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Qualitative behaviors of the high-order Lorenz-Stenflo chaotic system arising in mathematical physics describing the atmospheric acoustic-gravity waves

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Abstract

The boundedness of chaotic systems plays an important role in investigating the stability of the equilibrium, estimating the Lyapunov dimension of attractors, the Hausdorff dimension of attractors, the existence of periodic solutions, chaos control, and chaos synchronization. However, as far as the authors know, there are only a few papers dealing with bounds of high-order chaotic systems due to their complex algebraic structure. To sort this out, in this paper, we study the bounds of a high-order Lorenz-Stenflo system arising in mathematical physics. Based on Lyapunov stability theory, we show that there exists a globally exponential attractive set for this system. The innovation of the paper is that we not only prove that this system is globally bounded for all the parameters, but also give a family of mathematical expressions of global exponential attractive sets of this system with respect to its parameters. We also study some other dynamical characteristics of this chaotic system such as invariant sets and chaotic behaviors. To justify the theoretical analysis, we carry out detailed numerical simulations.

Keywords: High-order Lorenz-Stenflo system; Lyapunov exponents; Lyapunov stability; domain of attraction; nonlinear dynamics

1 Introduction

Chaos phenomena and chaotic systems have been extensively studied by many researchers due to their various applications in the fields of atmospheric dynamics, population dynamics, electric circuits, cryptology, fluid dynamics, lasers, engineering, stock exchanges, chemical reactions, and so on [1–11]. Most of the complex dynamical phenomena are characterized by chaotic and hyperchaotic systems of nonlinear ordinary differential equations [1–21].

Stenflo [22] obtained the Lorenz-Stenflo equation from the equations describing the atmospheric acoustic-gravity waves. The Lorenz-Stenflo equation is described by the following equations:

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) + sv, \\ \frac{dy}{dt} = -xz + rx - y, \\ \frac{dz}{dt} = xy - bz, \\ \frac{dv}{dt} = -x - \sigma v, \end{cases} \tag{1}$$

where x, y, z, v are state variables of the Lorenz-Stenflo equation (1), and σ, s, r, b are positive parameters of the system, σ is the Prandtl number, r is the generalized Rayleigh parameter, b is the geometric parameter, and s is the rotation parameter. The Lorenz-Stenflo equation is regarded as an extended Lorenz equation since it reduces to the Lorenz equation [1] when the rotation parameter s is zero, and it also can be obtained from the rotating thermal convection equations. The Lorenz-Stenflo system is a four-dimensional continuous-time dynamical system, derived to model atmospheric acoustic-gravity waves in a rotating atmosphere. Knowledge about acoustic gravity waves is important because they may be responsible both for minor local weather changes and for large-scale phenomena, for instance, storms. Many dynamical behaviors such as stability [23], bifurcation [24, 25], periodic solutions [26] and chaotic behaviors [27] of the Lorenz-Stenflo equations have been thoroughly studied for decades after Stenflo.

To improve the stability or predictability of the Lorenz-Stenflo system (1), Park et al. [26, 28] introduced the high-order Lorenz-Stenflo equations by including terms with higher vertical wave numbers:

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) + sv, \\ \frac{dy}{dt} = -xz + rx - y, \\ \frac{dz}{dt} = xy - xu - bz, \\ \frac{dv}{dt} = -x - \sigma v, \\ \frac{du}{dt} = xz - 2x\omega - (1+2b)u, \\ \frac{d\omega}{dt} = 2xu - 4b\omega, \end{cases} \tag{2}$$

where x, y, z, v, u, and ω are state variables, σ, s, r, and b are the same positive parameters as in the original Lorenz-Stenflo system (1). Various dynamical behaviors, such as stability, periodic and chaotic solutions, and Lyapunov exponents spectra of the high-order Lorenz-Stenflo equations (2), have been thoroughly studied [28, 29]. When $\sigma = 10$, $b = \frac{8}{3}$, $r = 40$, $s = 50$, system (2) has a chaotic attractor [28, 29]. When $\sigma = 10$, $b = \frac{8}{3}$, $r = 40$, $s = 50$, chaotic attractors of system (2) in $xOyz$ space are shown in Figure 1. Chaotic attractors of system (2) in yOz plane are shown in Figure 2. Chaotic attractors of system (2) in xOy plane are shown in Figure 3. Chaotic attractors of system (2) in xOz plane are shown in Figure 4.

