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Existence results for a class of generalized fractional boundary value problems

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Abstract
In this paper, we study a class of generalized fractional order three-point boundary value problems that involve fractional derivative defined in terms of weight and scale functions. Using several fixed point theorems, the existence and uniqueness results are obtained.

Keywords: fractional calculus; generalized fractional derivative; fractional boundary value problem; fixed point theorem; existence result

1 Introduction

Fractional calculus is the subject of studying fractional integrals and fractional derivatives, which means that the orders of integration and differentiation are not integers but non-integers, and even complex numbers. The history of fractional calculus is more than three hundreds years. However, only in the recent forty years, it was realized that these fractional integrals and derivatives may have many potential applications. Fractional differential equation is a differential equation which involves fractional derivatives, and it has been successfully used to model many real-world phenomena such as heat conduction [1], diffusion process [2], and quantum mechanics [3]. More applications can be seen in [4], Chapter 10.

Fractional boundary value problems (FBVPs) appear in many of these applications. In recent twenty years, considerable work has been done in the field of FBVPs. To verify the existence result of a solution and to study the behavior of the solution of FBVPs have become more and more popular. There are several methods to verify the existence of FBVPs, in which the topological degree method is one of the most effective techniques. By using the fixed point theorems, FBVPs with different types of boundary conditions have been studied. More specifically, in [5], the existence and multiplicity of positive solutions for a nonlinear FBVP with two-point boundary condition are studied. In [6], the existence of solutions for a class of three-point FBVPs involving nonlinear impulsive fractional differential equations is considered. In [7], the existence and uniqueness of solutions for a four-point nonlocal FBVP are derived. In [8], the positive solution of FBVP with integral boundary condition is obtained. In [9], the existence theory of FBVP with anti-periodic boundary condition is discussed. Furthermore, for the existence results of FBVPs with some mixed-type boundary conditions, the readers are referred to [10–16] and the references therein.

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The literature above only focuses on the FBVP with classical fractional derivatives, i.e., Riemann-Liouville or Caputo derivatives. Fractional derivative also has some limit, since it can be regarded as a convolution between a function and a fractional power kernel. The fractional power kernel puts much weight in the present and less weight in the past, which causes the nice property of fractional derivative called nonlocal property or short memory property. The short memory property is very effective in modeling some physical processes such as diffusion phenomenon in material with memory. However, some real-world phenomena cannot be modeled by such a fractional power kernel properly. For example, an old man and a child have different memory ability. The old man may remember things that happened several decades ago, but forget what happened yesterday. The child has an opposite ability, i.e., he may have no idea about things that happened in his early years, but remember almost everything in the recent week. To model this phenomenon, we need different kernels to weight the function differently. Hence in 2012, a new class of generalized fractional integrals and derivatives defined by using a weight function and a scale function was introduced in [17]. The new fractional operators contain many existing fractional integrals and derivatives as special cases. It is shown that many integral equations can be written and solved in an elegant way using the new operators. Therefore, using different weight functions and scale functions, many fractional problems are significantly generalized. It is also possible that the new generalized fractional integrals and derivatives will bring some interest in the near future, although the theoretical study and applications of them are in the very first stage right now.

Motivated by [18], in this present paper, we consider the following three-point FBVP:

$$\begin{cases} {}^{*}D^{\alpha}_{0+,[z;w]}u(t) = f(t,u(t)), & t \in [0,1], 1 < \alpha < 2, \\ u(0) = 0, & \gamma u(p) = u(1), \end{cases} \tag{1}$$

where $p \in (0,1)$, and γ is a positive constant. ${}^{*}D^{\alpha}_{0+,[z;w]}u(t)$ is the generalized fractional derivative of function u with respect to t, and its definition is given in the next section. f is a continuous function satisfying $\lim_{|u|\to+\infty} f(t,u) = 0$, and $u \in X, f : [0,1] \times X \to X$. Here $(X, \|\cdot\|)$ is a Banach space and $C = C([0,1],X)$ denotes the Banach space of all continuous functions from $[0,1]$ to X equipped with a topology of uniform convergence with the norm denoted by $\|\cdot\|$. Next, we will apply some fixed point theorems to study the existence and uniqueness results of this generalized fractional boundary value problem.

2 Preliminaries

We introduce the generalized fractional integral and derivative directly, and for more details about the classical fractional integral and derivative, such as Riemann-Liouville, Caputo, and Riesz operators, we refer to [19], Chapter 2.

Definition 1 ([17]) The left-sided generalized fractional integral of order $\alpha > 0$ of a function $u(t)$, with respect to a scale function $z(t)$ and a weight function $w(t)$, is defined as

$$\left(I^{\alpha}_{a+,[z;w]}u\right)(t) = \frac{[w(t)]^{-1}}{\Gamma(\alpha)} \int_a^t \frac{w(s)z'(s)u(s)}{[z(t)-z(s)]^{1-\alpha}}\,ds, \tag{2}$$

provided the integral exists.

Definition 2 ([17]) The left-sided generalized derivative of order 1 of a function $u(t)$, with respect to a scale function $z(t)$ and a weight function $w(t)$, is defined as

$$(D_{[z,w]}u)(t) = \left[w(t)\right]^{-1}\left[\left(\frac{1}{z'(t)}D_t\right)\left(w(t)u(t)\right)\right], \tag{3}$$

provided the right-hand side of equation is finite.

Definition 3 ([17]) The left-sided generalized fractional derivative of order m of a function $u(t)$, with respect to a scale function $z(t)$ and a weight function $w(t)$, is defined as

$$\left(D^m_{[z,w]}u\right)(t) = \left[w(t)\right]^{-1}\left[\left(\frac{1}{z'(t)}D_t\right)^m\left(w(t)u(t)\right)\right], \tag{4}$$

provided the right-hand side of equation is finite, where m is a positive integer.

