Yang and Ibrahim *Advances in Difference Equations* (2017) 2017:272
DOI 10.1186/s13662-017-1334-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Approximate controllability of fractional nonlocal evolution equations with multiple delays

He Yang* and Elyasa Ibrahim

*Correspondence:
yanghe256@163.com
College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, People's Republic of China

Abstract
This paper deals with the existence and approximate controllability for a class of fractional nonlocal control systems governed by abstract fractional evolution equations with multiple delays. Under some weaker assumptions, the existence as well as the approximate controllability is established by using fixed point theory. An example is given to illustrate the applicability of the abstract results.

MSC: 34A08; 34K35

Keywords: fractional evolution equations; nonlocal initial condition; multiple delays; compact analytic semigroup; approximate controllability

1 Introduction

Since the fractional differential equations have extensive physical background and realistic mathematical model, the theory has considerably developed in recent years; see [1–17] and the references therein. And the nonlocal initial conditions have better effects in applications than the classical ones; see [3, 8, 10, 16, 18–21] and the references therein. Therefore, the theory of fractional nonlocal differential equations has been a research field of focus in recent years.

Controllability of deterministic and stochastic dynamical control systems in infinite-dimensional spaces is well-developed in which the details can be found in various papers; see [3, 7–11, 13, 14, 22]. Several authors [3, 10, 14] investigated the exact controllability of control systems represented by nonlinear fractional evolution equations using fixed point approach. Debbouche and Baleanu [3] established the exact null controllability result for a class of fractional integro-differential control systems governed by nonlinear fractional evolution equations with nonlocal initial conditions in Banach spaces. Liang and Yang [10] investigated the exact controllability for a class of fractional integro-differential control systems represented by nonlinear fractional evolution equations involving specific nonlocal functions. Sakthivel *et al.* [14] studied the exact controllability for a class of fractional neutral control systems governed by abstract nonlinear fractional neutral evolution equations.

However, in infinite-dimensional spaces, the concept of exact controllability is usually too strong [22]. Therefore, it is necessary to present a weaker concept of controllability, namely approximate controllability for nonlinear control systems. In the recent literature, the approximate controllability of nonlinear fractional evolution systems has not yet been sufficiently studied. More precisely, there are limited papers regarding the approximate controllability of abstract nonlinear fractional evolution systems under different conditions [8, 9, 11, 13]. Kumar and Sukavanam [9] obtained a new set of sufficient conditions of approximate controllability for a class of semilinear fractional control systems involving delay. Mahmudov and Zorlu [11] established the sufficient conditions of approximate controllability for certain classes of abstract fractional evolution control systems. Sakthival *et al.* [13] investigated the approximate controllability for a class of nonlinear fractional dynamical systems governed by abstract fractional evolution equations with nonlocal conditions.

However, to the best of our knowledge, the approximate controllability for nonlinear fractional nonlocal control systems governed by abstract fractional evolution equations involving multiple delays and compact analysis semigroup has not been investigated yet and it is also the motivation of this paper. In this paper, we consider the existence and approximate controllability for fractional nonlocal control system

$$\begin{cases} D^q x(t) = Ax(t) + f(t, x(t), x(t-\tau_1), \dots, x(t-\tau_n)) + Bu(t), & t \in J := [0, T], \\ x(t) + g(x) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{1.1}$$

where D^q is Caputo fractional derivative of order $q \in (0,1)$, $T > 0$ is a constant, A generates a compact analytic semigroup $S(t)$ $(t \geq 0)$ of uniformly bounded linear operator, $u \in L^2(J, Y)$ is a control, Y is a Banach space, $B : Y \to X_\alpha$ is a linear bounded operator, $\tau_1, \tau_2, \dots, \tau_n$ are positive constants, $r = \max\{\tau_1, \tau_2, \dots, \tau_n\}$, $\varphi : [-r, 0] \to X_\alpha$ is continuous, f and g are given functions and will be specified later. The space X_α will be specified later.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given on fractional calculus and fractional power of generator of compact analytic semigroup. In Section 3, we study the existence of mild solutions for fractional nonlocal control system (1.1). The approximate controllability of fractional nonlocal control system (1.1) is discussed in Section 4. In Section 5, an example is given to illustrate the applicability of the abstract results.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. Throughout this paper, we always assume that $A : D(A) \subset X \to X$ generates a compact analytic semigroup $S(t)$ $(t \geq 0)$ of uniformly bounded linear operator in X. Then there exists a constant $M \geq 1$ such that $\|S(t)\| \leq M$ for all $t \geq 0$. Let $0 \in \rho(A)$. Then

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)\, dt$$

for some $0 < \alpha < 1$. Thus, we define A^α by $A^\alpha = (A^{-\alpha})^{-1}$ with $D(A^\alpha) = A^{-\alpha}(X)$. Let X_α be the Banach space of $D(A^\alpha)$ with norm $\|x\|_\alpha := \|A^\alpha x\|$ for any $x \in D(A^\alpha)$. Denote by $S_\alpha(t)$ the restriction of $S(t)$ to X_α for all $t \geq 0$. Then $S_\alpha(t)$ $(t \geq 0)$ is a C_0-semigroup in X_α and $\|S_\alpha(t)\|_\alpha \leq \|S(t)\|$ for all $t \geq 0$. To prove our main results, the following lemmas are needed.

Lemma 1 ([23]) *$A^{-\alpha}$ is a bounded linear operator in X for any $\alpha > 0$.*

Lemma 2 ([23]) *$A^\alpha S(t)$ is bounded in X for any $t > 0$ and there exists a constant $M_\alpha > 0$ such that $\|A^\alpha S(t)\| \le M_\alpha t^{-\alpha}$.*

Lemma 3 ([21]) *$S_\alpha(t)$ $(t \ge 0)$ is a compact semigroup in X_α, and hence it is norm-continuous.*

Denote by $C([-r,T],X_\alpha)$ the Banach space of all continuous X_α-valued functions on the interval $[-r,T]$ with norm $\|x\|_C = \max_{t\in[-r,T]}\|x(t)\|_\alpha$ for any $x \in C([-r,T],X_\alpha)$. Similarly, denote by $C([-r,0],X_\alpha)$ the Banach space of all continuous X_α-valued functions on the interval $[-r,0]$ with norm $\|x\|_{C[-r,0]} = \max_{t\in[-r,0]}\|x(t)\|_\alpha$ for any $x \in C([-r,0],X_\alpha)$. In this paper, we adopt the following definition of the mild solution of fractional nonlocal evolution equation (1.1).