Remark 1 An oscillation in a dynamical system can be localized numerically if the initial conditions from its open neighborhood lead to the long-time behavior that approaches the oscillation. Such an oscillation (or a set of oscillations) is called an attractor, and its attracting set is called a basin of attraction. Thus, from a point of view of the numerical

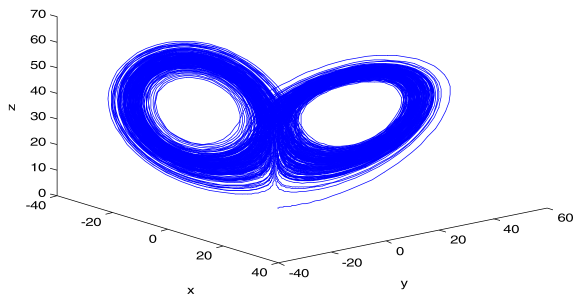


Figure 1 Chaotic attractors of system (2) in *xOyz* space.

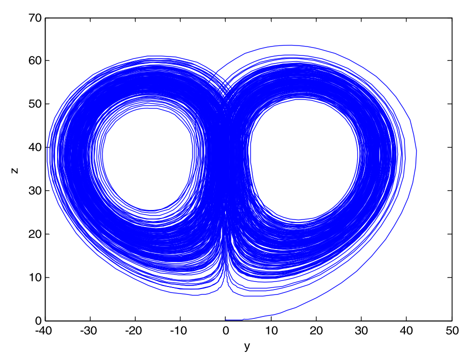


Figure 2 Chaotic attractors of system (2) in *yOz* plane.

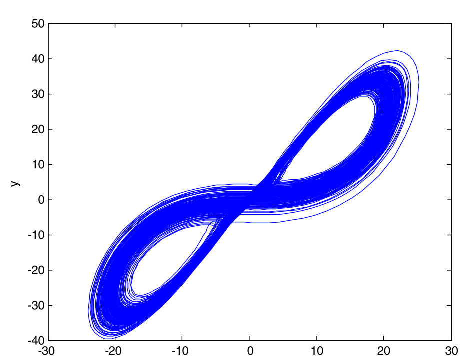


Figure 3 Chaotic attractors of system (2) in *xOy* plane.

analysis of nonlinear dynamical models, it is essential to classify an attractor as a self-excited or hidden attractor depending on simplicity of finding its basin of attraction [14, 30, 31]: An attractor is called a self-excited attractor if its basin of attraction intersects with an arbitrarily small open neighborhood of an unstable equilibrium; otherwise, it is called a hidden attractor (see [14] and [30, 31] for a detailed discussion of the attractors in dynamical systems). Although the authors in [28, 29] point out that system (2) has a attractor for $\sigma = 10$, $b = \frac{8}{3}$, $r = 40$, $s = 50$, they do not point out which type of attractor

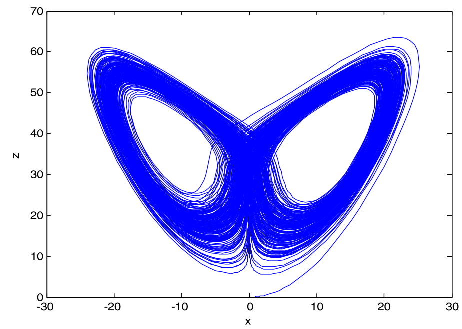


Figure 4 Chaotic attractors of system (2) in *xOz* plane.

system (2) has. It is necessary and interesting to discuss the classification of the attractors in system (2) in the future.

2 Some dynamics of high-order Lorenz-Stenflo system

2.1 Invariance

The positive z-axis, u-axis, and ω-axis are invariant under the flow, that is, they are positively invariant under the flow generated by system (2). However, this is not the case on the positive x-axis, y-axis, and v-axis for system (2) since they are all not positively invariant under the flow generated by system (2).

2.2 Ultimate bound set and domain of attraction

In this section, we further investigate the ultimate bound set and global domain of attraction of the high-order Lorenz-Stenflo system (2). The main result is described by the following theorems, Theorems 1 and 2.