Definition 4 ([17]) The left-sided Caputo type generalized fractional derivative of order $\alpha > 0$ of a function $u(t)$, with respect to a scale function $z(t)$ and a weight function $w(t)$, is defined as

$$\left({}^*D^\alpha_{a+,[z;w]}u\right)(t) = I^{m-\alpha}_{a+,[z;w]}\left(D^m_{[z,w]}u\right)(t), \tag{5}$$

provided the right-hand side of equation is finite, where $m - 1 < \alpha < m$, and m is a positive integer. Specifically, for $1 < \alpha < 2$,

$$\left({}^*D^\alpha_{a+,[z;w]}u\right)(t) = \frac{[w(t)]^{-1}}{\Gamma(2-\alpha)}\int_a^t \frac{[z(t)-z(s)]^{1-\alpha}}{z'(s)}\cdot\frac{d^2[w(s)u(s)]}{ds^2}\,ds. \tag{6}$$

Moreover, for $z(t) = t$ and $w(t) = 1$, the generalized fractional derivative reduces to the Caputo fractional derivative as

$$\left({}^cD^\alpha_{a+}u\right)(t) = \frac{1}{\Gamma(2-\alpha)}\int_a^t (t-s)^{1-\alpha}u''(s)\,ds.$$

In the above definitions, we only present the 'left-sided' sense of generalized fractional integrals and derivatives. The 'right-sided' sense of generalized fractional integrals and derivatives and their properties are discussed in [17]. We will not repeat them here since the derivative we use in this paper is defined in the left-sided sense. For simplicity, in what follows, we remove the term '$[z;w]$' from the subscript in equation (6).

Remark 1 To be more specific, we assume that the weight function is positive and the scale function $z(t)$ is monotone increasing over $[0,1]$. Moreover, both $w(t)$ and $z(t)$ are continuously differentiable.

Remark 2 Indeed, the generalized fractional derivatives have extended the classical Caputo fractional derivative. For example,

$$f(x) = \begin{cases} -1, & x \in [-1,0), \\ 1, & x \in (0,1], \end{cases}$$

is discontinuous, then $f \notin AC([-1,1])$, and hence it cannot have Caputo fractional derivative according to [19], Equations (2.4.17)-(2.4.18), p.92. Nevertheless, when we take $\alpha \in (0,1)$, $w(x) = |x|$, and $z(x) = x$ in $x \in [-1,0) \cup (0,1]$, then

$$\begin{aligned}\left({}^{*}D^{\alpha}_{(-1)+}f\right)(x) &= \frac{1}{|x|\Gamma(1-\alpha)}\int_{-1}^{x}\frac{s'}{[x-s]^{\alpha}}\,ds\\ &= \frac{(x+1)^{1-\alpha}}{|x|\Gamma(2-\alpha)}, \quad x \neq 0.\end{aligned}$$

Let us consider the following generalized fractional boundary value problem:

$$\begin{cases} {}^{*}D^{\alpha}_{0+}u(t) = \sigma(t), & t \in [0,1], 1 < \alpha < 2,\\ u(0) = 0, & \gamma u(p) = u(1), \end{cases} \tag{7}$$

where σ is a sufficiently smooth function, $p \in (0,1)$.

To solve problem (7), we have the following lemma.

Lemma 1 *Assume that $z(t)$ is strictly monotone increasing and $w(t)$ is positive on* $[0,1]$, *and*

$$L := \gamma\frac{z(p)}{w(p)} - \frac{z(1)}{w(1)} + \frac{z(0)}{w(1)} - \gamma\frac{z(0)}{w(p)} \neq 0, \tag{8}$$

then the solution of problem (7) *is given as*

$$u(t) = \frac{C_0}{w(t)} + \frac{C_1 z(t)}{w(t)} + \frac{[w(t)]^{-1}}{\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)\sigma(s)\,ds}{[z(t)-z(s)]^{1-\alpha}}, \tag{9}$$

where

$$C_0 = \frac{z(0)}{L\cdot\Gamma(\alpha)}\left\{\frac{\gamma}{w(p)}\int_0^p\frac{w(s)z'(s)\sigma(s)\,ds}{[z(p)-z(s)]^{1-\alpha}} - \frac{1}{w(1)}\int_0^1\frac{w(s)z'(s)\sigma(s)\,ds}{[z(1)-z(s)]^{1-\alpha}}\right\}$$

and

$$C_1 = \frac{1}{L\cdot\Gamma(\alpha)}\left\{\frac{1}{w(1)}\int_0^1\frac{w(s)z'(s)\sigma(s)\,ds}{[z(1)-z(s)]^{1-\alpha}} - \frac{\gamma}{w(p)}\int_0^p\frac{w(s)z'(s)\sigma(s)\,ds}{[z(p)-z(s)]^{1-\alpha}}\right\}.$$

Proof According to equation (6), we have

$$\begin{aligned}{}^{*}D^{\alpha}_{0+}u(t) &= \frac{[w(t)]^{-1}}{\Gamma(2-\alpha)}\int_a^t\frac{[z(t)-z(s)]^{1-\alpha}}{z'(s)}\cdot\frac{d^2[w(s)u(s)]}{ds^2}\,ds\\ &= \frac{[w(t)]^{-1}}{\Gamma(2-\alpha)}\int_{z(0)}^{z(t)}\left[z(t)-\xi\right]^{1-\alpha}\frac{d^2[w(z^{-1}(\xi))u(z^{-1}(\xi))]}{d\xi^2}\left[z'\left(z^{-1}(\xi)\right)\right]^2\,d\xi\\ &= \frac{1}{w(t)}{}^{c}D^{\alpha}_{z(0)+}\left[w\left(z^{-1}(\phi)\right)u\left(z^{-1}(\phi)\right)\right],\end{aligned} \tag{10}$$