Definition 1 By the mild solution of fractional nonlocal evolution equation (1.1), we mean a function $x \in C([-r,T],X_\alpha)$ satisfying initial condition $x(t) + g(x) = \varphi(t), t \in [-r,0]$ and integral equation

$$x(t) = U(t)\big(\varphi(0) - g(x)\big) + \int_0^t (t-s)^{q-1} V(t-s)\big[F(x)(s) + Bu(s)\big]\,ds, \quad t \in J,$$

where $F(x)(s) = f(s,x(s),x(s-\tau_1),\ldots,x(s-\tau_n))$, the operators $\{U(t)\}_{t\ge0}$ and $\{V(t)\}_{t\ge0}$ are defined by

$$U(t)x = \int_0^\infty \eta_q(\theta)S\big(t^q\theta\big)x\,d\theta, \qquad V(t)x = q\int_0^\infty \theta\eta_q(\theta)S\big(t^q\theta\big)x\,d\theta, \quad 0 < q < 1,$$

where

$$\eta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q\big(\theta^{-\frac{1}{q}}\big),$$
$$\rho_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty).$$

It is well known that $\eta_q(\theta) \ge 0$ for all $\theta \in (0,\infty)$ and

$$\int_0^\infty \eta_q(\theta)\,d\theta = 1, \qquad \int_0^\infty \theta\eta_q(\theta)\,d\theta = \frac{1}{\Gamma(1+q)}.$$

Lemma 4 ([10, 11]) *The operators $\{U(t)\}_{t\ge0}$ and $\{V(t)\}_{t\ge0}$ satisfy the following properties:*

(i) *For fixed $t \ge 0$ and any $x \in X_\alpha$, $\|U(t)x\|_\alpha \le M\|x\|_\alpha$, $\|V(t)x\|_\alpha \le \frac{M}{\Gamma(q)}\|x\|_\alpha$.*
(ii) *For fixed $t > 0$ and any $x \in X$, $\|V(t)x\|_\alpha \le C_\alpha t^{-q\alpha}\|x\|$, where $C_\alpha = \frac{M_\alpha q\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}$.*
(iii) *$U(t)$ and $V(t)$ are strongly continuous for all $t \ge 0$.*
(iv) *$U(t)$ and $V(t)$ are norm-continuous in X for $t > 0$.*
(v) *$U(t)$ and $V(t)$ are compact operators in X for $t > 0$.*
(vi) *For every $t > 0$, the restriction of $U(t)$ to X_α and the restriction of $V(t)$ to X_α are norm-continuous.*
(vii) *For every $t > 0$. the restriction of $U(t)$ to X_α and the restriction of $V(t)$ to X_α are compact operators.*

Let $x(T;u)$ be the state value of fractional nonlocal evolution equation (1.1) at terminal time T corresponding to control u. Introduce the set $\mathcal{R}(T)$ by $\mathcal{R}(T) := \{x(T;u) : u \in L^2(J, Y)\}$. $\overline{\mathcal{R}(T)}$ denotes its closure in X_α.

Definition 2 (Approximate controllability) The fractional nonlocal control system (1.1) is called approximately controllable on the interval $[-r, T]$ if $\overline{\mathcal{R}(T)} = X_\alpha$.

Consider the following linear fractional differential system:

$$\begin{cases} D^q x(t) = Ax(t) + Bu(t), & t \in [0, T], \\ x(0) = x_0 \in X_\alpha. \end{cases} \tag{2.1}$$

Let

$$\Gamma_0^T = \int_0^T (T-s)^{q-1} V(T-s)BB^* V^*(T-s)\,ds : X_\alpha \to X_\alpha,$$
$$R(\epsilon, \Gamma_0^T) = (\epsilon I + \Gamma_0^T)^{-1} : X_\alpha \to X_\alpha, \quad \epsilon > 0,$$

where B^* and $V^*(t)$ denote the adjoint of B and $V(t)$, respectively. It is well known that Γ_0^T is a linear bounded operator and $\|R(\epsilon, \Gamma_0^T)\| \leq \frac{1}{\epsilon}$.

Lemma 5 ([11]) *The linear fractional differential system* (2.1) *is approximately controllable on the interval* $[0, T]$ *if and only if* $\epsilon R(\epsilon, \Gamma_0^T) \to 0$ *as* $\epsilon \to 0^+$ *in the strong operator topology.*

Definition 3 Let E be a Banach space with norm $\|\cdot\|_E$. A mapping $Q : E \to E$ is called a nonlinear contraction if there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Qx - Qy\|_E \leq \phi(\|x - y\|_E)$$

for all $x, y \in E$ with $\phi(\tau) < \tau$ for $\tau > 0$.

Remark 1 It is clear if $\phi(\tau) \equiv k\tau$ for some $k \in (0, 1)$, the nonlinear contraction mapping degenerates into contraction mapping.