Theorem 1 *For any* $\lambda_1 > 0$, $m > 0$, $\sigma > 0$, $s > 0$, $r > 0$, $b > 0$, *there exists a positive number* $M > 0$ *such that*

$$\Psi = \left\{X \mid \lambda_1(x-m_2)^2 + my^2 + m(z-2\lambda_2)^2 + \lambda_1 s(v-m_3)^2 + mu^2 + m(\omega-\lambda_2)^2 \le M\right\}$$

is the ultimate bound and positively invariant set of the high-order Lorenz-Stenflo system (2), *where* $X(t) = (x(t), y(t), z(t), v(t), u(t), \omega(t))$.

Proof Define the following Lyapunov-like function

$$V(X) = \lambda_1(x-m_2)^2 + my^2 + m(z-2\lambda_2)^2 + \lambda_1 s(v-m_3)^2 + mu^2 + m(\omega-\lambda_2)^2, \tag{3}$$

where $\forall\lambda_1 > 0$, $\forall m > 0$, $\lambda_2 = \frac{\lambda_1\sigma+mr}{2m}$, $X(t) = (x(t), y(t), z(t), v(t), u(t), \omega(t))$, and $m_2 \in R$, $m_3 \in R$ are arbitrary constants.

We have

$$\begin{aligned} &\left.\frac{dV(X(t))}{dt}\right|_{(2)} \\ &\quad = 2\lambda_1(x-m_2)\frac{dx}{dt} + 2my\frac{dy}{dt} + 2m(z-2\lambda_2)\frac{dz}{dt} + 2\lambda_1 s(v-m_3)\frac{dv}{dt} \\ &\qquad + 2mu\frac{du}{dt} + 2m(\omega-\lambda_2)\frac{d\omega}{dt}, \end{aligned}$$

$$\begin{aligned}
&= 2\lambda_1(x-m_2)(\sigma y-\sigma x+sv)+2my(-xz+rx-y)+2m(z-2\lambda_2)(xy-xu-bz)\\
&\quad +2\lambda_1 s(v-m_3)(-x-\sigma v)+2mu\big[xz-2x\omega-(1+2b)u\big]+2m(\omega-\lambda_2)(2xu-4b\omega)\\
&= -2\lambda_1\sigma x^2+2\lambda_1\sigma m_2x+2\lambda_1 sm_3x-2my^2-2\lambda_1\sigma m_2y-2bmz^2+4bm\lambda_2 z-2\lambda_1 s\sigma v^2\\
&\quad -2\lambda_1 sm_2v+2\lambda_1 sm_3\sigma v-2m(1+2b)u^2-8bm\omega^2+8bm\lambda_2\omega.
\end{aligned}$$

Let $\frac{dV(X(t))}{dt}=0$. Then, we get that the surface

$$\Gamma:\left\{\begin{array}{l}
X \mid -2\lambda_1\sigma x^2+2\lambda_1\sigma m_2x+2\lambda_1 sm_3x-2my^2\\
\quad -2\lambda_1\sigma m_2y-2bmz^2+4bm\lambda_2z-2\lambda_1 s\sigma v^2\\
\quad -2\lambda_1 sm_2v+2\lambda_1 sm_3\sigma v-2m(1+2b)u^2-8bm\omega^2+8bm\lambda_2\omega=0
\end{array}\right\} \tag{4}$$

is an ellipsoid in R^6 $\forall\lambda_1>0, m>0, \sigma>0, s>0, r>0, b>0$. Outside Γ, $\frac{dV(X(t))}{dt}<0$, whereas inside Γ, $\frac{dV(X(t))}{dt}>0$. Thus, the ultimate boundedness for system (2) can only be reached on Γ. Since the Lyapunov-like function $V(X)$ is a continuous function and Γ is a bounded closed set, the function (3) can reach its maximum value $\max_{X\in\Gamma}V(X)=M$ on the surface Γ. Obviously, $\{X \mid V(X)\le\max_{X\in\Gamma}V(X)=M, X\in\Gamma\}$ contains solutions of system (2). It is obvious that the set Ψ is the ultimate bound set and positively invariant set for system (2).

This completes the proof. □

Theorem 1 points that the trajectories of system (2) are ultimately bounded. However, Theorem 1 does not give the rate of the trajectories of system (2) going from the exterior of the trapping set to the interior of the trapping set. The rate of the trajectories rate of system (2) is studied in the next theorem, Theorem 2.