where $\phi = z(t) \in [z(0), z(1)]$ and $\xi = z(s) \in [z(0), z(t)]$. We apply the mean value theorem to move $[z'(z^{-1}(\xi))]^2$ outside as a constant, i.e., $[z'(z^{-1}(\xi^*))]^2$, $\xi^* \in [z(0), z(1)]$, which is absorbed by the weight $\frac{1}{w(t)}$. Hence, problem (7) is transformed to

$$\begin{cases} {}^cD^{\alpha}_{z(0)+}[w(z^{-1}(\phi))u(z^{-1}(\phi))] = w(z^{-1}(\phi))\sigma(z^{-1}(\phi)), \\ u(z^{-1}(z(0))) = 0, \qquad \gamma u(z^{-1}(z(p))) = u(z^{-1}(z(1))). \end{cases} \tag{11}$$

Finally, it suffices to verify that (11) is solvable under assumptions in Remark 1.

According to [20], Section 5.5, the general solution of equation (11) is

$$w\big(z^{-1}(\phi)\big)u\big(z^{-1}(\phi)\big) = C_0 + C_1 z(t) + \frac{1}{\Gamma(\alpha)}\int_{z(0)}^{z(t)} \frac{w(z^{-1}(\xi))\sigma(z^{-1}(\xi))}{[z(t)-\xi]^{1-\alpha}}\,d\xi, \tag{12}$$

which implies

$$u(t) = C_0\frac{1}{w(t)} + C_1\frac{z(t)}{w(t)} + \frac{[w(t)]^{-1}}{\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)\sigma(s)}{[z(t)-z(s)]^{1-\alpha}}\,ds. \tag{13}$$

Imposing the initial and boundary conditions on equation (13) gives

$$C_0\frac{1}{w(0)} + C_1\frac{z(0)}{w(0)} = 0, \tag{14}$$

and

$$\begin{aligned} &\gamma\left[C_0\frac{1}{w(p)} + C_1\frac{z(p)}{w(p)} + \frac{[w(p)]^{-1}}{\Gamma(\alpha)}\int_0^p \frac{w(s)z'(s)\sigma(s)}{[z(p)-z(s)]^{1-\alpha}}\,ds\right] \\ &\quad = C_0\frac{1}{w(1)} + C_1\frac{z(1)}{w(1)} + \frac{[w(1)]^{-1}}{\Gamma(\alpha)}\int_0^1 \frac{w(s)z'(s)\sigma(s)}{[z(1)-z(s)]^{1-\alpha}}\,ds. \end{aligned} \tag{15}$$

Now we can solve equations (14) and (15) to get C_0 and C_1. Since $L := \gamma\frac{z(p)}{w(p)} - \frac{z(1)}{w(1)} + \frac{z(0)}{w(1)} - \gamma\frac{z(0)}{w(p)}$ is the determinant of the coefficient matrix of equations (14) and (15), when $L \neq 0$, equations (14) and (15) have a unique nonzero solution. This completes the proof. □

The following theorems play important roles in studying the existence and uniqueness of fractional boundary value problems.

Theorem 1 (Contraction mapping principle, see [21]) *Let E be a Banach space, $D \subset E$ be closed, and $F : D \to D$ be a strict contraction, i.e., $|Fx - Fy| \leq k|x - y|$ for some $k \in (0,1)$ and all $x, y \in D$. Then F has a unique fixed point x^*. Furthermore, the successive approximations $x_{n+1} = Fx_n = F^n x_0$, starting at any $x_0 \in D$, converge to x^* and satisfy $|x_n - x^*| \leq (1-k)^{-1}k^n|Fx_0 - x_0|$.*

Theorem 2 (Arzelà-Ascoli, see [21]) *If a sequence $\{x_n\}_{n=0}^{\infty}$ in a compact subset of X is uniformly bounded and equicontinuous, then it has a uniformly convergent subsequence.*

Theorem 3 ([22]) *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$, and let $T : \bar{\Omega} \to X$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\bar{\Omega}$.

Theorem 4 (Krasnosel'skii, see [22]) *Let M be a closed convex and nonempty subset of a Banach space X. Let A and B be two operators such that*:

(H1) $Ax + By \in M$, *wherever* $x, y \in M$;
(H2) *A is compact and continuous*; *and*
(H3) *B is a contraction mapping.*

Then there exists $z^* \in M$ such that $z^* = Az^* + Bz^*$.

3 Main results

In this section, we present some existence results of boundary value problem (1). Let $C = C([0,1], R)$ denote the Banach space of all continuous functions mapping $[0,1]$ to R equipped with the norm defined by $\|u\| = \sup_{0\le t\le 1}\{|u(t)|\}$.

Define the operator $T : C \to C$ as

$$\begin{aligned}(Tu)(t) = {} & \frac{L^{-1}z(0)}{w(t)\Gamma(\alpha)}\left[\int_0^p \frac{\gamma w(s)z'(s)f(s,u(s))\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}}\right.\\ & \left. - \int_0^1 \frac{w(s)z'(s)f(s,u(s))\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}}\right]\\ & + \frac{L^{-1}z(t)}{w(t)\Gamma(\alpha)}\left[\int_0^1 \frac{w(s)z'(s)f(s,u(s))\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}}\right.\\ & \left. - \int_0^p \frac{\gamma w(s)z'(s)f(s,u(s))\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}}\right]\\ & + \frac{[w(t)]^{-1}}{\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)f(s,u(s))}{[z(t)-z(s)]^{1-\alpha}}\,ds, \end{aligned} \tag{16}$$

where $0 < p < 1$, $0 < t < 1$.

If the operator $T : C \to C$ defined by equation (16) has a fixed point, then the fixed point coincides with the solution of fractional boundary problem (1). In what follows, we prove the complete continuity property of operator T.