Lemma 6 ([24]) *Let E be a Banach space and let $Q_1, Q_2 : E \to E$ be two operators satisfying*

(a) *Q_1 is a nonlinear contraction, and*
(b) *Q_2 is completely continuous.*

Then either

(i) *the operator equation $x = Q_1x + Q_2x$ has a solution, or*
(ii) *the set $\Sigma := \{x \in E : \lambda(Q_1x + Q_2x) = x, 0 < \lambda < 1\}$ is unbounded.*

Lemma 7 ([15]) *Let $\sigma > 0$ and let $a_1(t)$ be a nonnegative nondecreasing function locally integrable on $0 \leq t < b$ (some $b \leq +\infty$) and $a_2(t)$ be a nonnegative nondecreasing continuous* *function defined on* $0 \leq t < b, a_2(t) \leq \widetilde{M}$ *(constant). Suppose that* $x(t)$ *is nonnegative and local integrable on* $0 \leq t < b$ *with*

$$x(t) \leq a_1(t) + a_2(t) \int_0^t (t-s)^{\sigma-1} x(s)\, ds.$$

Then

$$x(t) \leq a_1(t) \sum_{k=0}^{\infty} \frac{(a_2(t)\Gamma(\sigma)t^{\sigma})^k}{\Gamma(k\sigma+1)}, \quad 0 \leq t < b.$$

3 Existence of mild solutions

We make the following assumptions:

(H_1) The function $f : J \times X_\alpha^{n+1} \to X$ is continuous and there exist positive constants $\beta_0, \beta_1, \ldots, \beta_n$ and $K \geq 0$ such that

$$\|f(t, \nu_0, \nu_1, \ldots, \nu_n)\| \leq \sum_{i=0}^{n} \beta_i \|\nu_i\|_\alpha + K, \quad t \in J, (\nu_0, \nu_1, \ldots, \nu_n) \in X_\alpha^{n+1}.$$

(H_2) The function $g : C([-r, T], X_\alpha) \to X_\alpha$ is continuous and there exists a constant $L \geq M$ such that

$$\|g(x) - g(y)\|_\alpha \leq \frac{\|x-y\|_C}{L + \|x-y\|_C}, \quad x, y \in C\big([-r, T], X_\alpha\big).$$

Remark 2 The condition (H_1) can be replaced by the condition

$(H_1)'$ The function $f : J \times X_\alpha^{n+1} \to X$ is continuous and there exist functions $\overline{\beta}_i \in L^1(J, \mathbb{R}^+), i = 0, 1, \ldots, n$ and nondecreasing function $\Phi : [0, +\infty) \to (0, +\infty)$ such that

$$\big\|f\big(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)\big)\big\| \leq \sum_{i=0}^{n} \overline{\beta}_i(t) \Phi\big(\big\|x(t-\tau_i)\big\|_\alpha\big),$$

for any $t \in J, x \in C([-r, T], X_\alpha)$, where $\tau_0 = 0$.

By $(H_1)'$, we have

$$\big\|f\big(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n)\big)\big\| \leq \sum_{i=0}^{n} \|\overline{\beta}_i\|_{L^1} \Phi\big(\big\|x(t-\tau_i)\big\|_\alpha\big).$$

Let $\Phi(r) = r$ for some $r > 0$. Then $(H_1)' \Rightarrow (H_1)$ with $\beta_i = \|\overline{\beta}_i\|_{L^1}$ and $K = 0$. Since $(H_1)'$ would not be an essential generalization, we only consider (H_1) in the following.

Define an operator $Q_1 : C([-r, T], X_\alpha) \to C([-r, T], X_\alpha)$ by

$$(Q_1 x)(t) = \begin{cases} \varphi(t) - g(x), & t \in [-r, 0], \\ U(t)(\varphi(0) - g(x)), & t \in J. \end{cases} \tag{3.1}$$

Lemma 8 *If the assumption* (H_2) *holds,* Q_1 *is a nonlinear contraction.*

Proof Let $x, y \in C([-r,T], X_\alpha)$. For $t \in [-r,0]$, we have

$$\|(Q_1x)(t) - (Q_1y)(t)\|_\alpha = \|g(x) - g(y)\|_\alpha \le \frac{\|x-y\|_C}{L + \|x-y\|_C} \le \frac{M\|x-y\|_C}{L + \|x-y\|_C}.$$

For $t \in [0,T]$, we have

$$\begin{aligned}\|(Q_1x)(t) - (Q_1y)(t)\|_\alpha &= \|U(t)(\varphi(0) - g(x)) - U(t)(\varphi(0) - g(y))\|_\alpha \\ &\le M\|g(x) - g(y)\|_\alpha \le \frac{M\|x-y\|_C}{L + \|x-y\|_C}.\end{aligned}$$

This implies that

$$\|Q_1x - Q_1y\|_C \le \frac{M\|x-y\|_C}{L + \|x-y\|_C}.$$

Let $\phi(r) = \frac{Mr}{L+r}$. Then $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and $\phi(r) < r$ for $r > 0$. Hence Q_1 is a nonlinear contraction in $C([-r,T], X_\alpha)$. This completes the proof. □

For any $\epsilon > 0$ and $h_1 \in X_\alpha$, define a control $u(t) := u(t;x)$ by

$$\begin{aligned}u(t;x) &= B^*V^*(T-t)R(\epsilon, \Gamma_0^T)\Big[h_1 - U(T)(\varphi(0) - g(x)) \\ &\quad - \int_0^T (T-s)^{q-1}V(T-s)F(x)(s)\,ds\Big].\end{aligned}$$

Then, from assumptions (H_1) and (H_2), we have

$$\|Bu(t;x)\|_\alpha \le L_u, \tag{3.2}$$

where

$$\begin{aligned}L_u &= \frac{1}{\epsilon}L_B\|A^{-\alpha}\|\big[\|h_1\|_\alpha + M\|\varphi\|_{C[-r,0]} + M(1 + \|g(0)\|_\alpha)\big] \\ &\quad + \frac{L_BMKT^q}{\epsilon\Gamma(q+1)} + \frac{L_BM}{\epsilon\Gamma(q)}\sum_{i=0}^n \int_0^T (T-s)^{q-1}\beta_i\|x(s-\tau_i)\|_\alpha\,ds, \\ L_B &= \|B\|_\alpha \sup_{t\in J}\|B^*V^*(T-t)\|.\end{aligned}$$