In the following section, we further investigate the globally attractive set of the high-order Lorenz-Stenflo system (2). We use the following Lyapunov-like function

$$V(X)=\lambda_1(x-m_2)^2+my^2+m(z-2\lambda_2)^2+\lambda_1 s(v-m_3)^2+mu^2+m(\omega-\lambda_2)^2 \tag{5}$$

which is obviously positive definite and radially unbounded. Here, $\forall\lambda_1>0$, $\forall m>0$, $\lambda_2=\frac{\lambda_1\sigma+mr}{2m}$, and $m_2\in R$, $m_3\in R$ are arbitrary constants.

Let $X(t)=(x(t),y(t),z(t),v(t),u(t),\omega(t))$ be an arbitrary solution of system (2). We have the following results for system (2).

Theorem 2 *Suppose that* $\forall\sigma>0, s>0, r>0, b>0$, *and let*

$$L^2=\frac{1}{\theta}\left[\frac{\lambda_1 s^2(m_3)^2}{\sigma}+\frac{(\lambda_1\sigma m_2)^2}{m}+\frac{\lambda_1 s(m_2)^2}{\sigma}+\lambda_1\sigma(m_2)^2+8bm(\lambda_2)^2+\lambda_1 s\sigma(m_3)^2\right],$$

$$\theta=\min(\sigma,b)>0.$$

Then, for system (2), *we have the estimate*

$$\big[V\big(X(t)\big)-L^2\big]\le\big[V\big(X(t_0)\big)-L^2\big]e^{-\theta(t-t_0)}. \tag{6}$$

Thus $\Omega=\{X \mid V(X)\le L^2\}$ *is a globally exponential attractive set of system* (2), *that is,* $\overline{\lim}_{t\to+\infty}V(X(t))\le L^2$.

Proof Define the following functions:

$$f(x) = -\lambda_1\sigma x^2 + 2\lambda_1 s m_3 x, \qquad h(y) = -my^2 - 2\lambda_1\sigma m_2 y, \qquad g(v) = -\lambda_1 s\sigma v^2 - 2\lambda_1 m_2 s v.$$

Then we have

$$\max_{x\in R} f(x) = \frac{\lambda_1 s^2 (m_3)^2}{\sigma}, \qquad \max_{y\in R} h(y) = \frac{(\lambda_1\sigma m_2)^2}{m}, \qquad \max_{v\in R} g(v) = \frac{\lambda_1 s(m_2)^2}{\sigma}.$$

Differentiating the Lyapunov-like function $V(X)$ in (5) with respect time t along the trajectory of system (2) yields

