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# Lyapunov type inequalities for the Riemann-Liouville fractional differential equations of higher order

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**Abstract**

In this paper, some new Lyapunov type inequalities will be presented for Riemann-Liouville fractional differential equations of the form

$$({}_aD^{\alpha}x)(t) + p(t)\big|x(t)\big|^{\mu-1}x(t) + q(t)\big|x(t)\big|^{\gamma-1}(t)x(t) = f(t),$$

where $\alpha \in (n-1, n]$ $(n \geq 3)$, $p, q, f$ are real-valued functions and $0 < \gamma < 1 < \mu < n$.

**Keywords:** Lyapunov type inequality; Riemann-Liouville fractional differential equation; Green's function; higher fractional order

## 1 Introduction

First, we consider the Hill equation

$$x''(t) + v(t)x(t) = 0 \tag{1.1}$$

with the boundary conditions

$$x(a) = x(b) = 0, \tag{1.2}$$

where $v : [a, b] \to R$ is a real-valued function. Lyapunov [1] discovered that if the boundary value problem (1.1)-(1.2) has a nontrivial solution, then

$$\int_a^b \big|v(s)\big|\,ds > \frac{4}{b-a}. \tag{1.3}$$

In [2], Wintner substituted the function '$|v(s)|$' with '$v^+(s)$' and he got the following inequality:

$$\int_a^b v^+(s)\,ds > \frac{4}{b-a}. \tag{1.4}$$

Inequality (1.4) was generalized by Hartman [3] as follows:

$$\int_a^b (b-s)(s-a)v^+(s)\,ds > b-a. \tag{1.5}$$

Lyapunov inequality is widely used in investigating the qualitative properties such as oscillation and spectral properties for differential equations and difference equations (see [4–13] for details). In recent years, there have been many literature works concerning the Lyapunov type inequality. On the one hand, some authors study Lyapunov type inequalities of integer-order linear differential equations, nonlinear differential equations or systems of differential equations. For example, Xianhua Tang and Meirong Zhang [14] studied the general linear Hamiltonian system

$$u'(t) = JH(t)u(t), \quad u \in R^{2n}, \tag{1.6}$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is the standard symplectic matrix and

$$H(t) = \begin{pmatrix} -C(t) & A^T(t) \\ A(t) & B(t) \end{pmatrix}$$

is a symplectic matrix-valued function which is locally Lebesgue integrable. They obtained corresponding Lyapunov type inequalities. On the other hand, Lyapunov type inequalities of the fractional differential equations are studied by more and more researchers, see [15–26] and the references cited therein for details. Recently, Cabrera *et al.* [21] studied the nonlocal fractional boundary value problem of order $\alpha \in (2,3]$

$$\begin{aligned} &D_a^\alpha x(t) + q(t)x(t) = 0, \quad a < t < b, \\ &x(a) = x'(a) = 0, \qquad x'(b) = x(\xi), \end{aligned}$$

where $a < \xi < b$, $0 \leq \beta(\xi - a)^{\alpha-1} < (\alpha-1)(b-a)^{\alpha-2}$, and $q : [a,b] \to R$ is a real-valued continuous function.

In 2017, Agarwal and Özbekler obtained Lyapunov type inequalities in [22] for the fractional forced nonlinear differential equations of order $\alpha \in (0,2]$

$$\left(D_a^\alpha\right)(t) + p(t)\left|x(t)\right|^{\mu-1}x(t) + q(t)\left|x(t)\right|^{\gamma-1}(t)x(t) = f(t), \tag{1.7}$$

subject to the Dirichlet (2-point) boundary conditions

$$x(a) = x(b) = 0, \tag{1.8}$$

where $p, q, f \in C[t_0, \infty)$ and the constants satisfy $0 < \gamma < 1 < \mu < 2$. Moreover, the function $p$, $q$ and the forcing term $f$ have no sign restrictions. They obtained that if $x(t) \neq 0$ in $(a,b)$, then

$$\left(\int_a^b \left[(b-t)(t-a)^{\alpha-1}\right]\left[\mu_0 p^+(t) + \gamma_0 q^+(t) + \left|f(t)\right|\right] dt\right) \times \left(\int_a^b \left[(b-t)(t-a)^{\alpha-1}\right]\left[p^+(t) + q^+(t)\right] dt\right) > \frac{\Gamma^2(\alpha)}{4}(b-a)^{2\alpha-2}, \tag{1.9}$$

where the constants $\mu_0$ and $\gamma_0$ are the same as in [24, Theorem 2.3].