Define an operator $Q_2 : C([-r,T], X_\alpha) \to C([-r,T], X_\alpha)$ by

$$(Q_2x)(t) = \begin{cases} 0, & t \in [-r,0], \\ \int_0^t (t-s)^{q-1}V(t-s)[F(x)(s) + Bu(s;x)]\,ds, & t \in J. \end{cases} \tag{3.3}$$

Lemma 9 *If the assumptions* (H_1) *and* (H_2) *hold,* Q_2 *is completely continuous.*

Proof By the assumptions (H_1) and (H_2), it is easy to prove that $Q_2 : C([-r,T], X_\alpha) \to C([-r,T], X_\alpha)$ is continuous. So, it remains to prove that Q_2 is a compact operator on

$C([-r,T],X_\alpha)$. The case $t\leq 0$ is trivial. Thus, let $t\in(0,T]$ be fixed. For each $\delta\in(0,t)$, $\rho>0$ and $x\in B_r:=\{x\in C([-r,T],X_\alpha):\|x\|_C\leq r\}$, $r>0$, we define $Q_2^{\delta,\rho}$ by

$$\begin{aligned}
\bigl(Q_2^{\delta,\rho}x\bigr)(t) &= \int_0^{t-\delta}(t-s)^{q-1}\int_\rho^\infty q\theta\eta_q(\theta)S\bigl((t-s)^q\theta\bigr)\bigl[F(x)(s)+Bu(s;x)\bigr]\,d\theta\,ds \\
&= S\bigl(\delta^q\rho\bigr)\int_0^{t-\delta}(t-s)^{q-1}\int_\rho^\infty q\theta\eta_q(\theta)S\bigl((t-s)^q\theta-\delta^q\rho\bigr) \\
&\quad\times\bigl[F(x)(s)+Bu(s;x)\bigr]\,d\theta\,ds.
\end{aligned}$$

Then the set $\{(Q_2^{\delta,\rho}x)(t):x\in B_r\}$ is relatively compact in X_α because of Lemma 3. By (H_1), (H_2) and (3.2), we have

$$\begin{aligned}
&\bigl\|(Q_2x)(t)-\bigl(Q_2^{\delta,\rho}x\bigr)(t)\bigr\|_\alpha \\
&\quad\leq\biggl\|\int_0^t(t-s)^{q-1}\int_0^\rho q\theta\eta_q(\theta)S\bigl((t-s)^q\theta\bigr)\bigl(F(x)(s)+Bu(s;x)\bigr)\,d\theta\,ds\biggr\|_\alpha \\
&\qquad+\biggl\|\int_{t-\delta}^t(t-s)^{q-1}\int_\rho^\infty q\theta\eta_q(\theta)S\bigl((t-s)^q\theta\bigr)\bigl(F(x)(s)+Bu(s;x)\bigr)\,d\theta\,ds\biggr\|_\alpha \\
&\quad\leq\frac{M_\alpha T^{q(1-\alpha)}}{1-\alpha}\Biggl(\sum_{i=0}^n\beta_ir+K\Biggr)\int_0^\rho\theta^{1-\alpha}\eta_q(\theta)\,d\theta \\
&\qquad+MT^qL_u\int_0^\rho\theta\eta_q(\theta)\,d\theta \\
&\qquad+C_\alpha\Biggl(\sum_{i=0}^n\beta_ir+K\Biggr)\int_{t-\delta}^t(t-s)^{q(1-\alpha)-1}\,ds \\
&\qquad+\frac{ML_u\delta^q}{\Gamma(q+1)}.
\end{aligned}$$

This implies that the set $\{(Q_2x)(t):x\in B_r\}$ is relatively compact in X_α for all $t\in(0,T]$. Hence, we obtain the relative compactness of $(Q_2B_r)(t)$ in X_α for all $t\in[-r,T]$. We further show that the operator Q_2 is equicontinuous in $C([-r,T],X_\alpha)$. For $x\in C([-r,T],X_\alpha)$, if $\nu\in[0,T)$, we have

$$\begin{aligned}
&\bigl\|(Q_2x)(\nu)-(Q_2x)(0)\bigr\|_\alpha \\
&\quad\leq C_\alpha\int_0^\nu(\nu-s)^{q(1-\alpha)-1}\sum_{i=0}^n\beta_i\bigl\|x(s-\tau_i)\bigr\|_\alpha\,ds \\
&\qquad+\frac{ML_u\nu^q}{\Gamma(q+1)}+\frac{C_\alpha K\nu^{q(1-\alpha)}}{q(1-\alpha)} \\
&\quad\to 0
\end{aligned}$$

as $\nu\to 0$. Hence, it is only necessary to consider the case $t>0$. For $0<t_1<t_2\leq T$, denote

$$\begin{aligned}
I_1 &= \biggl\|\int_0^{t_1}(t_1-s)^{q-1}\bigl[V(t_2-s)-V(t_1-s)\bigr]\bigl[F(x)(s)+Bu(s;x)\bigr]\,ds\biggr\|_\alpha, \\
I_2 &= \biggl\|\int_0^{t_1}\bigl[(t_2-s)^{q-1}-(t_1-s)^{q-1}\bigr]V(t_2-s)\bigl[F(x)(s)+Bu(s;x)\bigr]\,ds\biggr\|_\alpha,
\end{aligned}$$

$$I_3 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} V(t_2 - s)\big[F(x)(s) + Bu(s;x)\big]\,ds \right\|_\alpha .$$

By assumptions (H_1), (H_2) and (3.2), we can find

$$\begin{aligned} I_2 &\le C_\alpha \int_0^{t_1} \big|(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\big|(t_2 - s)^{-q\alpha} \big\|F(x)(s)\big\|\,ds \\ &\quad + \frac{ML_u[t_2^q - t_1^q - (t_2 - t_1)^q]}{\Gamma(q+1)}, \\ I_3 &\le C_\alpha \int_{t_1}^{t_2} (t_2 - s)^{q(1-\alpha)-1} \big\|F(x)(s)\big\|\,ds \\ &\quad + \frac{ML_u(t_2 - t_1)^q}{\Gamma(q+1)}. \end{aligned}$$