$$\begin{aligned}
&\left.\frac{dV(X(t))}{dt}\right|_{(2)} \\
&\quad = 2\lambda_1(x-m_2)\frac{dx}{dt} + 2my\frac{dy}{dt} + 2m(z-2\lambda_2)\frac{dz}{dt} + 2\lambda_1 s(v-m_3)\frac{dv}{dt} \\
&\qquad + 2mu\frac{du}{dt} + 2m(\omega-\lambda_2)\frac{d\omega}{dt} \\
&\quad = 2\lambda_1(x-m_2)(\sigma y - \sigma x + sv) + 2my(-xz + rx - y) + 2m(z-2\lambda_2)(xy - xu - bz) \\
&\qquad + 2\lambda_1 s(v-m_3)(-x-\sigma v) + 2mu\bigl[xz - 2x\omega - (1+2b)u\bigr] + 2m(\omega-\lambda_2)(2xu - 4b\omega) \\
&\quad = -2\lambda_1\sigma x^2 + 2\lambda_1\sigma m_2 x \\
&\qquad + 2\lambda_1 s m_3 x - 2my^2 - 2\lambda_1\sigma m_2 y - 2bmz^2 + 4bm\lambda_2 z - 2\lambda_1 s\sigma v^2 \\
&\qquad - 2\lambda_1 s m_2 v + 2\lambda_1 s m_3\sigma v - 2m(1+2b)u^2 - 8bm\omega^2 + 8bm\lambda_2\omega \\
&\quad = -\lambda_1\sigma x^2 + 2\lambda_1\sigma m_2 x - \lambda_1\sigma x^2 \\
&\qquad + 2\lambda_1 s m_3 x - my^2 - my^2 - 2\lambda_1\sigma m_2 y - 2bmz^2 + 4bm\lambda_2 z \\
&\qquad - \lambda_1 s\sigma v^2 + 2\lambda_1 s m_3\sigma v - \lambda_1 s\sigma v^2 - 2\lambda_1 s m_2 v - 2m(1+2b)u^2 - 8bm\omega^2 + 8bm\lambda_2\omega \\
&\quad = -\lambda_1\sigma x^2 + 2\lambda_1\sigma m_2 x + f(x) - my^2 + h(y) - 2bmz^2 + 4bm\lambda_2 z - \lambda_1 s\sigma v^2 + 2\lambda_1 s m_3\sigma v \\
&\qquad + g(v) - 2m(1+2b)u^2 - 8bm\omega^2 + 8bm\lambda_2\omega \\
&\quad \le -\lambda_1\sigma x^2 + 2\lambda_1\sigma m_2 x + f(x) - my^2 + h(y) - bmz^2 + 4bm\lambda_2 z - \lambda_1 s\sigma v^2 + 2\lambda_1 s m_3\sigma v \\
&\qquad + g(v) - 2m(1+2b)u^2 - 8bm\omega^2 + 8bm\lambda_2\omega \\
&\quad = -\lambda_1\sigma(x-m_2)^2 + \lambda_1\sigma(m_2)^2 + f(x) - my^2 + h(y) - bm(z-2\lambda_2)^2 + 4bm(\lambda_2)^2 \\
&\qquad - \lambda_1 s\sigma(v-m_3)^2 + \lambda_1 s\sigma(m_3)^2 + g(v) - 2m(1+2b)u^2 - 8bm\omega^2 + 8bm\lambda_2\omega \\
&\quad = -\lambda_1\sigma(x-m_2)^2 - my^2 - bm(z-2\lambda_2)^2 \\
&\qquad - \lambda_1 s\sigma(v-m_3)^2 - 2m(1+2b)u^2 - 8bm\omega^2 + 8bm\lambda_2\omega \\
&\qquad + f(x) + h(y) + g(v) + \lambda_1\sigma(m_2)^2 + 4bm(\lambda_2)^2 + \lambda_1 s\sigma(m_3)^2 \\
&\quad \le -\lambda_1\sigma(x-m_2)^2 - my^2 - bm(z-2\lambda_2)^2 - \lambda_1 s\sigma(v-m_3)^2 - mu^2 - 4bm\omega^2 + 8bm\lambda_2\omega \\
&\qquad + f(x) + h(y) + g(v) + \lambda_1\sigma(m_2)^2 + 4bm(\lambda_2)^2 + \lambda_1 s\sigma(m_3)^2 \\
&\quad = -\lambda_1\sigma(x-m_2)^2 - my^2 - bm(z-2\lambda_2)^2 - \lambda_1 s\sigma(v-m_3)^2 - mu^2 - 4bm(\omega-\lambda_2)^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + f(x) + h(y) + g(v) + \lambda_1\sigma(m_2)^2 + 4bm(\lambda_2)^2 + \lambda_1 s\sigma(m_3)^2 + 4bm(\lambda_2)^2 \\
&\le -\lambda_1\sigma(x-m_2)^2 - my^2 - bm(z-2\lambda_2)^2 - \lambda_1 s\sigma(v-m_3)^2 - mu^2 - bm(\omega-\lambda_2)^2 \\
&\quad + f(x) + h(y) + g(v) + \lambda_1\sigma(m_2)^2 + 8bm(\lambda_2)^2 + \lambda_1 s\sigma(m_3)^2 \\
&\le -\theta V(X) + \max_{x\in R} f(x) + \max_{y\in R} h(y) + \max_{v\in R} g(v) + \lambda_1\sigma(m_2)^2 + 8bm(\lambda_2)^2 + \lambda_1 s\sigma(m_3)^2 \\
&= -\theta V(X) + \frac{\lambda_1 s^2(m_3)^2}{\sigma} + \frac{(\lambda_1\sigma m_2)^2}{m} + \frac{\lambda_1 s(m_2)^2}{\sigma} \\
&\quad + \lambda_1\sigma(m_2)^2 + 8bm(\lambda_2)^2 + \lambda_1 s\sigma(m_3)^2 \\
&= -\theta\bigl[V\bigl(X(t)\bigr) - L^2\bigr].
\end{aligned}
$$