Lemma 2 *The operator $T : C \to C$ defined by equation* (16) *is completely continuous.*

Proof Let $\Omega \subset C$ be a bounded set, then for any $t \in [0,1]$ and $u \in \Omega$, since $f(t,u)$ is continuous on $[0,1] \times \mathbb{R}$, there exists a positive constant L_1 such that $|f(t,u)| \le L_1$. Thus one can have

$$\begin{aligned} & \bigl|(Tu)(t)\bigr| \\ & \quad \le \frac{|L^{-1}|z(0)}{w(t)\Gamma(\alpha)}\left[\int_0^p \frac{\gamma w(s)z'(s)|f(s,u(s))|\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}}\right.\\ & \qquad \left. + \int_0^1 \frac{w(s)z'(s)|f(s,u(s))|\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}}\right] + \frac{|L^{-1}|z(t)}{w(t)\Gamma(\alpha)}\left[\int_0^1 \frac{w(s)z'(s)|f(s,u(s))|\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}}\right.\\ & \qquad \left. + \int_0^p \frac{\gamma w(s)z'(s)|f(s,u(s))|\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}}\right] + \frac{[w(t)]^{-1}}{\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)|f(s,u(s))|}{[z(t)-z(s)]^{1-\alpha}}\,ds \\ & \quad \le \frac{z(0)L_1 w_{\max}}{w_{\min}\Gamma(\alpha)|L|}\left[\int_0^p \frac{\gamma z'(s)\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}}\right. \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^1 \frac{z'(s)\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}}\Bigg] + \frac{z(t)L_1 w_{\max}}{w_{\min}\Gamma(\alpha)|L|}\Bigg[\int_0^1 \frac{z'(s)\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}} \\
&+ \int_0^p \frac{\gamma z'(s)\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}}\Bigg] + \frac{L_1 w_{\max}}{w_{\min}\Gamma(\alpha)}\int_0^t \frac{z'(s)}{[z(t)-z(s)]^{1-\alpha}}\,ds \\
= {} & \frac{z(0)L_1 w_{\max}}{w_{\min}\Gamma(\alpha+1)|L|}\left\{\frac{\gamma}{w(p)}\big[z(p)-z(0)\big]^{\alpha} + \frac{1}{w(1)}\big[z(1)-z(0)\big]^{\alpha}\right\} \\
&+ \frac{z(t)L_1 w_{\max}}{w_{\min}\Gamma(\alpha+1)|L|}\left\{\frac{1}{w(1)}\big[z(1)-z(0)\big]^{\alpha} + \frac{\gamma}{w(p)}\big[z(p)-z(0)\big]^{\alpha}\right\} \\
&+ \frac{L_1 w_{\max}}{w_{\min}\Gamma(\alpha+1)}\big[z(t)-z(0)\big]^{\alpha} \\
\le {} & \frac{z(0)L_1 w_{\max}}{w_{\min}\Gamma(\alpha+1)|L|}\left\{\frac{\gamma}{w(p)}\big[z(p)-z(0)\big]^{\alpha} + \frac{1}{w(1)}\big[z(1)-z(0)\big]^{\alpha}\right\} \\
&+ \frac{z(1)L_1 w_{\max}}{w_{\min}\Gamma(\alpha+1)|L|}\left\{\frac{1}{w(1)}\big[z(1)-z(0)\big]^{\alpha}\right. \\
&\left.+ \frac{\gamma}{w(p)}\big[z(p)-z(0)\big]^{\alpha}\right\} + \frac{L_1 w_{\max}}{w_{\min}\Gamma(\alpha+1)}\big[z(1)-z(0)\big]^{\alpha} \\
:= {} & L_2,
\end{aligned}
\tag{17}
$$

where $w_{\max} = \max_{0\le t\le 1}\{w(t)\}$, $w_{\min} = \min_{0\le t\le 1}\{w(t)\}$, and L_2 is a positive constant. Equation (17) implies that $\|Tu\| \le L_2$. Moreover, for the derivative of T, we have

$$
\begin{aligned}
\big|(Tu)'(t)\big| = {} & \Bigg|\frac{-L^{-1}z(0)w'(t)}{w^2(t)\Gamma(\alpha)}L_{31} + \frac{L^{-1}[z'(t)w(t)-z(t)w'(t)]}{w^2(t)\Gamma(\alpha)}L_{32} \\
&- \frac{[w'(t)]^{-1}}{w^2(t)\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)f(s,u(s))}{[z(t)-z(s)]^{1-\alpha}}\,ds \\
&+ \frac{[w(t)]^{-1}}{\Gamma(\alpha-1)}\int_0^t \frac{w(s)z'(s)z'(t)|f(s,u(s))|}{[z(t)-z(s)]^{2-\alpha}}\,ds\Bigg| \\
\le {} & \frac{|L^{-1}|z(0)w'_{\max}}{w_{\min}^2\Gamma(\alpha)}|L_{31}| + \frac{|L^{-1}|(z'_{\max}w_{\max}+z_{\max}w'_{\max})}{w_{\min}^2\Gamma(\alpha)}|L_{32}| \\
&+ \frac{w'_{\max}w_{\max}L_1}{w_{\min}^2\Gamma(\alpha)}\big[z(1)-z(0)\big]^{\alpha} \\
&+ \frac{w_{\max}z'_{\max}z'_{\max}L_1}{w_{\min}\Gamma(\alpha-1)}\big[z(1)-z(0)\big]^{\alpha-1} \\
:= {} & L_3,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
L_{31} &= \int_0^p \frac{\gamma w(s)z'(s)f(s,u(s))\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}} - \int_0^1 \frac{w(s)z'(s)f(s,u(s))\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}}, \\
L_{32} &= \int_0^1 \frac{w(s)z'(s)f(s,u(s))\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}} - \int_0^p \frac{\gamma w(s)z'(s)f(s,u(s))\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}},
\end{aligned}
$$

are constants, i.e., $L_{31}+L_{32}=0$, and $w'_{\max}$, $z'_{\max}$ indicate the maximum values of the derivative of functions $w(t)$, $z(t)$ on $[0,1]$, respectively.