This implies that $I_i \to 0, i = 2, 3$ as $t_2 - t_1 \to 0$. For $t_1 > 0$ and $\eta \in (0, t_1)$ small enough, we have

$$\begin{aligned} I_1 &\le \int_0^{t_1-\eta} (t_1 - s)^{q-1} \big\|\big[V(t_2 - s) - V(t_1 - s)\big]\big[F(x)(s) + Bu(s;x)\big]\big\|_\alpha\,ds \\ &\quad + \int_{t_1-\eta}^{t_1} (t_1 - s)^{q-1} \big\|\big[V(t_2 - s) - V(t_1 - s)\big]\big[F(x)(s) + Bu(s;x)\big]\big\|_\alpha\,ds \\ &\le \int_0^{t_1-\eta} (t_1 - s)^{q-1} \big\|F(x)(s)\big\|\,ds \sup_{s\in[0,t_1-\eta]} \big\|V(t_2 - s) - V(t_1 - s)\big\|_\alpha \\ &\quad + \frac{L_u(t_1^q - \eta^q)}{q} \sup_{s\in[0,t_1-\eta]} \big\|V(t_2 - s) - V(t_1 - s)\big\|_\alpha \\ &\quad + C_\alpha \int_{t_1-\eta}^{t_1} (t_1 - s)^{q-1}(t_2 - s)^{-q\alpha} \big\|F(x)(s)\big\|\,ds \\ &\quad + C_\alpha \int_{t_1-\eta}^{t_1} (t_1 - s)^{q(1-\alpha)-1} \big\|F(x)(s)\big\|\,ds \\ &\quad + \frac{2ML_u\eta^q}{\Gamma(q+1)}. \end{aligned}$$

Since compact semigroup is equicontinuous semigroup, it follows that $I_1 \to 0$ as $t_2 - t_1 \to 0$ and $\eta \to 0$. Therefore, from the inequality

$$\big\|(Q_2x)(t_2) - (Q_2x)(t_1)\big\|_\alpha \le I_1 + I_2 + I_3,$$

we see that the operator Q_2 is equicontinuous in $C([-r, T], X_\alpha)$. Hence, by the Ascoli-Arzela theorem, Q_2 is a compact operator in $C([-r, T], X_\alpha)$. This completes the proof. □

Theorem 1 *Assume that the conditions* (H_1) *and* (H_2) *hold. Then the fractional nonlocal control system* (1.1) *has at least one mild solution.*

Proof Define two operators $Q_1, Q_2 : C([-r, T], X_\alpha) \to C([-r, T], X_\alpha)$ as in (3.1) and (3.3). By Lemma 8 and 9, it follows that all the conditions of Lemma 6 are satisfied and a direct application of Lemma 6 shows that either the conclusion (i) or the conclusion (ii) holds.

We next show that the conclusion (ii) is not possible. Equivalently, we prove that the set $\Sigma := \{x \in C([-r,T],X_\alpha) : \lambda(Q_1x + Q_2x) = x, 0 < \lambda < 1\}$ is bounded.

Let $x \in C([-r,T],X_\alpha)$ satisfy the operator equation $x = \lambda(Q_1x + Q_2x)$ for some $\lambda \in (0,1)$. Then, for any $t \in [-r,0]$, by assumption (H_2), we have

$$\|x(t)\|_\alpha \le \|\varphi\|_{C[-r,0]} + \|g(x)\|_\alpha \le \|\varphi\|_{C[-r,0]} + 1 + \|g(0)\|_\alpha \triangleq \overline{M}_1.$$

For $t \ge 0$, by assumptions (H_1) and (H_2), we have

$$\begin{aligned}\|x(t)\|_\alpha &\le \|U(t)(\varphi(0) - g(x))\|_\alpha + \left\|\int_0^t (t-s)^{q-1}V(t-s)F(x)(s)\,ds\right\|_\alpha \\ &\quad + \left\|\int_0^t (t-s)^{q-1}V(t-s)Bu(s;x)\,ds\right\|_\alpha \\ &\le C + C_\alpha \sum_{i=0}^n \int_0^t (t-s)^{q(1-\alpha)-1}\beta_i \|x(s-\tau_i)\|_\alpha\,ds,\end{aligned}$$

where $\tau_0 = 0$ and $C = M\|\varphi\|_{C[-r,0]} + M(1 + \|g(0)\|_\alpha) + \frac{KC_\alpha T^{q(1-\alpha)}}{q(1-\alpha)} + \frac{ML_u T^q}{\Gamma(q+1)}$.

Let

$$\psi(t) = \max_{s\in[-r,t]} \|x(s)\|_\alpha, \quad t \in [-r,T].$$

Then $\psi \in C([-r,T],\mathbb{R}^+)$ and $\|x(t)\|_\alpha \le \psi(t)$ for $t \in [-r,T]$. For every $t \ge 0$, by the definition of ψ, there exists $\theta_t \in [-r,t]$ such that $\psi(t) = \|x(\theta_t)\|_\alpha$.

If $-r \le \theta_t \le 0$, we have

$$\begin{aligned}\psi(t) &= \|x(\theta_t)\|_\alpha \le \|\varphi(\theta_t)\|_\alpha + 1 + \|g(0)\|_\alpha \\ &\le M\|\varphi\|_{C[-r,0]} + M(1 + \|g(0)\|_\alpha) \\ &\le C + C_\alpha \sum_{i=0}^n \beta_i \int_0^t (t-s)^{q(1-\alpha)-1}\psi(s)\,ds.\end{aligned}$$

If $\theta_t > 0$, we have

$$\begin{aligned}\psi(t) &= \|x(\theta_t)\|_\alpha \le C + C_\alpha \int_0^{\theta_t} (\theta_t - s)^{q(1-\alpha)-1} \sum_{i=0}^n \beta_i \|x(s-\tau_i)\|_\alpha\,ds \\ &\le C + C_\alpha \sum_{i=0}^n \beta_i \int_0^{\theta_t} (\theta_t - s)^{q(1-\alpha)-1}\psi(s)\,ds \\ &\le C + C_\alpha \sum_{i=0}^n \beta_i \int_0^t (t-s)^{q(1-\alpha)-1}\psi(s)\,ds.\end{aligned}$$

