}{m(z)}\exp^{2qc+2p} + \frac{m(z)}{m(z+c)}\exp^{2p}\right| \\ &\leq \left|\frac{m(z+c)}{m(z)}\exp^{2qc+2p}\right| + \left|\frac{m(z)}{m(z+c)}\exp^{2p}\right| \leq M_1 \end{aligned}$$

outside a finite logarithmic measure, where M_1 is a finite constant. This is impossible, since $\deg A(z) > 0$.

Case 2. Suppose that $\rho(g) > 1$. Then

$$g(z) = m(z)\exp^{l(z)}, \tag{3.11}$$

where $l(z)$ is a polynomial such that $\rho(g) = \deg l(z) > 1$, and $m(z)$ is a meromorphic function such that $\rho(m) < \rho(g)$. Substituting (3.11) into (2.1), we obtain

$$\bigl(m(z+c)\exp^{l(z+c)} - m(z)\exp^{l(z)}\bigr)^2 = A(z)m(z+c)\exp^{l(z+c)} m(z)\exp^{l(z)}. \tag{3.12}$$

Let

$$l(z) = p_k z^k + p_{k-1}z^{k-1} + \cdots + p_1 z + p_0,$$

where $p_k \neq 0$. Then

$$l(z+c) = p_k z^k + (ckp_k + p_{k-1})z^{k-1} + Q(z), \tag{3.13}$$

$$l(z+c) - l(z) = (ckp_k)z^{k-1} + Q_1(z), \tag{3.14}$$

where $Q(z)$ and $Q_1(z)$ are polynomials of degree at most $k-2$. Equalities (3.12) and (3.14) imply that

$$m^2(z+c)\exp^{2ckp_k z^{k-1}+2Q_1(z)} + m^2(z) = \bigl(A(z)+2\bigr)m(z+c)m(z)\exp^{ckp_k z^{k-1}+Q_1(z)},$$

that is,

$$\begin{aligned} \bigl|\exp^{2ckp_k z^{k-1}+2Q_1(z)}\bigr| &= \left|-\frac{m^2(z)}{m^2(z+c)} + \bigl(A(z)+2\bigr)\frac{m(z)}{m(z+c)}\exp^{ckp_k z^{k-1}+Q_1(z)}\right| \\ &\leq \left|\frac{m^2(z)}{m^2(z+c)}\right| + \left|\bigl(A(z)+2\bigr)\frac{m(z)}{m(z+c)}\exp^{ckp_k z^{k-1}+Q_1(z)}\right|. \end{aligned} \tag{3.15}$$

By Lemma 3.2 we obtain

$$\left|\frac{m(z)}{m(z+c)}\right| \leq \exp\bigl(r^{\rho(m)-1+\varepsilon}\bigr) \tag{3.16}$$

outside a possible set of finite logarithmic measure E. As $|z| = r \notin E \cup [0,1]$, and $r \to \infty$, we deduce by (3.15) and (3.16) that

$$\begin{aligned} &\bigl|\exp^{2ckp_k z^{k-1}+2Q_1(z)}\bigr| \\ &\quad \leq \exp\bigl(2r^{\rho(m)-1+\varepsilon}\bigr) + \bigl|r^M \exp\bigl(r^{\rho(m)-1+\varepsilon}\bigr)\exp^{ckp_k z^{k-1}+Q_1(z)}\bigr|, \end{aligned} \tag{3.17}$$

where M is a positive constant.

We can find a sequence $\{z_k\}$ ($|z_k| \to \infty$) such that $|z_k| = r_k \notin E \cup [0,1]$, and $cp_k z_k^{k-1} = |cp_k| r_k^{k-1}$ as $r_k \to \infty$. We obtain

$$\bigl|\exp^{2ckp_k z_k^{k-1}+2Q_1(z_k)}\bigr| = \exp^{2k|cp_k|r_k^{k-1}}\bigl|\exp^{Q_1(z_k)}\bigr| \geq \exp^{\frac{3}{2}k|cp_k|r_k^{k-1}}. \tag{3.18}$$

By (3.17) and (3.18), for any given ε ($0 < \varepsilon < \frac{k-\rho(m)}{2}$), we obtain

$$\exp^{\frac{3}{2}k|cp_k|r_k^{k-1}} \leq \exp\bigl(2r_k^{\rho(m)-1+\varepsilon}\bigr) + r_k^M \exp\bigl(r_k^{\rho(m)-1+\varepsilon}\bigr)\exp^{ckp_k r_k^{k-1}} \leq \exp^{\frac{6}{5}k|cp_k|r_k^{k-1}},$$

which is impossible. Hence we proved that $\max\{\lambda(g), \lambda(\frac{1}{g})\} = \rho(g)$. Theorem 2.1 is thus proved.

4 Proof of Theorem 2.2

Suppose that (2.2) has a rational solution $g(z)$ and has poles $l_1, l_2, \ldots, l_k$. Hence $g(z)$ can be represented as

$$g(z) = \frac{H(z)}{K(z)} = \sum_{j=1}^{k}\left[\frac{t_{js_j}}{(z-l_j)^{s_j}} + \cdots + \frac{t_{j1}}{(z-l_j)}\right] + b_0 + b_1 z + \cdots + b_r z^r, \tag{4.1}$$

where $b_0, \ldots, b_r, t_{js_j}, \ldots, t_{j1}$ are constants.