Therefore, for all $0 \le t_1 \le t_2 \le 1$,

$$\bigl|(Tu)(t_2) - (Tu)(t_1)\bigr| \le \int_{t_1}^{t_2} \bigl|(Tu)'(s)\bigr|\, ds \le L_3(t_2 - t_1), \tag{19}$$

which implies that the operator T is equicontinuous on $[0,1]$. Hence, by the Arzelà-Ascoli theorem, the operator $T : C \to C$ is completely continuous. □

Remark 3 The absolute value of L_{31} (or L_{32}) has the following upper-bound estimation:

$$\begin{aligned}
|L_{31}| &= \left| \int_0^p \frac{\gamma w(s)z'(s)f(s,u(s))\, ds}{w(p)[z(p)-z(s)]^{1-\alpha}} - \int_0^1 \frac{w(s)z'(s)f(s,u(s))\, ds}{w(1)[z(1)-z(s)]^{1-\alpha}} \right| \\
&\le \left| \int_0^p \frac{\gamma w(s)z'(s)f(s,u(s))\, ds}{w(p)[z(p)-z(s)]^{1-\alpha}} \right| + \left| \int_0^1 \frac{w(s)z'(s)f(s,u(s))\, ds}{w(1)[z(1)-z(s)]^{1-\alpha}} \right| \\
&\le \frac{\gamma L_1 w_{\max}}{w(p)} \int_0^p \frac{z'(s)\, ds}{[z(p)-z(s)]^{1-\alpha}} + \frac{L_1 w_{\max}}{w(1)} \int_0^1 \frac{z'(s)\, ds}{[z(1)-z(s)]^{1-\alpha}} \\
&= \frac{L_1 w_{\max}}{\alpha} \left[\frac{\gamma}{w(p)} \bigl[z(p)-z(0)\bigr]^\alpha + \frac{1}{w(1)} \bigl[z(1)-z(0)\bigr]^\alpha \right].
\end{aligned}$$

Denote

$$L_4 = L_1 w_{\max} \left[\frac{\gamma}{w(p)} \bigl[z(p)-z(0)\bigr]^\alpha + \frac{1}{w(1)} \bigl[z(1)-z(0)\bigr]^\alpha \right],$$

we have

$$\alpha |L_{31}| \le L_4.$$

Theorem 5 *Assume that* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *and* $\lim_{u\to 0} f(t,u) = 0$. *Then the boundary value problem* (1) *has at least one solution.*

Proof Since $\lim_{u\to 0} f(t,u) = 0$, then there exist constants $d > 0$ and $d_1 > 0$ such that for all $0 < |u| < d$, we have $|f(t,u)| \le d_1|u|$, where d_1 is such that

$$\begin{aligned}
&\max_{t\in[0,1]} \left\{ \left(\frac{|L^{-1}|z(0)}{w(t)\Gamma(\alpha+1)} + \frac{|L^{-1}|z(t)}{w(t)\Gamma(\alpha+1)} \right) \left[\frac{\gamma w_{\max}}{w(p)} \bigl[z(p)-z(0)\bigr]^\alpha \right.\right. \\
&\quad \left.\left. + \frac{w_{\max}}{w(1)} \bigl[z(1)-z(0)\bigr]^\alpha \right] + \frac{w_{\max}}{w(t)\Gamma(\alpha+1)} \bigl[z(t)-z(0)\bigr]^\alpha \right\} \cdot d_1 \le 1.
\end{aligned} \tag{20}$$

Define $\Omega_1 = \{u \in C : |u| \le d\}$. Choose $u_0 \in C$ such that $|u_0| = d$, which means that

$$u_0 \in \partial\Omega_1.$$

By Lemma 2, the operator T is completely continuous, and by equation (20), we have

$$\begin{aligned}
\bigl|(Tu_0)(t)\bigr| \le \max_{t\in[0,1]} &\left\{ \left(\frac{|L^{-1}|z(0)}{w(t)\Gamma(\alpha+1)} + \frac{|L^{-1}|z(t)}{w(t)\Gamma(\alpha+1)} \right) \right. \\
&\times \left[\frac{\gamma w_{\max}}{w(p)} \bigl[z(p)-z(0)\bigr]^\alpha + \frac{w_{\max}}{w(1)} \bigl[z(1)-z(0)\bigr]^\alpha \right]
\end{aligned}$$

$$\begin{aligned}
&+ \frac{w_{\max}}{w(t)\Gamma(\alpha+1)}\bigl[z(t)-z(0)\bigr]^{\alpha}\biggr\} \cdot d_1|u_0| \\
&\leq |u_0|. \qquad (21)
\end{aligned}$$

Therefore, by Theorem 3, the operator T has at least one fixed point, which implies that the boundary value problem (1) has at least one solution. □

Theorem 6 *Let $f : [0,1] \times X \to X$ be a jointly continuous function satisfying the Lipschitz condition*

$$\bigl\|f(t,u_1)-f(t,u_2)\bigr\| \leq L_5\|u_1-u_2\|, \quad \forall t \in [0,1], u_1, u_2 \in X.$$

Then the boundary value problem (1) *has a unique solution provided* $\Delta < 1$, *where*

$$\begin{aligned}
\Delta = {} & \frac{L_5}{\mu} \cdot \frac{w_{\max}}{w_{\min}\Gamma(\alpha+1)}\biggl\{\frac{z(0)+z(1)}{|L|}\biggl[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha} \\
&+ \frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\biggr] + \bigl[z(1)-z(0)\bigr]^{\alpha}\biggr\},
\end{aligned}$$

and $\Delta \leq \mu < 1$.

