Gegenhasi *Advances in Difference Equations* (2017) 2017:277
DOI 10.1186/s13662-017-1347-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The modified two-dimensional Toda lattice with self-consistent sources

Gegenhasi*

*Correspondence:
gegen@amss.ac.cn
School of Mathematical Sciences, Inner Mongolia University, No. 235 West College Road, Hohhot, Inner Mongolia 010021, China

Abstract

In this paper, we derive the Grammian determinant solutions to the modified two-dimensional Toda lattice, and then we construct the modified two-dimensional Toda lattice with self-consistent sources via the source generation procedure. We show the integrability of the modified two-dimensional Toda lattice with self-consistent sources by presenting its Casoratian and Grammian structure of the N-soliton solution. It is also demonstrated that the commutativity between the source generation procedure and Bäcklund transformation is valid for the two-dimensional Toda lattice.

MSC: 37K10; 37K40

Keywords: modified two-dimensional Toda lattice equation; source generation procedure; Grammian determinant; Casorati determinant

1 Introduction

The two-dimensional Toda lattice, which can be regarded as a spatial discretization of the KP equation, takes the following form:

$$\frac{\partial^2}{\partial x \partial s}\ln(V_n+1)=V_{n+1}+V_{n-1}-2V_n, \tag{1}$$

where V_n denotes $V(n,x,s)$. We use the above notation throughout the paper. Under the dependent variable transformation

$$V_n=\frac{\partial^2}{\partial x \partial s}\ln f_n, \tag{2}$$

equation (1) is transformed into the bilinear form [1, 2]:

$$D_xD_sf_n\cdot f_n=2\left(e^{D_n}f_n\cdot f_n-f_n^2\right), \tag{3}$$

where the bilinear operators are defined by [2]

$$D_x^mD_t^nf\cdot g=\frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial s^n}f(x+y,t+s)g(x-y,t-s)\bigg|_{s=0,y=0},$$

$$e^{D_n}f_n\cdot g_n=f_{n+1}g_{n-1}.$$

It is shown in [2, 3] that the two-dimensional Toda lattice equation possesses the following bilinear Bäcklund transformation:

$$D_x f_{n+1} \cdot f'_n = -\frac{1}{\lambda} f_n f'_{n+1} + \nu f_{n+1} f'_n, \tag{4}$$

$$D_s f_n \cdot f'_n = \lambda f_{n+1} f'_{n-1} - \mu f_n f'_n, \tag{5}$$

where λ, μ, ν are arbitrary constants. Equations (4)-(5) are transformed into the following nonlinear form:

$$\frac{\partial}{\partial x} u_n = (\mu + u_n)(v_n - v_{n+1}), \tag{6}$$

$$\frac{\partial}{\partial s} v_n = (\nu + v_n)(u_{n-1} - u_n), \tag{7}$$

through the dependent variable transformation $u_n = \frac{\partial}{\partial s} \ln(\frac{f_n}{f'_n})$, $v_n = -\frac{\partial}{\partial x} \ln(\frac{f_n}{f'_{n-1}})$. Equations (4)-(5) or (6)-(7) are called the modified two-dimensional Toda lattice [2, 3]. The solutions V_n of the two-dimensional Toda lattice (1) and u_n, v_n of the modified two-dimensional Toda lattice (6)-(7) are connected through a Miura transformation [2].

The soliton equations with self-consistent sources can model a lot of important physical processes. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves [4]. The KP equation with self-consistent sources describes the interaction of a long wave with a short-wave packet propagating on the x, y plane at an angle to each other [5, 6]. Since the pioneering work of Mel'nikov [7], lots of soliton equations with self-consistent sources have been studied via inverse scattering methods [7–11], Darboux transformation methods [12–17], Hirota's bilinear method and the Wronskian technique [18–24].

In [25], a new algebraic method, called the source generation procedure, is proposed to construct and solve the soliton equations with self-consistent sources both in continuous and discrete cases. The source generation procedure has been successfully applied to many $(2+1)$-dimensional continuous and discrete soliton equations such as the Ishimori-I equation [26], the semi-discrete Toda equation [27], the modified discrete KP equation [28], and others. The purpose of this paper is to construct the modified two-dimensional Toda lattice with self-consistent sources via the source generation procedure and clarify the determinant structure of N-soliton solution for the modified two-dimensional Toda lattice with self-consistent sources.

The paper is organized as follows. In Section 2, we derive the Grammian solution to the modified two-dimensional Toda lattice equation and then construct the two-dimensional Toda lattice equations with self-consistent sources. In Section 3, the Casoratian formulation of N-soliton solution for the modified two-dimensional Toda lattice with self-consistent is given. Section 4 is devoted to showing that the commutativity of the source generation procedure and Bäcklund transformation is valid for two-dimensional Toda lattice. We end this paper with a conclusion and discussion in Section 5.

2 The modified two-dimensional Toda lattice equation with self-consistent sources

The N-soliton solution in Casoratian form for the modified two-dimensional Toda lattice equation (4)-(5) is given in [2] and [29]. In this section, we first derive the Grammian formulation of the N-soliton solution for the modified two-dimensional Toda lattice equation, and then we construct the modified two-dimensional Toda lattice equation with self-consistent sources via the source generation procedure.

If we choose $\lambda = 1$, $\nu = \mu = 0$, then the modified two-dimensional Toda lattice (4)-(5) becomes

$$\left(D_x e^{\frac{1}{2}D_n} + e^{-\frac{1}{2}D_n}\right) f_n \cdot f_n' = 0, \tag{8}$$

$$\left(D_s - e^{D_n}\right) f_n \cdot f_n' = 0. \tag{9}$$

Proposition 1 *The modified two-dimensional Toda lattice* (8)-(9) *has the following Grammian determinant solution*:

$$f_n = \det\left| c_{ij} + (-1)^n \int_{-\infty}^{x} \phi_i(n)\psi_j(-n)\,dx \right|_{1\le i,j\le N} = |M|, \tag{10}$$

$$f_n'(n,x,s) = \begin{vmatrix} M & \Phi(n) \\ \Psi(n)^T & -\phi_{N+1}(n) \end{vmatrix}, \tag{11}$$

where

$$\Phi(n) = \left(-\phi_1(n), \ldots, -\phi_N(n)\right)^T, \tag{12}$$

$$\begin{aligned}\Psi(n) = \Bigg(& c_{N+1,1} + (-1)^n \int_{-\infty}^{x} \phi_{N+1}(n)\psi_1(-n)\,dx, \ldots, \\ & c_{N+1,N} + \int_{-\infty}^{x} (-1)^n \phi_{N+1}(n)\psi_N(-n)\,dx \Bigg)^T,\end{aligned} \tag{13}$$

in which the $\phi_i(n)$ *denote* $\phi_i(n,x,s)$ *and the* $\psi_i(-n)$ *denote* $\psi_i(-n,x,s)$ *for* $i = 1, \ldots, N+1$. *In addition*, c_{ij} $(1 \le i,j \le N+1)$ *are arbitrary constants and* $\phi_i(n)$, $\psi_i(-n)$ $(i = 1, \ldots, N+1)$ *satisfy the following dispersion relations*:

$$\frac{\partial \phi_i(n)}{\partial x} = \phi_i(n+1), \qquad \frac{\partial \psi_i(-n)}{\partial x} = \psi_i(-n+1), \tag{14}$$

$$\frac{\partial \phi_i(n)}{\partial s} = -\phi_i(n-1), \qquad \frac{\partial \psi_i(-n)}{\partial s} = -\psi_i(-n-1). \tag{15}$$