(i) If $d > f$ and $d - f$ is an even number, then (2.2) and (4.1) imply that

$$\left(\frac{H(z+c)}{K(z+c)} - \frac{H(z)}{K(z)}\right)^2 - \frac{H(z+c)}{K(z+c)}\frac{H(z)}{K(z)} = \frac{D(z)}{F(z)}. \tag{4.2}$$

Let $\deg H(z) = h$ and $\deg K(z) = k$. Suppose $h < k$. Then

$$\lim_{z\to\infty}\frac{H(z+c)}{K(z+c)} = 0, \qquad \lim_{z\to\infty}\frac{H(z)}{K(z)} = 0. \tag{4.3}$$

From (4.2) with (4.3) we obtain

$$\frac{D(z)}{F(z)} = \left(\frac{H(z+c)}{K(z+c)} - \frac{H(z)}{K(z)}\right)^2 - \frac{H(z+c)}{K(z+c)}\frac{H(z)}{K(z)} \to 0.$$

This is impossible, since $\frac{D(z)}{F(z)} \to \infty$ as $z \to \infty$.

Suppose $h = k$. Then

$$\lim_{z\to\infty} \frac{H(z+c)}{K(z+c)} = \beta, \qquad \lim_{z\to\infty} \frac{H(z)}{K(z)} = \beta, \tag{4.4}$$

where $\beta \in \mathbb{C} \setminus \{0\}$. Relations (4.2) and (4.4) yield that

$$\frac{D(z)}{F(z)} = \left(\frac{H(z+c)}{K(z+c)} - \frac{H(z)}{K(z)}\right)^2 - \frac{H(z+c)}{K(z+c)}\frac{H(z)}{K(z)} \to -\beta^2,$$

a contradiction, since $\frac{D(z)}{F(z)} \to \infty$ as $z \to \infty$. Hence, we obtain that $h > k$. So $b_r \neq 0$ $(r \geq 1)$. As $z \to \infty$, we have

$$\begin{aligned} &g(z) = b_r z^s\bigl(1+o(1)\bigr), \qquad g(z+c) = b_r z^s\bigl(1+o(1)\bigr), \\ &\frac{D(z)}{F(z)} = \alpha z^{d-f}\bigl(1+o(1)\bigr), \end{aligned} \tag{4.5}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$. As $z \to \infty$, by (4.2) and (4.5) we can deduce

$$-b_r^2 z^{2r}\bigl(1+o(1)\bigr) = \alpha z^{d-f}\bigl(1+o(1)\bigr). \tag{4.6}$$

Relation (4.6) implies that

$$h - k = r = \frac{d-f}{2}.$$

If $d = f$, then, as $z \to \infty$, we obtain

$$\frac{D(z)}{F(z)} = \alpha\bigl(1+o(1)\bigr),$$

where $\alpha \in \mathbb{C} \setminus \{0\}$. If $h < k$, then using the similar method as before, we can obtain a contradiction. If $h > k$, then $b_r \neq 0$ $(r \geq 1)$. By (4.2), as $z \to \infty$, we obtain

$$-b_r^2 z^{2r}\bigl(1+o(1)\bigr) = \alpha z^{d-f}\bigl(1+o(1)\bigr) = \alpha\bigl(1+o(1)\bigr), \tag{4.7}$$

a contradiction. Hence $h = k$, that is,

$$h - k = 0 = \frac{d-f}{2}.$$

(ii) We next consider the case $d < f$. Suppose that $h > k$. Then $b_r \neq 0$ $(r \geq 1)$. Using the similar method as before, as $z \to \infty$, by (4.2) we obtain that

$$3b_r^2 z^{2r}\bigl(1+o(1)\bigr) = 0,$$

a contradiction.

If $h = k$, then using the similar method as before, we obtain

$$\frac{D(z)}{F(z)} = \left(\frac{H(z+c)}{K(z+c)} - \frac{H(z)}{K(z)}\right)^2 - \frac{H(z+c)}{K(z+c)}\frac{H(z)}{K(z)} \to \beta^2 \neq 0 \quad \text{as } z \to \infty,$$

which is a contradiction, since $\frac{D(z)}{F(z)} \to 0$ as $z \to \infty$. Hence $h < k$. Substituting $g(z) = \frac{H(z)}{K(z)}$ into (2.2), we have

$$\begin{aligned} &F(z)H^2(z+c)K^2(z) - 3F(z)H(z)H(z+c)K(z)K(z+c) + F(z)H^2(z)K^2(z+c) \\ &\quad = D(z)K^2(z)K^2(z+c). \end{aligned} \tag{4.8}$$

Since

$$\begin{aligned} &\deg\bigl(F(z)H^2(z+c)K^2(z) - 3F(z)H(z)H(z+c)K(z)K(z+c) + F(z)H^2(z)K^2(z+c)\bigr) \\ &\quad = f + 2h + 2k, \\ &\deg\bigl(D(z)K^2(z)K^2(z+c)\bigr) = d + 4k. \end{aligned}$$

From this and from (4.8) we have

$$h - k = \frac{d - f}{2}.$$

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The author declares that he has no competing interests.

Author's contributions
Author read and approved the final manuscript.

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