Proof First of all, we verify that T maps a bounded ball into itself. Denote $L_6 = \sup_{t\in[0,1]} |f(t,0)|$, and select

$$\begin{aligned}
r \geq {} & \frac{L_6}{1-\mu} \cdot \frac{w_{\max}}{w_{\min}\Gamma(\alpha+1)}\biggl\{\frac{z(0)+z(1)}{|L|}\biggl[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha} \\
&+ \frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\biggr] + \bigl[z(1)-z(0)\bigr]^{\alpha}\biggr\}, \qquad (22)
\end{aligned}$$

where μ is a real number satisfying $\Delta \leq \mu < 1$. We define a closed ball as $B_r = \{u \in C : \|u\| \leq r\}$, then

$$\begin{aligned}
&\bigl\|(Tu)(t)\bigr\| \\
&\quad \leq \frac{|L^{-1}|z(0)}{w(t)\Gamma(\alpha)}\biggl\{\int_0^p \frac{\gamma w(s)z'(s)|f(s,u(s))|}{w(p)[z(p)-z(s)]^{1-\alpha}}\,ds \\
&\qquad + \int_0^1 \frac{w(s)z'(s)|f(s,u(s))|}{w(1)[z(1)-z(s)]^{1-\alpha}}\,ds\biggr\} \\
&\qquad + \frac{|L^{-1}|z(t)}{w(t)\Gamma(\alpha)}\biggl\{\int_0^1 \frac{w(s)z'(s)|f(s,u(s))|}{w(1)[z(1)-z(s)]^{1-\alpha}}\,ds \\
&\qquad + \int_0^p \frac{\gamma w(s)z'(s)|f(s,u(s))|}{w(p)[z(p)-z(s)]^{1-\alpha}}\,ds\biggr\} \\
&\qquad + \frac{1}{w(t)\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)|f(s,u(s))|}{[z(t)-z(s)]^{1-\alpha}}\,ds \\
&\quad \leq \frac{|L^{-1}|z(0)}{w(t)\Gamma(\alpha)}\biggl\{\int_0^p \frac{\gamma w(s)z'(s)\{|f(s,u(s))-f(s,0)|+|f(s,0)|\}}{w(p)[z(p)-z(s)]^{1-\alpha}}\,ds \\
&\qquad + \int_0^1 \frac{w(s)z'(s)\{|f(s,u(s))-f(s,0)|+|f(s,0)|\}}{w(1)[z(1)-z(s)]^{1-\alpha}}\,ds\biggr\}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|L^{-1}|z(t)}{w(t)\Gamma(\alpha)}\left\{\int_0^1 \frac{w(s)z'(s)\{|f(s,u(s))-f(s,0)|+|f(s,0)|\}}{w(1)[z(1)-z(s)]^{1-\alpha}}\,ds\right.\\
&\left.+\int_0^p \frac{\gamma w(s)z'(s)\{|f(s,u(s))-f(s,0)|+|f(s,0)|\}}{w(p)[z(p)-z(s)]^{1-\alpha}}\,ds\right\}\\
&+\frac{1}{w(t)\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)\{|f(s,u(s))-f(s,0)|+|f(s,0)|\}}{[z(t)-z(s)]^{1-\alpha}}\,ds\\
\le{}& \frac{(L_5 r+L_6)|L^{-1}|z(0)}{w(t)\Gamma(\alpha+1)}\left\{\frac{\gamma w_{\max}}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}+\frac{w_{\max}}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\right\}\\
&+\frac{(L_5 r+L_6)|L^{-1}|z(t)}{w(t)\Gamma(\alpha+1)}\left\{\frac{\gamma w_{\max}}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}+\frac{w_{\max}}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\right\}\\
&+\frac{(L_5 r+L_6)w_{\max}}{w_{\min}\Gamma(\alpha+1)}\bigl[z(t)-z(0)\bigr]^{\alpha}\\
\le{}& \frac{(L_5 r+L_6)w_{\max}}{w_{\min}\Gamma(\alpha+1)}\left\{\bigl[z(1)-z(0)\bigr]^{\alpha}\right.\\
&\left.+\frac{z(0)+z(1)}{|L|}\left[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}+\frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\right]\right\}\\
\le{}& (\mu\Delta+1-\mu)r\\
\le{}& r,
\end{aligned}
$$

which implies that $T(B_r) \subset B_r$. Next, for any $u_1, u_2 \in C$ and for each $t \in [0,1]$, one can obtain

$$
\begin{aligned}
&\bigl\|(Tu_1)(t)-(Tu_2)(t)\bigr\|\\
&\quad\le \frac{|L^{-1}|z(0)}{w(t)\Gamma(\alpha)}\left\{\int_0^p \frac{\gamma w(s)z'(s)\|f(s,u_1(s))-f(s,u_2(s))\|}{w(p)[z(p)-z(s)]^{1-\alpha}}\,ds\right.\\
&\qquad\left.+\int_0^1 \frac{w(s)z'(s)\|f(s,u_1(s))-f(s,u_2(s))\|}{w(1)[z(1)-z(s)]^{1-\alpha}}\,ds\right\}\\
&\qquad+\frac{|L^{-1}|z(t)}{w(t)\Gamma(\alpha)}\left\{\int_0^1 \frac{w(s)z'(s)\|f(s,u_1(s))-f(s,u_2(s))\|}{w(1)[z(1)-z(s)]^{1-\alpha}}\,ds\right.\\
&\qquad\left.+\int_0^p \frac{\gamma w(s)z'(s)\|f(s,u_1(s))-f(s,u_2(s))\|}{w(p)[z(p)-z(s)]^{1-\alpha}}\,ds\right\}\\
&\qquad+\frac{1}{w(t)\Gamma(\alpha)}\int_0^t \frac{w(s)z'(s)\|f(s,u_1(s))-f(s,u_2(s))\|}{[z(t)-z(s)]^{1-\alpha}}\,ds\\
&\quad\le \frac{L_5\|u_1-u_2\|w_{\max}}{w_{\min}\Gamma(\alpha+1)}\left\{\bigl[z(1)-z(0)\bigr]^{\alpha}\right.\\
&\qquad\left.+\frac{z(0)+z(1)}{|L|}\left[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}+\frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\right]\right\}\\
&\quad\le \mu\Delta\|u_1-u_2\|\\
&\quad\le \|u_1-u_2\|,
\end{aligned}
$$