Proof The Grammian determinants f_n in (10) and f_n' in (11) can be expressed in terms of the following Pfaffians:

$$f_n = \left(a_1, \ldots, a_N, a_N^*, \ldots, a_1^*\right) = (\star), \tag{16}$$

$$f_n' = \left(a_1, \ldots, a_{N+1}, d_0^*, a_N^*, \ldots, a_1^*\right) = \left(a_{N+1}, d_0^*, \star\right), \tag{17}$$

where the Pfaffian elements are defined by

$$\left(a_i, a_j^*\right)_n = c_{ij} + (-1)^n \int_{-\infty}^{x} (-1)^n \phi_i(n)\psi_j(-n)\,dx, \tag{18}$$

$$\left(d_m^*, a_i\right) = \phi_i(n+m), \left(d_m, a_j^*\right) = (-1)^{n+m}\psi_j(-n+m), \tag{19}$$

$$(a_i, a_j)_n = \left(a_i^*, a_j^*\right)_n = (d_m, d_k) = \left(d_m, d_k^*\right) = \left(d_m^*, d_k^*\right) = 0, \tag{20}$$

in which $i, j = 1, \ldots, N+1$ and k, m are integers.

Using the dispersion relations (14)-(15), we can compute the following differential and difference formula for the Pfaffians (16)-(17):

$$f_{n+1,x} = \left(d_{-1}, d_1^*, \star\right), \qquad f_{n+1} = (\star) + \left(d_{-1}, d_0^*, \star\right), \tag{21}$$

$$f_{ns} = \left(d_{-1}, d_{-1}^*, \star\right), \qquad f'_{nx} = \left(a_{N+1}, d_1^*, \star\right), \qquad f'_{n-1} = \left(a_{N+1}, d_{-1}^*, \star\right) \tag{22}$$

$$f'_{n+1} = \left(a_{N+1}, d_1^*, \star\right) + \left(a_{N+1}, d_{-1}, d_o^*, d_1^*, \star\right), \tag{23}$$

$$f'_{ns} = \left(a_{N+1}, d_{-1}, d_{-1}^*, d_0^*, \star\right) - \left(a_{N+1}, d_{-1}^*, \star\right). \tag{24}$$

Substituting equations (21)-(24) into the modified two-dimensional Toda lattice (8)-(9) gives the following two Pfaffian identities:

$$\left(d_{-1}, d_1^*, \star\right)\left(a_{N+1}, d_0^*, \star\right) - \left(d_{-1}, d_0^*, \star\right)\left(a_{N+1}, d_1^*, \star\right) + (\star)\left(a_{N+1}, d_{-1}, d_0^*, d_1^*, \star\right) = 0,$$
$$\left(d_{-1}, d_0^*, \star\right)\left(a_{N+1}, d_{-1}^*, \star\right) - \left(d_{-1}, d_{-1}^*, \star\right)\left(a_{N+1}, d_0^*, \star\right) + (\star)\left(a_{N+1}, d_{-1}, d_{-1}^*, d_0^*, \star\right) = 0. \qquad \square$$

In order to construct the modified two-dimensional Toda lattice with self-consistent sources, we change the Grammian determinant solutions (10)-(11) into the following form:

$$f(n,x,s) = \det\left|\gamma_{ij}(s) + (-1)^n \int_{-\infty}^{x} (-1)^n \phi_i(n)\psi_j(-n)\,dx\right|_{1\le i,j\le N} = |F|, \tag{25}$$

$$f'_n(n,x,s) = \begin{vmatrix} F & \Phi(n) \\ \Psi(n)^T & -\phi_{N+1}(n) \end{vmatrix}, \tag{26}$$

where Nth column vectors $\Phi(n)$, $\Psi(n)$ are given in (12)-(13) and $\phi_i(n)$, $\psi_i(-n)$ ($i = 1, \ldots, N+1$) also satisfy the dispersion relations (14)-(15). In addition, $\gamma_{ij}(s)$ satisfies

$$\gamma_{ij}(s) = \begin{cases} \gamma_i(s), & i = j \text{ and } 1 \le i \le K \le N, \\ c_{ij}, & \text{otherwise}, \end{cases} \tag{27}$$

with $\gamma_i(s)$ being an arbitrary function of s and K being a positive integer.

The Grammian determinants f_n in (25) and f'_n in (26) can be expressed by means of the following Pfaffians:

$$f_n = \left(1, \ldots, N, N^*, \ldots, 1^*\right) = (\cdot), \tag{28}$$

$$f'_n = \left(1, \ldots, N+1, d_0^*, N^*, \ldots, 1^*\right) = \left(N+1, d_0^*, \cdot\right), \tag{29}$$

where the Pfaffian elements are defined by

$$\left(i, j^*\right)_n = \gamma_{ij}(s) + (-1)^n \int_{-\infty}^{x} (-1)^n \phi_i(n)\psi_j(-n)\,dx, \qquad \left(i^*, j^*\right)_n = 0, \tag{30}$$

$$\left(d_m^*, i\right) = \phi_i(n+m), \qquad \left(d_m, j^*\right) = (-1)^{n+m}\psi_j(-n+m), \qquad (i,j)_n = 0, \tag{31}$$

$$(d_m, i) = \left(d_m^*, j^*\right) = (d_m, d_k) = \left(d_m, d_k^*\right) = \left(d_m^*, d_k^*\right) = 0, \tag{32}$$

in which $i, j = 1, \ldots, N+1$ and k, m are integers.

It is easy to show that the functions $f(n,x,s), f'(n,x,s)$ given in (28)-(29) still satisfy equation (8). However, they will not satisfy (9), and they satisfy the following new equation:

$$D_s f_n \cdot f_n' - f_{n+1} f_{n-1}' = -\sum_{j=1}^{K} g_n^{(j)} h_n^{(j)}, \tag{33}$$

where the new functions $g_n^{(j)}$, $h_n^{(j)}$ are given by

$$g_n^{(j)} = \sqrt{\dot{\gamma}_j(t)}\left(1, \ldots, N, d_0^*, N^*, \ldots, \hat{j}^*, \ldots, 1^*\right), \tag{34}$$

$$h_n^{(j)} = \sqrt{\dot{\gamma}_j(t)}\left(1, \ldots, \hat{j}, \ldots, N+1, N^*, \ldots, 1^*\right), \tag{35}$$

where $j = 1, \ldots, K$ and the dot denotes the derivative of $\gamma_j(t)$ with respect to t. Furthermore, we can show that $f_n, f_n', g_n^{(j)}, h_n^{(j)}$ $(j = 1, \ldots, K)$ satisfy the following $2K$ equations:

$$\left(D_x e^{\frac{1}{2}D_n} + e^{-\frac{1}{2}D_n}\right) f \cdot g_n^{(j)} = 0, \quad j = 1, \ldots, K, \tag{36}$$

$$\left(D_x e^{\frac{1}{2}D_n} + e^{-\frac{1}{2}D_n}\right) h_n^{(j)} \cdot f_n' = 0, \quad j = 1, \ldots, K. \tag{37}$$