Using the well-known singular version of Gronwall inequality [12], we can deduce that there exists a constant $\overline{M}_2 > 0$ such that $\psi(t) \le \overline{M}_2$. Thus, for any $t \ge 0$, we have

$$\|x(t)\|_\alpha \le \psi(t) \le \overline{M}_2.$$

Consequently,

$$\|x\|_C = \max_{t\in[-r,T]} \|x(t)\|_\alpha \le \overline{M}_1 + \overline{M}_2 \triangleq \overline{M}.$$

This implies that the set Σ is bounded. Therefore, by Lemma 6, the operator equation $x = Q_1x + Q_2x$ has at least one fixed point which is the mild solution of the fractional control system (1.1) on $C([-r,T],X_\alpha)$. This completes the proof. □

Remark 3 Even if $g(x) \equiv 0$ and without control u in the fractional nonlocal control system (1.1), Theorem 1 is still new.

The condition (H_2) can be replaced by the following condition:

$(H_2)'$ The function $g : C([-r,T],X_\alpha) \to X_\alpha$ is Lipschitz continuous with constant $L_1 \in (0,\frac{1}{M})$, that is, for any $x,y \in C([-r,T],X_\alpha)$, we have

$$\|g(x) - g(y)\|_\alpha \le L_1\|x - y\|_C.$$

Theorem 2 *Let the conditions* (H_1) *and* $(H_2)'$ *hold. Then the fractional nonlocal control system* (1.1) *has at least one mild solution.*

Proof By the condition $(H_2)'$, similar to the proof as in Lemma 8, we obtain

$$\|Q_1x - Q_1y\|_C \le ML_1\|x - y\|_C, \quad \forall x,y \in C\bigl([-r,T],X_\alpha\bigr).$$

Let $\phi(r) = ML_1r$. Then $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and $\phi(r) < r$ for $r > 0$. Hence Q_1 is a nonlinear contraction on $C([-r,T],X_\alpha)$. The remaining proof is similar to the proof of Theorem 1, we omit it here. This completes the proof. □

Remark 4 In some existing literature, see [2, 14, 16, 17], the authors always assume that $f(t,x) \le m(t)$ with some functions $m \in L^1(J,\mathbb{R}^+)$ independent of x, and g is either completely continuous or Lipschitz continuous and the coefficients satisfy some inequality conditions. But in Theorems 1 and 2, we only assume that the conditions (H_1) and (H_2) (or $(H_2)'$) hold. Hence, Theorems 1 and 2 greatly extend the main results of [2, 13, 14, 16, 17].

4 Approximate controllability

To prove the approximate controllability of (1.1), the following assumption is required:

(H_5) The function $f : J \times X_\alpha^{n+1} \to X$ is bounded.

(H_6) The linear fractional control system (2.1) is approximately controllable.

Theorem 3 *In addition to the assumptions of Theorem* 1, *suppose that the conditions* (H_5) *and* (H_6) *hold. Then the fractional nonlocal control system* (1.1) *is approximately controllable.*

Proof Let x_ϵ be a fixed point of the operator $Q_1 + Q_2$ on $C([-r,T],X_\alpha)$. By Theorem 1, x_ϵ is a mild solution of fractional nonlocal control system (1.1) with control

$$u(t;x_\epsilon) = B^*V^*(T-t)R\bigl(\epsilon,\Gamma_0^T\bigr)P(x_\epsilon), \quad t \in J,$$

where

$$P(x_\epsilon) = h_1 - U(T)\big(\varphi(0) - g(x_\epsilon)\big) - \int_0^T (T-s)^{q-1} V(T-s) F(x_\epsilon)(s)\,ds,$$
$$F(x_\epsilon)(t) = f\big(t, x_\epsilon(t), x_\epsilon(t-\tau_1), \ldots, x_\epsilon(t-\tau_n)\big), \quad t \in J.$$

A direct calculation shows that x_ϵ satisfies

$$x_\epsilon(T) = h_1 - \epsilon R\big(\epsilon, \Gamma_0^T\big) P(x_\epsilon). \tag{4.1}$$

Moreover, by the assumption (H_5), there exists a constant $N > 0$ such that

$$\int_0^T \big\|F(x_\epsilon)(s)\big\|^2\,ds = \int_0^T \big\|f\big(s, x_\epsilon(s), x_\epsilon(s-\tau_1), \ldots, x_\epsilon(s-\tau_n)\big)\big\|^2\,ds \le TN^2.$$

Consequently, there is a sequence still denoted by $\{F(x_\epsilon)(s)\}$ weakly converging to, say, $\{F(s)\}$ in $L^2(J,X)$. Denote

$$\rho = h_1 - U(T)\big(\varphi(0) - g(x_\epsilon)\big) - \int_0^T (T-s)^{q-1} V(T-s) F(s)\,ds.$$

Now, we have

$$\begin{aligned}\big\|P(x_\epsilon) - \rho\big\|_\alpha &\le \int_0^T (T-s)^{q-1} \big\|V(T-s)\big(F(x_\epsilon)(s) - F(s)\big)\big\|_\alpha\,ds \\ &\le C_\alpha \int_0^T (T-s)^{q(1-\alpha)-1} \big\|F(x_\epsilon)(s) - F(s)\big\|\,ds.\end{aligned}$$