which implies that T is a contraction as $\mu\Delta < 1$. Therefore, by the contraction mapping principle (i.e., Banach fixed point theorem), the boundary value problem (1) has a unique solution. □

Theorem 7 *Assume that* $f:[0,1]\times X\to X$ *is a jointly continuous function and further*:

(H1) $|f(t,u_1)-f(t,u_2)|\le L_5|u_1-u_2|$, $u_1,u_2\in X$,
(H2) $|f(t,u)|\le\lambda(t)$, $\forall(t,u)\in[0,1]\times X$, *and* $\lambda\in L^1([0,1],R^+)$.

If

$$\begin{aligned}&\frac{w_{\max}L_5}{w_{\min}\Gamma(\alpha+1)}\Bigl\{\bigl[z(1)-z(0)\bigr]^{\alpha}\\&\quad+\frac{z(0)+z(1)}{|L|}\biggl[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}+\frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\biggr]\Bigr\}<1,\end{aligned}$$

then the boundary value problem (1) *has at least one solution*.

Proof Let

$$\begin{aligned}r\ge{}&\|\lambda\|_{L^1[0,1]}\times\frac{w_{\max}}{w_{\min}\Gamma(\alpha+1)}\Bigl\{\bigl[z(1)-z(0)\bigr]^{\alpha}\\&+\frac{z(0)+z(1)}{|L|}\biggl[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}+\frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\biggr]\Bigr\},\end{aligned}$$

and consider $B_r=\{u\in X:\|u\|\le r\}$. We define the operators

$$\begin{aligned}(\Phi u)(t)&=\frac{[w(t)]^{-1}}{\Gamma(\alpha)}\int_0^t\frac{w(s)z'(s)f(s,u(s))}{[z(t)-z(s)]^{1-\alpha}}\,ds,\\(\Psi u)(t)&=\biggl[\frac{L^{-1}z(0)}{w(t)\Gamma(\alpha)}-\frac{L^{-1}z(t)}{w(t)\Gamma(\alpha)}\biggr]\\&\quad\times\biggl[\int_0^1\frac{w(s)z'(s)f(s,u(s))\,ds}{w(1)[z(1)-z(s)]^{1-\alpha}}-\int_0^p\frac{\gamma w(s)z'(s)f(s,u(s))\,ds}{w(p)[z(p)-z(s)]^{1-\alpha}}\biggr].\end{aligned}$$

For $u_1,u_2\in B_r$, simple calculation yields

$$\begin{aligned}&\bigl\|(\Phi u_1)(t)+(\Psi u_2)(t)\bigr\|\\&\quad\le\|\lambda\|_{L^1[0,1]}\times\frac{w_{\max}}{w_{\min}\Gamma(\alpha+1)}\Bigl\{\bigl[z(1)-z(0)\bigr]^{\alpha}\\&\qquad+\frac{z(0)+z(1)}{|L|}\biggl[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}+\frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\biggr]\Bigr\}\\&\quad\le r.\end{aligned}$$

Therefore, $(\Phi u_1)(t)+(\Psi u_2)(t)\in B_r$. Moreover, by (H1), it is easy to verify that Ψ is a contraction mapping for

$$\begin{aligned}&\frac{w_{\max}L_5}{w_{\min}\Gamma(\alpha+1)}\Bigl\{\bigl[z(1)-z(0)\bigr]^{\alpha}+\frac{z(0)+z(1)}{|L|}\biggl[\frac{\gamma}{w(p)}\bigl[z(p)-z(0)\bigr]^{\alpha}\\&\quad+\frac{1}{w(1)}\bigl[z(1)-z(0)\bigr]^{\alpha}\biggr]\Bigr\}<1.\end{aligned}$$

Since the weight function $w(t)$, scale function $z(t)$ and f are continuous, Φ is also continuous. Furthermore, Φ is uniformly bounded in B_r since

$$\begin{aligned}\|\Phi u\| &\le \|\lambda\|_{L^1} \frac{[w(t)]^{-1}}{\Gamma(\alpha)} \int_0^t \frac{w(s)z'(s)}{[z(t)-z(s)]^{1-\alpha}}\,ds \\ &\le \|\lambda\|_{L^1} \cdot \frac{w_{\max}}{w_{\min}\Gamma(\alpha+1)} \cdot \bigl[z(t)-z(0)\bigr]^{\alpha} \\ &\le \frac{w_{\max}}{w_{\min}\Gamma(\alpha+1)} \cdot \bigl[z(1)-z(0)\bigr]^{\alpha} \cdot \|\lambda\|_{L^1}.\end{aligned}$$

Next, we prove the compactness of the operator Φ. Let $E = [0,1] \times B_r$, and denote $f_{\max} = \sup_{(t,u)\in E} |f(t,u)|$, then

$$\bigl|(\Phi u)(t_2) - (\Psi u)(t_1)\bigr| \le \frac{w_{\max} f_{\max}}{w_{\min}\Gamma(\alpha+1)} \bigl\{\bigl[z(t_2)-z(0)\bigr]^{\alpha} + \bigl[z(t_1)-z(0)\bigr]^{\alpha}\bigr\},$$

which is independent of u. Thus, Φ is equicontinuous. Since Φ maps bounded subsets into relatively compact subsets, one can deduce that $\Phi(C_{bs})$ is relatively compact in X for every t, where C_{bs} is a bounded subset of C. Therefore, Φ is relatively compact on B_r, and hence, by the Arzelà-Ascoli theorem, Φ is compact on B_r and conditions (H1) and (H2) are satisfied. Consequently, by Theorem 4, the boundary value problem (1) has at least one solution. □

4 Examples

We present three examples to demonstrate the main results discussed in the last section.