In fact, we have the following differential and difference formula for f_n in (28), f_n' in (29) and $g_n^{(j)}$, $h_n^{(j)}$ $(j = 1, \ldots, K)$ by employing the dispersion relations (14)-(15):

$$\begin{aligned} f_{ns} &= \left(d_{-1}, d_{-1}^*, \cdot\right) \\ &\quad + \sum_{j=1}^{K} \dot{\gamma}_j(s)\left(1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right), \end{aligned} \tag{38}$$

$$\begin{aligned} f_{ns}' &= \left(N+1, d_{-1}, d_{-1}^*, d_0^*, \cdot\right) - \left(N+1, d_{-1}^*, \cdot\right) \\ &\quad + \sum_{i=1}^{K} \dot{\gamma}_i(s)\left(N+1, d_0^*, 1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i}^*, \ldots, 1^*\right), \end{aligned} \tag{39}$$

$$f_{n+1} = (\cdot) + \left(d_{-1}, d_0^*, \cdot\right), \qquad f_{n-1}' = \left(N+1, d_{-1}^*, \cdot\right), \tag{40}$$

$$g_{n-1}^{(j)} = \sqrt{\dot{\gamma}_j(t)}\left(1, \ldots, N, d_{-1}^*, N^*, \ldots, \hat{j}^*, \ldots, 1^*\right), \tag{41}$$

$$\begin{aligned} g_{n-1,x}^{(j)} &= \sqrt{\dot{\gamma}_j(t)}\left[\left(1, \ldots, N, d_0^*, N^*, \ldots, \hat{j}^*, \ldots, 1^*\right)\right. \\ &\quad \left. + \left(1, \ldots, N, d_0, d_0^*, d_{-1}^*, N^*, \ldots, \hat{j}^*, \ldots, 1^*\right)\right], \end{aligned} \tag{42}$$

$$f_{n-1} = (\cdot) - \left(d_0, d_{-1}^*, \cdot\right), \qquad f_{nx} = \left(d_0, d_0^*, \ldots\right), \tag{43}$$

$$\begin{aligned} h_{n+1}^{(j)} &= \sqrt{\dot{\gamma}_j(t)}\left[\left(1, \ldots, \hat{j}, \ldots, N+1, N^*, \ldots, 1^*\right)\right. \\ &\quad \left. + \left(1, \ldots, \hat{j}, \ldots, N+1, d_{-1}, d_0^* N^*, \ldots, 1^*\right)\right], \end{aligned} \tag{44}$$

$$h_{n+1,x}^{(j)} = \sqrt{\dot{\gamma}_j(t)}\left(1, \ldots, \hat{j}, \ldots, N+1, d_{-1}, d_1^*, N^*, \ldots, 1^*\right), \tag{45}$$

$$f'_{nx} = \left(N+1, d_1^*, \cdot\right), \tag{46}$$

$$f'_{n+1} = \left(N+1, d_1^*, \cdot\right) + \left(N+1, d_{-1}, d_0^*, d_1^*, \cdot\right), \tag{47}$$

where ˆ indicates deletion of the letter under it.

Substitution of equations (38)-(47) into equations (33), (36)-(37) gives the following Pfaffian identities:

$$\begin{aligned}
&\bigl[\left(d_{-1}, d_{-1}^*, \cdot\right)\left(N+1, d_0^*, \cdot\right) - (\cdot)\left(N+1, d_{-1}, d_{-1}^*, d_0^*, \cdot\right) - \left(d_{-1}, d_0^*, \cdot\right)\left(N+1, d_{-1}^*, \cdot\right)\bigr], \\
&\quad + \sum_{j=1}^{K} \dot{\gamma}_j(s)\bigl[\left(1, \ldots, N+1, d_0^*, N^*, \ldots, 1^*\right)\left(1, \ldots, \hat{i}, \ldots, N, N^*, \ldots, \hat{i^*}, \ldots, 1^*\right) \\
&\quad - (\cdot)\left(1, \ldots, \hat{i}, \ldots, N+1, d_0^*, N^*, \ldots, i^*, \ldots, 1^*\right) \\
&\quad + \left(1, \ldots, N, d_0^*, N^*, \ldots, \hat{i^*}, \ldots, 1^*\right)\left(1, \ldots, \hat{i}, \ldots, N+1, N^*, \ldots, 1^*\right)\bigr] = 0, \\
&\left(d_0, d_0^*, \cdot\right)\left(1, \ldots, N, d_{-1}^*, N^*, \cdot, \hat{j^*}, \ldots, 1^*\right) \\
&\quad - (\cdot)\left(1, \ldots, N, d_0, d_0^*, d_{-1}^*, N^*, \cdot, \hat{j^*}, \ldots, 1^*\right) \\
&\quad - \left(d_0, d_{-1}^*, \cdot\right)\left(1, \ldots, N, d_0^*, N^*, \cdot, \hat{j^*}, \ldots, 1^*\right) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left(N+1, d_0^*, \cdot\right)\left(1, \ldots, \hat{i}, \ldots, N+1, d_{-1}, d_1^*, N^*, \ldots, 1^*\right) \\
&\quad - \left(N+1, d_1^*, \cdot\right)\left(1, \ldots, \hat{i}, \ldots, N+1, d_{-1}, d_0^*, N^*, \ldots, 1^*\right) \\
&\quad + \left(N+1, d_{-1}, d_0^*, d_1^*, \cdot\right)\left(1, \ldots, \hat{i}, \ldots, N+1, N^*, \ldots, 1^*\right) = 0,
\end{aligned}$$

respectively. Therefore, equations (8), (33), (36)-(37) constitute the modified two-dimensional Toda lattice with self-consistent sources, and it possesses the Grammian determinant solution (28)-(29), (34)-(35).

Through the dependent variable transformation

$$u_n = \frac{f_{n+1}f'_{n-1}}{f_n f'_n}, \qquad v_n = -\frac{\partial}{\partial x}\ln\left(\frac{f_n}{f'_{n-1}}\right), \qquad G_n^{(j)} = \frac{g_n^{(j)}}{f_n}, \qquad H_n^{(j)} = \frac{h_n^{(j)}}{f'_n}, \tag{48}$$

the bilinear modified two-dimensional Toda lattice with self-consistent sources (8, 33, 36)-(37) can be transformed into the following nonlinear form:

$$\frac{\partial}{\partial x}u_n = u_n(v_n - v_{n+1}), \tag{49}$$

$$\frac{\partial}{\partial s}v_n = v_n(u_{n-1} - u_n) + v_n\sum_{j=1}^{K}\bigl[u_n G_n^{(j)}H_n^{(j)} - u_{n-1}G_{n-1}^{(j)}H_{n-1}^{(j)}\bigr], \tag{50}$$

$$\frac{\partial}{\partial x}G_{n-1}^{(j)} + G_n^{(j)}u_n v_n = 0, \quad j = 1, \ldots, K, \tag{51}$$

$$\frac{\partial}{\partial x}H_{n+1}^{(j)} + H_n^{(j)}u_n v_{n+1} = 0, \quad j = 1, \ldots, K. \tag{52}$$

When we take $G_n^{(j)} = H_n^{(j)} = 0$, $j = 1,\dots,K$ in (49)-(52), the nonlinear modified two-dimensional Toda lattice with self-consistent sources (49)-(52) is reduced to the nonlinear modified two-dimensional Toda lattice (6)-(7) with $\lambda = 1$, $\nu = \mu = 0$.