By using infinite-dimensional version of the Ascoli-Arzela theorem, we see that the operator $\ell(\cdot) \to \int_0^{\cdot} (\cdot - s)^{q(1-\alpha)-1} \ell(s)\,ds : L^2(J,X) \to C(J,X)$ is compact. Hence, $\|P(x_\epsilon) - \rho\|_\alpha \to 0$ as $\epsilon \to 0^+$. Moreover, by (4.1), we have

$$\begin{aligned}\big\|x_\epsilon(T) - h_1\big\|_\alpha &\le \big\|\epsilon R\big(\epsilon, \Gamma_0^T\big)(\rho)\big\|_\alpha + \big\|\epsilon R\big(\epsilon, \Gamma_0^T\big)\big\| \big\|P(x_\epsilon) - \rho\big\|_\alpha \\ &\le \big\|\epsilon R\big(\epsilon, \Gamma_0^T\big)(\rho)\big\|_\alpha + \big\|P(x_\epsilon) - \rho\big\|_\alpha.\end{aligned}$$

Hence, by Lemma 5 and (H_5), we have

$$\big\|x_\epsilon(T) - h_1\big\|_\alpha \to 0$$

as $\epsilon \to 0^+$. This proves the approximate controllability of fractional nonlocal control system (1.1). This completes the proof. □

Theorem 4 *In addition to the assumptions of Theorem* 2, *suppose that the conditions* (H_5) *and* (H_6) *hold. Then the fractional nonlocal control system* (1.1) *is approximately controllable.*

Proof The proof is similar to the proof of Theorem 3, and we omit it here. □

Remark 5 Compared with [11], the present paper studies the fractional control system (1.1) with nonlocal condition and delays, and we suppose that the function f maps $J \times X_\alpha^{n+1}$ to X. Hence, our results extend some existing results.

5 Existence and approximate controllability of delay parabolic equations

Let $\Omega \in \mathbb{R}^N$ be a bounded domain, whose boundary $\partial\Omega$ is sufficiently smooth. Let

$$A(z,D)x = \sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\left(\phi_{ij}(z)\frac{\partial x}{\partial z_j}\right) - \phi_0(z)x$$

be a uniformly elliptic differential operator of divergence form in $\overline{\Omega}$. We assume that following conditions are satisfied:

(A1) $\phi_{ij} \in C^{1+\mu}(\overline{\Omega})$ $(i,j = 1,2,\ldots,N)$ for some $\mu \in (0,1)$, $[\phi_{ij}(z)]_{N\times N}$ is a positive definite symmetric matrix for $z \in \overline{\Omega}$ and there exists a constant $\ell > 0$ such that

$$\sum_{i,j=1}^{N} \phi_{ij}(z)\eta_i\eta_j \geq \ell|\eta|^2, \quad \forall \eta = (\eta_1,\eta_2,\ldots,\eta_N) \in \mathbb{R}^N, z \in \overline{\Omega}.$$

(A2) $\phi_0 \in C^\mu(\overline{\Omega})$ for some $\mu \in (0,1)$, and $\phi_0(z) \geq 0$ on $\overline{\Omega}$.

Let $X = L^2(\Omega)$. Define an operator $A : D(A) \subset X \to X$ by

$$Ax = A(z,D)x, \qquad D(A) = H^2(\Omega) \cap H_0^1(\Omega). \tag{5.1}$$

Then, from [25], A generates a compact analysis semigroup $S(t)$ $(t \geq 0)$ in X and there exists a constant $M \geq 1$ such that $\|S(t)\| \leq M$. It is well-known that $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$.

Under above assumptions we discuss the approximate controllability of delay parabolic boundary value problem

$$\begin{cases} \frac{\partial^q}{\partial t^q}x(z,t) = A(z,D)x(z,t) + F(z,t,x(z,t),x(z,t-\tau_1),\ldots,x(z,t-\tau_n)) \\ \qquad\qquad + \omega u(z,t), \quad z \in \Omega, t \in [0,T], \\ x|_{\partial\Omega} = 0, \\ x(z,t) = \varphi(z,t) - g(x), \quad z \in \Omega, t \in [-r,0], \end{cases} \tag{5.2}$$

where $\frac{\partial^q}{\partial t^q}$ is the Caputo fractional partial derivative of order $q \in (0,1)$, $\omega > 0$ is a constant, $\tau_1,\ldots,\tau_n$ are positive constants, $r = \max\{\tau_1,\ldots,\tau_n\}, \varphi : \Omega \times [-r,0] \to \mathbb{R}$.

By Theorem 1, we have the following existence result.

Theorem 5 *Assume that the following conditions are satisfied*:

(P_1) *The function* $F : \overline{\Omega} \times [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$ *is continuous and there exist positive constants* $\beta_0,\beta_1,\ldots,\beta_n$ *and* K *such that*

$$\left|F(z,t,\xi_0,\xi_1,\ldots,\xi_n)\right| \leq \sum_{i=0}^{n} \beta_i|\xi_i| + K, \quad z \in \Omega, t \in [0,T].$$

(P_2) *The function* g *is continuous and there exists a constant* $L \geq M$ *such that*

$$\left|g(x) - g(y)\right| \leq \frac{|x-y|}{L + |x-y|},$$

for all $x,y \in \mathbb{R}$.

Then the delay parabolic boundary value problem (5.2) *has at least one mild solution*.

By Theorem 3, we have the following approximate controllability result.

Theorem 6 *In addition to the assumptions of Theorem* 5, *suppose that the function F is bounded and the following condition holds*:

(P_3) *The linear fractional control system corresponding to* (5.2) *is approximately controllable*.

Then the delay parabolic boundary value problem (5.2) *is approximately controllable*.

By Theorem 4, we obtain the following theorem.

Theorem 7 *Assume that the conditions* (P_1), (P_3) *and*

(P_4) *The function $g : C([-r,T],X_\alpha) \to X_\alpha$ is Lipschitz continuous with constant $L_1 \in (0, \frac{1}{M})$*.

Then the delay parabolic boundary value problem (5.2) *is approximately controllable*.