Thus, we have

$$\bigl[V\bigl(X(t)\bigr) - L^2\bigr] \le \bigl[V\bigl(X(t_0)\bigr) - L^2\bigr]e^{-\theta(t-t_0)}.$$

Therefore,

$$\overline{\lim_{t\to+\infty}} V\bigl(X(t)\bigr) \le L^2,$$

which clearly shows that $\Omega = \{X \mid V(X) \le L^2\}$ is a globally exponential attractive set of system (2). The proof is complete. □

Remark 2 (i) Let us take $\lambda_1 = 1$, $m = 1$, $m_2 = 0$, $m_3 = 0$ in Theorem 2. Then we get that

$$\Delta = \left\{(x,y,z,v,u,w) \mid x^2 + y^2 + (z-\sigma-r)^2 + sv^2 + u^2 + \left(\omega - \frac{\sigma+r}{2}\right)^2 \le \frac{2b(\sigma+r)^2}{\min(\sigma,b)}\right\} \quad (7)$$

is a globally exponential attractive set of system (2) according to Theorem 2.

(ii) Taking $\sigma = 10$, $b = \frac{8}{3}$, $r = 40$, $s = 50$, we get that

$$\Delta = \bigl\{(x,y,z,v,u,w) \mid x^2 + y^2 + (z-50)^2 + 50v^2 + u^2 + (\omega-25)^2 \le (50\sqrt{2})^2\bigr\} \quad (8)$$

is a globally exponential attractive set of system (2) according to Theorem 2. Figure 5 shows chaotic attractors of system (2) in the (x,y,z) space defined by Δ in (8). Figure 6 shows

Figure 5 Chaotic attractors of (2) in the (x,y,z) space defined by Δ.

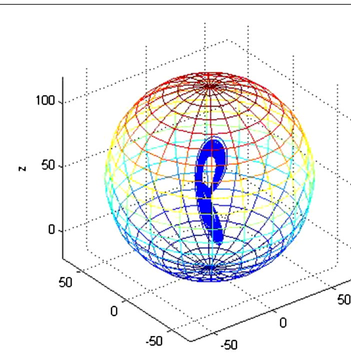


Figure 6 Chaotic attractors of (2) in the (x, y, v) space defined by Δ.

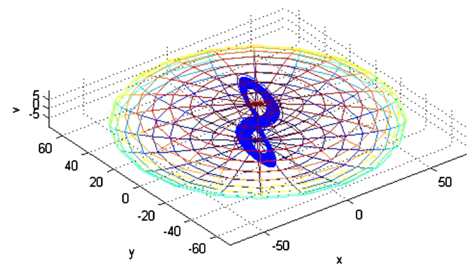


Figure 7 Chaotic attractors of (2) in the (x, y, u) space defined by Δ.

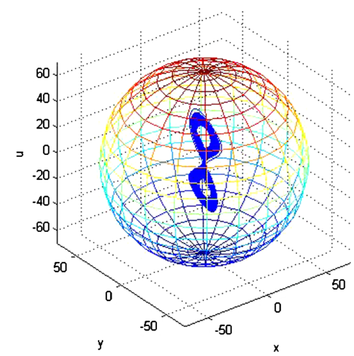
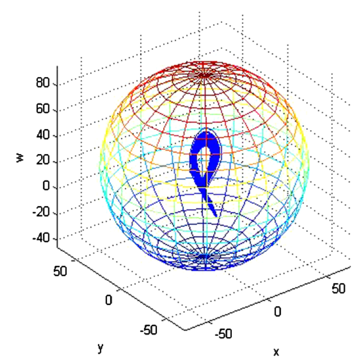


Figure 8 Chaotic attractors of (2) in the (x, y, ω) space defined by Δ.



chaotic attractors of system (2) in the (x, y, v) space defined by Δ in (8). Figure 7 shows chaotic attractors of system (2) in the (x, y, u) space defined by Δ in (8). Figure 8 shows chaotic attractors of system (2) in the (x, y, ω) space defined by Δ in (8).

3 Conclusions

By means of Lyapunov-like functions, we have studied some dynamical behaviors of a high-order Lorenz-Stenflo system using theoretical analysis and numerical simulations. The obtained results show that this system has complex dynamics and this system deserves a further detailed investigation. The results of this paper are useful in many engineering applications such as chaos synchronization, chaos cryptology, coding information, and information compression.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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