If we choose

$$\begin{aligned}&\phi_i(n) = e^{\xi_i}, \qquad \psi_i(-n) = (-1)^n e^{\eta_i},\\ &\xi_i = e^{q_i}x + q_i n - e^{-q_i}t, \qquad \eta_i = -e^{Q_i}x - Q_i n + e^{-Q_i}t,\end{aligned} \tag{53}$$

where $i = 1, 2, \dots, N+1$ in the Grammian determinants (25)-(26), (34)-(35), then we obtain the N-soliton solution of the modified two-dimensional Toda lattice with self-consistent sources (8), (33), (36)-(37). Here q_i, Q_i $(i = 1, 2, \dots, N+1)$ are arbitrary constants.

For example, if we take $K = 1$, $N = 1$ and

$$\phi_1(n) = e^{\xi_1}, \qquad \phi_2(n) = e^{\xi_2}, \qquad \psi_1(n) = e^{\eta_1}, \qquad \gamma_1(t) = \frac{e^{2a(t)}}{e^{q_1} - e^{Q_1}}, \qquad c_{21} = 0, \tag{54}$$

where $a(t)$ is an arbitrary function of t, then we have

$$f_n(x,n,t) = \frac{e^{2a(t)}}{e^{q_1} - e^{Q_1}}\bigl(1 + e^{\xi_1+\eta_1-2a(t)}\bigr), \tag{55}$$

$$f'_n(x,n,t) = -\frac{e^{2a(t)+\xi_2}}{e^{q_1} - e^{Q_1}}\left(1 + \frac{e^{q_2} - e^{q_1}}{e^{q_2} - e^{Q_1}}e^{\xi_1+\eta_1-2a(t)}\right), \tag{56}$$

$$g_n^{(1)}(x,n,t) = -\sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1} - e^{Q1}}}e^{\xi_1+a(t)}, \tag{57}$$

$$h_n^{(1)}(x,n,t) = \sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1} - e^{Q1}}}\frac{1}{e^{q_2} - e^{Q_1}}e^{\xi_2-\eta_1+a(t)}. \tag{58}$$

Therefore, the one-soliton solution of the nonlinear modified two-dimensional Toda lattice with self-consistent sources (49)-(52) is given by

$$u_n(x,n,t) = \frac{e^{-q_2}(1 + e^{q_1-Q_1}e^{\xi_1+\eta_1-2a(t)})(1 + \frac{e^{q_2}-e^{q_1}}{e^{q_2}-e^{Q_1}}e^{Q_1-q_1}e^{\xi_1+\eta_1-2a(t)})}{(1 + e^{\xi_1+\eta_1-2a(t)})(1 + \frac{e^{q_2}-e^{q_1}}{e^{q_2}-e^{Q_1}}e^{\xi_1+\eta_1-2a(t)})}, \tag{59}$$

$$v_n(x,n,t) = -\frac{\partial}{\partial x}\ln\left(\frac{1 + e^{\xi_1+\eta_1-2a(t)}}{-e^{\xi_2}(1 + \frac{e^{q_2}-e^{q_1}}{e^{q_2}-e^{Q_1}}e^{Q_1-q_1}e^{\xi_1+\eta_1-2a(t)})}\right), \tag{60}$$

$$G_n^{(1)}(x,n,t) = -\sqrt{2\dot{a}(t)\bigl(e^{q_1} - e^{Q1}\bigr)}\frac{e^{\xi_1-a(t)}}{1 + e^{\xi_1+\eta_1-2a(t)}}, \tag{61}$$

$$H_n^{(1)}(x,n,t) = \frac{-\sqrt{2\dot{a}(t)(e^{q_1} - e^{Q1})}}{e^{q_2} - e^{Q_1}}\frac{e^{-\eta_1-a(t)}}{1 + \frac{e^{q_2}-e^{q_1}}{e^{q_2}-e^{Q_1}}e^{\xi_1+\eta_1-2a(t)}}. \tag{62}$$

If we take $K = 1$, $N = 2$ and

$$\begin{aligned}&\phi_1(n) = e^{\xi_1}, \qquad \phi_2(n) = e^{\xi_2}, \qquad \phi_3(n) = e^{\xi_3}, \qquad \psi_1(n) = e^{\eta_1}, \qquad \psi_2(n) = e^{\eta_2},\\ &\gamma_1(t) = \frac{e^{2a(t)}}{e^{q_1} - e^{Q_1}}, \qquad \gamma_2(t) = \frac{1}{e^{q_2} - e^{Q_2}}, \qquad c_{12} = 0, \qquad c_{21} = 0, \qquad c_{31} = 0,\\ &c_{32} = 0,\end{aligned}$$

we derive

$$
\begin{aligned}
f_n(x,n,t) &= \frac{e^{2a(t)}}{(e^{q_1}-e^{Q_1})(e^{q_2}-e^{Q_2})}\Bigg(1+e^{\xi_1+\eta_1-2a(t)}+e^{\xi_2+\eta_2} \\
&\quad + \frac{(e^{q_1}-e^{q_2})(e^{Q_1}-e^{Q_2})}{(e^{q_1}-e^{Q_2})(e^{Q_1}-e^{q_2})}e^{\xi_1+\eta_1+\xi_2+\eta_2-2a(t)}\Bigg),
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
f_n'(x,n,t) &= -\frac{e^{\xi_3+2a(t)}}{(e^{q_1}-e^{Q_1})(e^{q_2}-e^{Q_2})}\Bigg(1+\frac{e^{q_3}-e^{q_1}}{e^{q_3}-e^{Q_1}}e^{\xi_1+\eta_1-2a(t)}+\frac{e^{q_3}-e^{q_2}}{e^{q_3}-e^{Q_2}}e^{\xi_2+\eta_2} \\
&\quad + \frac{(e^{q_1}-e^{q_2})(e^{Q_2}-e^{Q_1})(e^{q_3}-e^{q_2})(e^{q_3}-e^{q_1})}{(e^{q_1}-e^{Q_2})(e^{q_2}-e^{Q_1})(e^{q_3}-e^{Q_2})(e^{q_3}-e^{Q_1})}e^{\xi_1+\eta_1+\xi_2+\eta_2-2a(t)}\Bigg),
\end{aligned} \tag{64}
$$

$$
g_n^{(1)}(x,n,t) = \sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q_1}}}\frac{e^{\xi_1+a(t)}}{e^{q_2}-e^{Q_2}}\left(1+\frac{e^{q_1}-e^{q_2}}{e^{q_1}-e^{Q_2}}e^{\xi_2+\eta_2}\right), \tag{65}
$$

$$
\begin{aligned}
h_n^{(1)}(x,n,t) &= -\sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q_1}}}\frac{e^{\xi_3+\eta_1+a(t)}}{(e^{q_2}-e^{Q_2})(e^{q_3}-e^{Q_1})} \\
&\quad \times\left(1+\frac{(e^{q_2}-e^{q_3})(e^{Q_1}-e^{Q_2})}{(e^{Q_2}-e^{q_3})(e^{Q_1}-e^{q_2})}e^{\xi_2+\eta_2}\right).
\end{aligned} \tag{66}
$$

Substituting functions (63)-(66) into the dependent variable transformations (48), we obtain two-soliton solution of the nonlinear modified two-dimensional Toda lattice with self-consistent sources (49)-(52).