Example 1 Consider the approximate controllability of delay parabolic boundary value problem

$$\begin{cases} \frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} x(z,t) = A(z,D)x(z,t) + \sin^3 \frac{t}{2} + x(z,t) - \frac{1}{3}x(z,t-\tau_1) - \frac{2}{3}x(z,t-\tau_2) \\ \qquad\qquad + 4u(z,t), \quad z \in \Omega, t \in [0,T], \\ x|_{\partial\Omega} = 0, \\ x(z,t) = \varphi(z,t) - \sum_{k=1}^{m} (-1)^k \gamma_k x(z,k), \quad z \in \Omega, t \in [-r,0], \end{cases} \tag{5.3}$$

where $m \le T$ and $\gamma_k \in \mathbb{R}, k = 1,2,\ldots,m$ with $\sum_{k=1}^{m} |\gamma_k| \le \frac{1}{M}$.

Let

$$F\bigl(z,t,x(z,t),x(z,t-\tau_1),z(z,t-\tau_2)\bigr) = \sin^3 \frac{t}{2} + x(z,t) - \frac{1}{3}x(z,t-\tau_1) - \frac{2}{3}x(z,t-\tau_2),$$

$$g(x) = \sum_{k=1}^{m} (-1)^k \gamma_k x(z,k).$$

Then

$$\bigl|F\bigl(z,t,x(z,t),x(z,t-\tau_1),z(z,t-\tau_2)\bigr)\bigr| \le 1 + \sum_{k=0}^{2} \frac{k}{3} \bigl|x(z,t-\tau_k)\bigr|,$$

$$\bigl|g(x) - g(y)\bigr| \le \sum_{k=1}^{m} |\gamma_k| \bigl|x(z,k) - y(z,k)\bigr|,$$

where $\tau_0 = 0$.

Hence, by Theorem 7, if the linear fractional control system corresponding to (5.3) is approximately controllable, then the delay parabolic boundary value problem (5.3) is approximately controllable.

6 Conclusion

In this paper, the approximate controllability of a nonlocal control system governed by fractional multi-delay evolution equation is investigated. By using fixed point theory and also the Gronwall-Bellman inequality of fractional order, sufficient conditions of approximate controllability are obtained. As an example, the approximate controllability of fractional multi-delay parabolic boundary value problem is discussed and the abstract result is applied to a special fractional multi-delay parabolic function.

Acknowledgements
The research is supported by the NSF (No. 11661071).

Competing interests
None of the authors have any competing interests in the manuscript.

Authors' contributions
All authors contributed equally in writing this paper. All authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 April 2017 Accepted: 26 August 2017 Published online: 07 September 2017

References

1. Ahmad, B, Nieto, J: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **58**, 1838-1843 (2009)
2. Agarwal, R, Zhou, Y, He, Y: Existence of fractional neutral functional differential equations. Comput. Math. Appl. **59**, 1095-1100 (2010)
3. Debbouche, A, Baleanu, D: Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. Comput. Math. Appl. **62**, 1442-1450 (2011)
4. El-Borai, M: Exact solutions for some nonlinear fractional parabolic partial differential equations. Appl. Math. Comput. **206**, 150-153 (2008)
5. Glockle, W, Nonnenmacher, T: A fractional calculus approach of self-similar protein dynamics. J. Biophys. **68**, 46-53 (1995)
6. Hilfer, R: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
7. Kailasavalli, S, Baleanu, D, Suganya, S, Arjunan, M: Exact controllability of fractional neutral integro-differential systems with state-dependent delay in Banach spaces. An. Ştiinţ. Univ. 'Ovidius' Constanţa **24**, 29-55 (2016)
8. Kerboua, M, Debbouche, A, Baleanu, D: Approximate controllability of Sobolev type fractional stochastic nonlocal nonlinear differential equations in Hilbert spaces. Electron. J. Qual. Theory Differ. Equ. **2014**, 58 (2014)
9. Kumar, S, Sukavanam, N: Approximate controllability of fractional onder semilinear systems with bounded deday. J. Differ. Equ. **251**, 6163-6174 (2012)
10. Liang, J, Yang, H: Controllability of fractional integro-differential evolution equations with nonlocal conditions. Appl. Math. Comput. **254**, 20-29 (2015)
11. Mahmudov, N, Zorlu, S: On the approximate controllability of fractional evolution equations with compact analytic semigroup. J. Comput. Appl. Math. **259**, 194-204 (2014)
12. Ravichandran, C, Baleanu, D: On the controllability of fractional functional integro-differential systems with an infinite delay in Banach spaces. Adv. Differ. Equ. **2013**, 291 (2013)
13. Sakthivel, R, Ganesh, R, Ren, Y, Anthoni, S: Approximate controllability of nonlinear fractional dynamical systems. Commun. Nonlinear Sci. Numer. Simul. **18**, 3498-3508 (2013)
14. Sakthivel, R, Mahmudov, N, Nieto, J: Controllability for a class of fractional-order neutral evolution control systems. Appl. Math. Comput. **218**, 10334-10340 (2012)
15. Ye, HP, Gao, JM, Ding, YS: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**, 1075-1081 (2007)
16. Zhou, Y, Jiao, F: Nonlocal Cauchy problem for fractional evolution equations. Nonlinear Anal. **11**, 4465-4475 (2010)
17. Zhou, Y, Jiao, F: Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. **59**, 1063-1077 (2010)
18. Byszewski, L: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**, 419-437 (1991)
19. Byszewski, L, Lakshmikantham, V: Theorems about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Appl. Anal. **40**, 11-19 (1991)
20. Liang, J, Van Casteren, J, Xiao, TJ: Nonlocal Cauchy problems for semilinear evolution equations. Nonlinear Anal. **50**, 173-189 (2002)
21. Liu, H, Chang, JC: Existence for a class of partial differential equations with nonlocal conditions. Nonlinear Anal. **70**, 3076-3083 (2009)
22. Mahmudov, N: Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces. SIAM J. Control Optim. **42**, 1604-1622 (2003)
23. Pazy, A: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)

24. Dhage, B: A nonlinear alternative with applications to nonlinear perturbed differential equations. Nonlinear Stud. **13**, 343-354 (2006)
25. Li, YX: Existence and asymptotic stability of periodic solution for evolution equations with delays. J. Funct. Anal. **261**, 1309-1324 (2011)