Example 1 Consider the generalized fractional boundary value problem

$$\begin{cases} {}^{*}D_{0+}^{\alpha} u(t) = f(t,u(t)), & t \in [0,1], 1 < \alpha < 2, \\ u(0) = 0, & \gamma u(p) = u(1), \end{cases} \tag{23}$$

where $\alpha = 1.85$, $\gamma = 1/2$, $p = 3/4$, $w(t) = e^t$, $z(t) = t^2$, and $f(t,u) = u^2 \sin(0.5\pi t)$. Since $\lim_{u\to 0} \frac{f(t,u)}{u} = 0$ and taking $0 < |u| < 1/5$ such that $|f(t,u)| \le |u^2| \le \frac{1}{5}|u|$ and

$$\begin{aligned}&\max_{t\in[0,1]} \biggl\{ \biggl(\frac{L^{-1}z(0)}{w(t)\Gamma(\alpha+1)} + \frac{L^{-1}z(t)}{w(t)\Gamma(\alpha+1)} \biggr) \biggl[\frac{\gamma w_{\max}}{w(p)} \bigl[z(p)-z(0)\bigr]^{\alpha} \\ &\quad + \frac{w_{\max}}{w(1)} \bigl[z(1)-z(0)\bigr]^{\alpha} \biggr] + \frac{w_{\max}}{w(t)\Gamma(\alpha+1)} \bigl[z(t)-z(0)\bigr]^{\alpha} \biggr\} \cdot d_1 \\ &< 4.5246 \times \frac{1}{5} < 1.\end{aligned}$$

Therefore, by Theorem 5, the boundary value problem (23) has at least one solution.

Example 2 Consider the generalized fractional boundary value problem

$$\begin{cases} {}^{*}D_{0+}^{\alpha} u(t) = f(t,u(t)), & t \in [0,1], 1 < \alpha < 2, \\ u(0) = 0, & \gamma u(p) = u(1), \end{cases} \tag{24}$$

where $\alpha = 1.92$, $\gamma = 1/3$, $p = 2/5$, $w(t) = t^3 + 1$, $z(t) = t^{1.5}$, and $f(t,u) = \frac{\|u\|}{(\|u\|+1)(t+5)^3}$ such that

$$\bigl\|f(t,u_1) - f(t,u_2)\bigr\| \leq \frac{1}{125}\|u_1 - u_2\|,$$

and

$$\begin{aligned}\Delta = {} & \frac{L_5}{\mu} \cdot \frac{w_{\max}}{w_{\min}\Gamma(\alpha+1)} \times \left\{ \frac{z(0)+z(1)}{|L|} \left[\frac{\gamma}{w(p)} \bigl[z(p) - z(0)\bigr]^\alpha \right.\right. \\ & \left.\left. + \frac{1}{w(1)} \bigl[z(1) - z(0)\bigr]^\alpha \right] + \bigl[z(1) - z(0)\bigr]^\alpha \right\} = 0.0052 < 1,\end{aligned}$$

where we take $\mu = 0.5$. Therefore, by Theorem 6, the boundary value problem (23) has a unique solution.

Example 3 Consider the generalized fractional boundary value problem

$$\begin{cases} {}^*D_{0+}^\alpha u(t) = f(t,u(t)), & t \in [0,1], 1 < \alpha < 2, \\ u(0) = 0, & \gamma u(p) = u(1), \end{cases} \tag{25}$$

where $\alpha = 1.73$, $\gamma = 1/4$, $p = 1/2$, $w(t) = e^t + 1$, $z(t) = t^{1.2}$, and $f(t,u) = \frac{\|u\|}{(\|u\|+1)(t+3)^4}$ such that

$$\bigl\|f(t,u_1) - f(t,u_2)\bigr\| \leq \frac{1}{81}\|u_1 - u_2\|,$$

and $|f(t,u)| \leq \lambda(t) = \frac{1}{(t+3)^4} \in L^1([0,1],R^+)$, and

$$\begin{aligned} & \frac{w_{\max} L_5}{w_{\min}\Gamma(\alpha+1)} \left\{ \bigl[z(1) - z(0)\bigr]^\alpha + \frac{z(0)+z(1)}{|L|} \left[\frac{\gamma}{w(p)} \bigl[z(p) - z(0)\bigr]^\alpha \right.\right. \\ & \quad \left.\left. + \frac{1}{w(1)} \bigl[z(1) - z(0)\bigr]^\alpha \right] \right\} = 0.3310 < 1.\end{aligned}$$

Therefore, by Theorem 7, the boundary value problem (23) has at least one solution.

5 Conclusion remark

The existence results of generalized fractional boundary value problem are discussed in this paper by using several fixed point theorems. The generalized fractional derivative is defined upon a weight function and a scale function, which contains many fractional derivatives in the literature as special cases. Hence, the boundary value problems studied in this paper are more general, and it is important to develop certain methods for investigating the existence results of them. In fact, equation (10) provides us with an effective transform, under which the generalized FBVP can be regarded as a regular FBVP defined in a general time scale $z(t)$ and weighted by a weight function $w(t)$. We hope that our work will bring much attention into this field in the near future.

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