3 Casorati determinant solution to the modified two-dimensional Toda lattice equation with self-consistent sources

In Section 2, we derived that the modified two-dimensional Toda lattice with self-consistent sources (8), (33), (36)-(37) possess the Grammian determinant solution (25), (26), (34), (35). In this section, we derive the Casoratian formulation of the N-soliton for the modified two-dimensional Toda lattice with self-consistent sources (8), (33), (36)-(37).

Proposition 2 *The modified two-dimensional Toda lattice with self-consistent sources* (8), (33), (36)-(37) *has the following Casorati determinant solution*:

$$
f_n = \det\left|\psi_i(n+j-1)\right|_{1\le i,j\le N} = (d_0,\ldots,d_{N-1},N,\ldots,1), \tag{67}
$$

$$
f_n' = \det\left|\psi_i(n+j-1)\right|_{1\le i,j\le N+1} = (d_0,\ldots,d_N,N+1,\ldots,1), \tag{68}
$$

$$
g_n^{(j)} = \sqrt{\dot{\gamma}_j(t)}(d_0,\ldots,d_N,N,\ldots,1,\alpha_j), \tag{69}
$$

$$
h_n^{(j)} = \sqrt{\dot{\gamma}_j(t)}(d_0,\ldots,d_{N-1},N+1,\ldots,\hat{j},\ldots,1), \tag{70}
$$

where $\psi_i(n+m) = \phi_{i1}(n+m)+(-1)^{i-1}C_i(s)\phi_{i2}(n+m)$ $(m=0,\ldots,N)$ *and*

$$
C_i(s) = \begin{cases} \gamma_i(s), & 1\le i\le K\le N+1, \\ \gamma_i, & \textit{otherwise}, \end{cases} \tag{71}
$$

with $\gamma_i(s)$ being an arbitrary function of s and K, N being positive integers. In addition, $\phi_{i1}(n)$, $\phi_{i2}(n)$ satisfy the following dispersion relations:

$$\frac{\partial\phi_{ij}(n)}{\partial x} = \phi_{ij}(n+1), \qquad \frac{\partial\phi_{ij}(n)}{\partial s} = -\phi_{ij}(n-1), \quad j = 1,2, \tag{72}$$

and the Pfaffian elements are defined by

$$(d_m, i) = \psi_i(n+m), \qquad (d_m, \alpha_i) = \phi_{i2}(n+m), \tag{73}$$

$$(d_m, d_l) = (i,j) = 0, \qquad (\alpha_i, j) = (\alpha_i, \alpha_j) = 0, \tag{74}$$

in which $i,j = 1,\dots,N+1$ and m, l are integers.

Proof We can derive the following dispersion relation for $\psi_i(n)$ ($i = 1,\dots,N+1$) from equations (72):

$$\frac{\partial\psi_i(n)}{\partial x} = \phi_i(n+1), \tag{75}$$

$$\frac{\partial\psi_i(n)}{\partial s} = -\psi_i(n-1) + (-1)^{i-1}\dot{C_i}(t)\phi_{i2}(n). \tag{76}$$

Applying the dispersion relation (75)-(76), we can calculate the following differential and difference formula for the Casorati determinants (67)-(70):

$$f_{n+1,x} = (d_1,\dots,d_{N-1},d_{N+1},N,\dots,1), \tag{77}$$

$$f_{n+1} = (d_1,\dots,d_N,N,\dots,1), \qquad f_{n-1} = (d_{-1},\dots,d_{N-2},N,\dots,1) \tag{78}$$

$$f'_{nx} = (d_0,\dots,d_{N-1},d_{N+1},N+1,\dots,1), \tag{79}$$

$$\begin{aligned} f_{n,s} = {} & -(d_{-1},d_1,\dots,d_{N-1},N,\dots,1) \\ & + \sum_{j=1}^{K} \dot{\gamma}_j(t)(d_0,\dots,d_{N-1},N,\dots,\hat{j},\dots,1,\alpha_j), \end{aligned} \tag{80}$$

$$\begin{aligned} f'_{n,s} = {} & -(d_{-1},d_1,\dots,d_N,N+1,\dots,1) \\ & + \sum_{j=1}^{K} \dot{\gamma}_j(t)(d_0,\dots,d_N,N+1,\dots,\hat{j},\dots,1,\alpha_j), \end{aligned} \tag{81}$$

$$f'_{n+1} = (d_1,\dots,d_{N+1},N+1,\dots,1), \tag{82}$$

$$f'_{n-1} = (d_{-1},d_1,\dots,d_{N-1},N+1,\dots,1), \tag{83}$$

$$g_n^{(j)} = \sqrt{\dot{\gamma}_j(t)}(d_{-1},\dots,d_N,N,\dots,1,\alpha_j), \tag{84}$$

$$h_{n+1}^{(j)} = \sqrt{\dot{\gamma}_j(t)}(d_1,\dots,d_N,N+1,\dots,\hat{j},\dots,1), \tag{85}$$

$$f_{nx} = (d_0,\dots,d_{N-2},d_N,N,\dots,1), \tag{86}$$

$$g_{n,x}^{(j)} = \sqrt{\dot{\gamma}_j(t)}(d_{-1},\dots,d_{N-2},d_N,N,\dots,1,\alpha_j), \tag{87}$$

$$h_{n+1,x}^{(j)} = \sqrt{\dot{\gamma}_j(t)}(d_1,\dots,d_{N-1},d_{N+1},N+1,\dots,\hat{j},\dots,1). \tag{88}$$

By substituting equations (77)-(88) into the modified two-dimensional Toda lattice with self-consistent sources (8), (33), (36)-(37), we obtain the following Pfaffian identities, respectively:

$$\begin{aligned}
&(d_1,\dots,d_{N-1},d_{N+1},N,\dots,1)(d_0,\dots,d_N,N+1,\dots,1)\\
&\quad-(d_1,\dots,d_N,N,\dots,1)(d_0,\dots,d_{N-1},d_{N+1},N+1,\dots,1)\\
&\quad+(d_0,\dots,d_{N-1},N,\dots,1)(d_1,\dots,d_{N+1},N+1,\dots,1)=0,\\
&\bigl[-(d_{-1},d_1,\dots,d_{N-1},N,\dots,1)(d_0,\dots,d_N,N+1,\dots,1)\\
&\quad+(d_0,\dots,d_{N-1},N,\dots,1)(d_{-1},d_1,\dots,d_N,N+1,\dots,1)\\
&\quad-(d_1,\dots,d_N,N,\dots,1)(d_{-1},\dots,d_{N-1},N+1,\dots,1)\bigr]\\
&\quad+\sum_{j=1}^{K}\dot{\gamma}_j(s)\bigl[(d_0,\dots,d_{N-1},N,\dots,\hat{j},\dots,1,\alpha_j)(d_0,\dots,d_N,N+1,\dots,1)\\
&\quad-(d_0,\dots,d_N,N+1,\dots,\hat{j},\dots,1,\alpha_j)(d_0,\dots,d_{N-1},N,\dots,1)\\
&\quad+(d_0,\dots,d_N,N,\dots,1,\alpha_j)(d_0,\dots,d_{N-1},N+1,\dots,\hat{j},\dots,1)\bigr]=0,\\
&(d_0,\dots,d_{N-2},d_N,N,\dots,1)(d_{-1},\dots,d_{N-1},N,\dots,1,\alpha_j)\\
&\quad-(d_0,\dots,d_{N-1},N,\dots,1)(d_{-1},\dots,d_{N-2},d_N,N,\dots,1,\alpha_j)\\
&\quad+(d_{-1},\dots,d_{N-2},N,\dots,1)(d_0,\dots,d_N,N,\dots,1,\alpha_j)=0,
\end{aligned}$$

and

$$\begin{aligned}
&(d_1,\dots,d_{N-1},d_{N+1},N+1,\dots,\hat{j},\dots,1)(d_0,\dots,d_N,N+1,\dots,1)\\
&\quad-(d_1,\dots,d_N,N+1,\dots,\hat{j},\dots,1)(d_0,\dots,d_{N-1},d_{N+1},N+1,\dots,1)\\
&\quad+(d_0,\dots,d_{N-1},N+1,\dots,\hat{j},\dots,1)(d_1,\dots,d_{N+1},N+1,\dots,1)=0,
\end{aligned}$$

respectively. □

In order to obtain the one-soliton solution of the nonlinear modified two-dimensional Toda lattice with self-consistent sources (49)-(52), we take $N=1$, $K=1$ and

$$\begin{aligned}
&\phi_{11}=\frac{e^{\xi_1}}{e^{q_1}-e^{Q_1}},\qquad \phi_{12}=e^{-\eta_1},\qquad \phi_{21}=-\frac{e^{\xi_2}}{e^{q_2}-e^{Q_1}},\\
&\gamma_1(t)=\frac{e^{a(t)}}{e^{q_1}-e^{Q_1}},\qquad \gamma_2=0,
\end{aligned}$$

in the Casoratian determinants (67)-(70). Here ξ_i, η_i ($i=1,2$) are given in (53) and $a(t)$ is an arbitrary function of t. Hence we obtain

$$f_n(x,n,t)=\frac{e^{2a(t)-\eta1}}{e^{q_1}-e^{Q_1}}\bigl(1+e^{\xi_1+\eta_1-2a(t)}\bigr),\tag{89}$$

$$f_n'(x,n,t)=-\frac{e^{2a(t)+\xi_2-\eta_1}}{e^{q_1}-e^{Q_1}}\left(1+\frac{e^{q_2}-e^{q_1}}{e^{q_2}-e^{Q_1}}e^{\xi_1+\eta_1-2a(t)}\right),\tag{90}$$

$$g_n^{(1)}(x,n,t)=\sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q1}}}e^{\xi_1-\eta_1+a(t)}, \tag{91}$$

$$h_n^{(1)}(x,n,t)=-\sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q1}}}\frac{e^{\xi_2+a(t)}}{e^{q_2}-e^{Q_1}}. \tag{92}$$

Substituting functions (89)-(92) into the dependent variable transformations (48), we get a one-soliton solution of the nonlinear modified two-dimensional Toda lattice with self-consistent sources (49)-(52) given in (59)-(62).

If we take $N=2$, $K=1$ and

$$\begin{aligned}
&\phi_{11}=\frac{e^{\xi_1}}{e^{q_1}-e^{Q_1}},\qquad \phi_{12}=e^{-\eta_1},\qquad \phi_{21}=-\frac{e^{\xi_2}}{e^{q_2}-e^{Q_1}},\\
&\phi_{22}=e^{\eta_2},\qquad \phi_{31}=\frac{e^{\xi_3}}{e^{q_3}-e^{Q_1}},\\
&\gamma_1(t)=\frac{e^{a(t)}}{e^{q_1}-e^{Q_1}},\qquad \gamma_2=-\frac{1}{e^{q_2}-e^{Q_1}},\qquad \gamma_3=0,
\end{aligned}$$

in the Casoratian determinants (67)-(70), we get

$$\begin{aligned}
f_n(x,n,t)={}&\frac{(e^{Q_1}-e^{Q_2})e^{2a(t)-\eta 1-\eta 2}}{(e^{q_2}-e^{Q_1})(e^{q_1}-e^{Q_1})}\biggl(1+\frac{e^{q_1}-e^{Q_2}}{e^{Q_1}-e^{Q_2}}e^{\xi_1+\eta_1-2a(t)}+\frac{e^{Q_1}-e^{q_2}}{e^{Q_1}-e^{Q_2}}e^{\xi_2+\eta_2}\\
&+\frac{e^{q_1}-e^{q_2}}{e^{Q_1}-e^{Q_2}}e^{\xi_1+\eta_1+\xi_2+\eta_2-2a(t)}\biggr),
\end{aligned} \tag{93}$$

$$\begin{aligned}
f_n'(x,n,t)={}&\frac{(e^{Q_1}-e^{Q_2})(e^{q_3}-e^{Q_2})e^{2a(t)+\xi_3-\eta_1-\eta_2}}{(e^{q_2}-e^{Q_1})(e^{q_1}-e^{Q_1})}\biggl(1+\frac{(e^{q_1}-e^{Q_2})(e^{q_1}-e^{q_3})}{(e^{Q_1}-e^{Q_2})(e^{Q_1}-e^{q_3})}e^{\xi_1+\eta_1-2a(t)}\\
&+\frac{(e^{q_2}-e^{Q_1})(e^{q_2}-e^{q_3})}{(e^{Q_1}-e^{Q_2})(e^{q_3}-e^{Q_2})}e^{\xi_2+\eta_2}\\
&+\frac{(e^{q_1}-e^{q_2})(e^{q_1}-e^{q_3})(e^{q_3}-e^{q_2})}{(e^{Q_1}-e^{Q_2})(e^{Q_1}-e^{q_3})(e^{q_3}-e^{Q_2}))}e^{\xi_1+\eta_1+\xi_2+\eta_2-2a(t)}\biggr),
\end{aligned} \tag{94}$$

$$g_n^{(1)}(x,n,t)=\sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q_1}}}e^{a(t)+\xi_1-\eta_1-\eta_2}\biggl(\bigl(e^{q_1}-e^{q_2}\bigr)e^{\xi_2+\eta_2}+\frac{(e^{Q_2}-e^{Q_1})(e^{q_1}-e^{Q_2})}{e^{q_2}-e^{Q_1}}\biggr), \tag{95}$$

$$\begin{aligned}
h_n^{(1)}(x,n,t)={}&\sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q_1}}}e^{a(t)-\eta_1-\eta_2}\biggl(\frac{e^{Q_2}-e^{q_3}}{(e^{q_3}-e^{Q_1})(e^{q_2}-e^{Q_1})}e^{\xi_3+\eta_1}\\
&+\frac{e^{q_2}-e^{q_3}}{(e^{q_3}-e^{Q_1})(e^{q_2}-e^{Q_1})}e^{\xi_2+\xi_3+\eta_1+\eta_2}\biggr).
\end{aligned} \tag{96}$$

We introduce five constants δ_1, δ_2, δ_3, ϵ_1, ϵ_2 satisfying

$$e^{\delta_1}=e^{Q_2}-e^{q_1},\qquad e^{\epsilon_1}=\frac{1}{e^{Q_2}-e^{Q_1}},\qquad e^{\delta_3}=e^{Q_2}-e^{q_3},\qquad e^{\delta_2+\epsilon_2}=\frac{e^{Q_1}-e^{q_2}}{e^{Q_1}-e^{Q_2}},$$

and take

$$\tilde{\xi}_1=\xi_1+\delta_1,\qquad \tilde{\xi}_2=\xi_2+\delta_2,\qquad \tilde{\xi}_3=\xi_3+\delta_3,\qquad \tilde{\eta}_1=\eta_1+\epsilon_1,\qquad \tilde{\eta}_2=\eta_2+\epsilon_2,$$

then equations (93)-(96) become

$$f_n(x,n,t) = \frac{(e^{Q_1}-e^{Q_2})e^{\epsilon_1+\epsilon_2}e^{-\tilde{\eta}_1-\tilde{\eta}_2+2a(t)}}{(e^{q_2}-e^{Q_1})(e^{q_1}-e^{Q_1})}\left(1+e^{\tilde{\xi}_1+\tilde{\eta}_1-2a(t)}+e^{\tilde{\xi}_2+\tilde{\eta}_2}\right.$$
$$\left.+\frac{(e^{q_1}-e^{q_2})(e^{Q_1}-e^{Q_2})}{(e^{q_1}-e^{Q_2})(e^{Q_1}-e^{q_2})}e^{\tilde{\xi}_1+\tilde{\eta}_1+\tilde{\xi}_2+\tilde{\eta}_2-2a(t)}\right), \tag{97}$$

$$f'_n(x,n,t) = -\frac{(e^{Q_1}-e^{Q_2})e^{\epsilon_1+\epsilon_2}e^{\tilde{\xi}_3-\tilde{\eta}_1-\tilde{\eta}_2+2a(t)}}{(e^{q_2}-e^{Q_1})(e^{q_1}-e^{Q_1})}\left(1+\frac{e^{q_3}-e^{q_1}}{e^{q_3}-e^{Q_1}}e^{\tilde{\xi}_1+\tilde{\eta}_1-2a(t)}+\frac{e^{q_3}-e^{q_2}}{e^{q_3}-e^{Q_2}}e^{\tilde{\xi}_2+\tilde{\eta}_2}\right.$$
$$\left.+\frac{(e^{q_1}-e^{q_2})(e^{Q_2}-e^{Q_1})(e^{q_3}-e^{q_2})(e^{q_3}-e^{q_1})}{(e^{q_1}-e^{Q_2})(e^{q_2}-e^{Q_1})(e^{q_3}-e^{Q_2})(e^{q_3}-e^{Q_1})}e^{\tilde{\xi}_1+\tilde{\eta}_1+\tilde{\xi}_2+\tilde{\eta}_2-2a(t)}\right), \tag{98}$$

$$g_n^{(1)}(x,n,t) = \sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q_1}}}\,\frac{e^{\epsilon_1+\epsilon_2}(e^{Q_1}-e^{Q_2})e^{\tilde{\xi}_1-\tilde{\eta}_1-\tilde{\eta}_2+a(t)}}{e^{q_2}-e^{Q_1}}\left(1+\frac{e^{q_1}-e^{q_2}}{e^{q_1}-e^{Q_2}}e^{\tilde{\xi}_2+\tilde{\eta}_2}\right), \tag{99}$$

$$h_n^{(1)}(x,n,t) = \sqrt{\frac{e^{2\dot{a}(t)}}{e^{q_1}-e^{Q_1}}}\,\frac{(e^{Q_2}-e^{Q_1})e^{\epsilon_1+\epsilon_2}e^{\tilde{\xi}_3-\tilde{\eta}_2+a(t)}}{(e^{q_2}-e^{Q_1})(e^{q_3}-e^{Q_1})}$$
$$\times\left(1+\frac{(e^{q_2}-e^{q_3})(e^{Q_1}-e^{Q_2})}{(e^{Q_2}-e^{q_3})(e^{Q_1}-e^{q_2})}e^{\tilde{\xi}_2+\tilde{\eta}_2}\right). \tag{100}$$

We rederive the two-soliton solution of the nonlinear modified two-dimensional Toda lattice with self-consistent sources (49)-(52) obtained in Section 2, substituting the above functions in equations (97)-(100) into the dependent variable transformation (48).

4 Commutativity of the source generation procedure and Bäcklund transformation

In this section, we show that the commutativity of the source generation procedure and Bäcklund transformation holds for the two-dimensional Toda lattice. For this purpose, we derive another form of the modified two-dimensional Toda lattice with self-consistent sources which is the Bäcklund transformation for the two-dimensional Toda lattice with self-consistent sources given in [25].

We have shown that the Casorati determinants $f_n, f'_n, g_n^{(j)}, h_n^{(j)}$ given in (67)-(70) satisfy the modified two-dimensional Toda lattice with self-consistent sources (8), (33), (36)-(37). Now we take

$$F_n = f_n = \det\left|\psi_i(n+j-1)\right|_{1\le i,j\le N} = (d_0,\ldots,d_{N-1},N,\ldots,1), \tag{101}$$

$$F'_n = f'_{n-1} = \det\left|\psi_i(n+j-1)\right|_{1\le i,j\le N+1}$$
$$= (d_{-1},\ldots,d_{N-1},N+1,\ldots,1), \tag{102}$$

$$G_n^{(j)} = \sqrt{2}g_{n-1}^{(j)} = \sqrt{2\dot{\gamma}_j(t)}(d_{-1},\ldots,d_{N-1},N,\ldots,1,\alpha_j),$$
$$j=1,\ldots,K, \tag{103}$$

$$H_n^{\prime(j)} = \sqrt{2}h_n^{(j)} = \sqrt{2\dot{\gamma}_j(t)}(d_0,\ldots,d_{N-1},N+1,\ldots,\hat{j},\ldots,1),$$
$$j=1,\ldots,K, \tag{104}$$

and we introduce two new fields

$$G_n'^{(j)} = \sqrt{2\dot{\gamma}_j(t)}(d_{-2},\ldots,d_{N-1},N+1,\ldots,1,\alpha_j), \quad j=1,\ldots,K, \tag{105}$$

$$H_n^{(j)} = \sqrt{2\dot{\gamma}_j(t)}(d_1,\ldots,d_{N-1},N,\ldots,\hat{j},\ldots,1), \quad j=1,\ldots,K, \tag{106}$$

where the Pfaffian elements are defined in (67)-(74).

In [25], the authors prove that the Casorati determinant F_n, $G_n^{(j)}$, $H_n^{(j)}$ solves the following two-dimensional Toda lattice with self-consistent sources [25]:

$$\big(D_xD_s - 2e^{D_n} + 2\big)F_n \cdot F_n = -\sum_{j=1}^{K} e^{D_n} G_n^{(j)} H_n^{(j)}, \tag{107}$$

$$\big(D_x + e^{-D_n}\big)F_n \cdot G_n^{(j)} = 0, \quad j=1,\ldots,K, \tag{108}$$

$$\big(D_x + e^{-D_n}\big)H_n^{(j)} \cdot F_n = 0, \quad j=1,\ldots,K. \tag{109}$$

It is not difficult to show that the Casorati determinant with F_n', $G_n'^{(j)}$, $H_n'^{(j)}$ is another solution to the two-dimensional Toda lattice with self-consistent sources (107)-(109).

Furthermore, we can verify that the Casorati determinants F_n, F_n', $G_n^{(j)}$, $G_n'^{(j)}$, $H_n^{(j)}$, $H_n'^{(j)}$ given in (101)-(106) satisfy the following bilinear equations:

$$2\big(D_se^{-1/2D_n} - e^{1/2D_n}\big)F_n \cdot F_n' = -\sum_{j=1}^{K} e^{1/2D_n} G_n^{(j)} \cdot H_n'^{(j)}, \tag{110}$$

$$\big(D_x + e^{-D_n}\big)F_n \cdot F_n' = 0, \quad j=1,\ldots,K, \tag{111}$$

$$\big(D_x + e^{-D_n}\big)H_n^{(j)} \cdot H_n'^{(j)} = 0, \quad j=1,\ldots,K, \tag{112}$$

$$\big(D_x + e^{-D_n}\big)G_n^{(j)} \cdot G_n'^{(j)} = 0, \quad j=1,\ldots,K, \tag{113}$$

$$\begin{aligned} e^{1/2D_n}F_n \cdot H_n'^{(j)} &= e^{-1/2D_n}F_n \cdot H_n'^{(j)} - e^{-1/2D_n}H_n^{(j)} \cdot F_n', \\ &\quad j=1,\ldots,K, \end{aligned} \tag{114}$$

$$\begin{aligned} e^{1/2D_n}G_n^{(j)} \cdot F_n' &= e^{-1/2D_n}G_n^{(j)} \cdot F_n' - e^{-1/2D_n}F_n \cdot G_n'^{(j)}, \\ &\quad j=1,\ldots,K, \end{aligned} \tag{115}$$

which is another form of the modified two-dimensional Toda lattice with self-consistent sources. It is proved in [30] that equations (110)-(115) constitute the Bäcklund transformation for the two-dimensional Toda lattice with self-consistent sources (107)-(109). Therefore, the commutativity of source generation procedure and Bäcklund transformation is valid for the two-dimensional Toda lattice.

5 Conclusion and discussion

In this paper, Grammian solutions to the modified two-dimensional Toda lattice are presented. From the Grammian solutions, the modified two-dimensional Toda lattice with self-consistent sources (8), (33), (36)-(37) are produced via the source generation procedure. We show that the modified two-dimensional Toda lattice with self-consistent sources (8), (33), (36)-(37) are resolved into the determinant identities by presenting its

Grammian and Casorati determinant solutions. We also construct another form of the modified discrete KP equation with self-consistent sources (110)-(115) which is the Bäcklund transformation for the two-dimensional Toda lattice with self-consistent sources derived in [25].

Now we show that the modified two-dimensional Toda lattice has a continuum limit into the mKP equation [2, 31], and the modified two-dimensional Toda lattice with self-consistent sources (8, 33, 36)-(37) yields the mKP equation with self-consistent sources derived in [32] through a continuum limit. For this purpose, we take

$$\begin{aligned} &D_n = 2\epsilon D_X - 2\epsilon^2 D_Y, \qquad D_x = \epsilon^2 D_Y + \frac{3}{2}\epsilon D_X, \qquad D_s = -\frac{16}{3}\epsilon^3 D_T, \\ &f(n,x,s) = F(X,Y,T), \qquad f'(n,x,s) = F'(X,Y,T), \end{aligned}$$

in the modified two-dimensional Toda lattice (8)-(9), and compare the ϵ^2 order in (8), and the ϵ^3 order in (9), then we obtain the mKP equation [2, 31]:

$$\begin{aligned} &\left(D_Y + D_X^2\right) F \cdot F' = 0, \\ &\left(D_X^3 - 4D_T - 3D_X D_Y\right) F \cdot F' = 0, \end{aligned}$$

where F, F' denote $F(X,Y,T)$, $F'(X,Y,T)$, respectively.

By taking

$$\begin{aligned} &D_n = 2\epsilon D_X - 2\epsilon^2 D_Y, \qquad D_x = \epsilon^2 D_Y + \frac{3}{2}\epsilon D_X, \qquad D_s = \frac{4}{3}\epsilon^3 D_T, \\ &f(n,x,s) = F(X,Y,T), \qquad g^{(j)}(n,x,s) = \frac{2\sqrt{3}}{3}\epsilon^{\frac{3}{2}} G_j(X,Y,T), \\ &f'(n,x,s) = F'(X,Y,T), \qquad h^{(j)}(n,x,s) = \frac{2\sqrt{3}}{3}\epsilon^{\frac{3}{2}} H_j(X,Y,T), \end{aligned}$$

for $j = 1, \ldots, K$ in the modified two-dimensional Toda lattice with self-consistent sources (8, 33, 36)-(37), and comparing the ϵ^2 order in (8), (36)-(37), and the ϵ^3 order in (33), we obtain the mKP equation with self-consistent sources [32]:

$$\begin{aligned} &\left(D_Y + D_X^2\right) F \cdot F' = 0, \\ &\left(D_T - 3D_X D_Y + D_X^3\right) F \cdot F' = -\sum_{j=1}^{K} G_j H_j, \\ &\left(D_Y + D_X^2\right) F \cdot G_j = 0, \quad j = 1, \ldots, K, \\ &\left(D_Y + D_X^2\right) H_j \cdot F' = 0, \quad j = 1, \ldots, K, \end{aligned}$$

where F, F', G_j, H_j denote $F(X,Y,T)$, $F'(X,Y,T)$, $G_j(X,Y,T)$, $H_j(X,Y,T)$ for $j = 1, \ldots, K$, respectively.

Recently, generalized Wronskian (Casorati) determinant solutions are constructed for continuous and discrete soliton equations [33–39]. Besides soliton solutions, a broader class of solutions such as rational solutions, negatons, positons and complexitons solutions are obtained from the generalized Wronskian (Casorati) determinant solutions [33–38]. In [39], a general Casoratian formulation is presented for the two-dimensional Toda lattice equation from which various examples of Casoratian type solutions are derived. It is interesting for us to construct the two-dimensional Toda lattice equation with self-consistent sources having a generalized Casorati determinant solution via the source generation procedure. This will bring us a broader class of solutions such as negatons, positons, and complexiton type solutions of the two-dimensional Toda lattice equation with self-consistent sources.

Acknowledgements
This work was supported by the Natural Science Foundation of Inner Mongolia Autonomous Region (Grant no. 2016MS0115), the National Natural Science Foundation of China (Grants no. 11601247 and 11605096).

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author has contributed solely to the writing of this paper. She read and approved the manuscript.

Publisher's Note

Received: 21 June 2017 Accepted: 31 August 2017 Published online: 11 September 2017

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