In this paper, we consider the Riemann-Liouville fractional differential equation with mixed nonlinearities of order $\alpha \in (n-1, n]$ for $n \geq 3$

$$\left(D_a^\alpha x\right)(t) + p(t)\left|x(t)\right|^{\mu-1}x(t) + q(t)\left|x(t)\right|^{\gamma-1}(t)x(t) = f(t), \tag{1.10}$$

where $p, q, f \in C[t_0, \infty)$ and the constants satisfy $0 < \gamma < 1 < \mu < n$ ($n \geq 3$). Equation (1.10) subjects to the following two kinds of boundary conditions, respectively:

$$x(a) = x'(a) = x''(a) = \cdots = x^{(n-2)}(a) = x(b) = 0, \tag{1.11}$$

and the boundary conditions

$$x(a) = x'(a) = x''(a) = \cdots = x^{(n-2)}(a) = x'(b) = 0. \tag{1.12}$$

Obviously, it is easy to see that equation (1.10) has two special forms; one is the forced 'sub-linear' ($p(t) = 0$) fractional equation

$$\left(D_a^\alpha x\right)(t) + q(t)\left|x(t)\right|^{\gamma-1}(t)x(t) = f(t); \quad 0 < \gamma < 1, \tag{1.13}$$

and the other is the forced 'super-linear' ($q(t) = 0$) fractional equation

$$\left(D_a^\alpha x\right)(t) + p(t)\left|x(t)\right|^{\mu-1}x(t) = f(t); \quad 1 < \mu < n. \tag{1.14}$$

Besides, from boundary conditions (1.11), it is noted that $a < b$ and $a$, $b$ are consecutive zeros.

To our best knowledge, there has been no such papers relating to equation (1.10) with higher order $\alpha \in (n-1, n]$ ($n \geq 3$). We will give Lyapunov type inequalities for the fractional differential equations (1.10), (1.13) and (1.14) under the boundary conditions (1.11) and (1.12) with the help of the Green's function. The results relating to the boundary conditions (1.11) and (1.12) are a new type of Lyapunov type inequalities.

We first give some preliminary results about fractional calculus and some lemmas corresponding to the boundary conditions (1.11) and (1.12) in Section 2. In Section 3, we provide two lemmas that are essential in the proof of our results. In addition, we state and prove Lyapunov type inequalities for equations (1.10), (1.13) and (1.14) under the boundary conditions (1.11) or (1.12), respectively. To make our paper more rigorous, we discuss the case when $n$ ($n \geq 3$) is a positive even integer and obtain corresponding results. Besides, we give an example about an eigenvalue problem. Finally, Section 4 is devoted to concluding remarks.

## 2 Preliminaries

At first, we give the concept of fractional integral defined on $[a,b]$.

**Definition 2.1** Let $\alpha \geq 0$ and $f$ be a real function defined on $[a,b]$. The Riemann-Liouville integral of order $\alpha$ is defined by $(I_a^0 f)(t) = f(t)$ and

$$\left(I_a^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds$$

for $t \in [a,b]$, where $\alpha$ is a positive constant.

**Definition 2.2** The Riemann-Liouville fractional derivative of order $\alpha \geq 0$ is defined by

$$\left(D_a^0 f\right)(t) = f(t)$$

and

$$\left(D_a^\alpha f\right)(t) = \left(D_a^m I^{m-\alpha} f\right)(t)$$

for $\alpha > 0$, where $m$ is the smallest integer greater than or equal to $\alpha$.

To obtain our results, we introduce the following lemmas.

**Lemma 2.1** ([21]) *Assume that $f \in C(a,b) \cap L^1(a,b)$. Then*

$$I_a^\alpha D_a^\alpha f(t) = f(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n}$$

*for $t \in [a,b]$, where $c_i \in R$, $i = 1,2,\ldots,n$, and $n = [\alpha] + 1$.*

Corresponding to the boundary conditions (1.11), the following lemmas are essential.

**Lemma 2.2** ([26]) *A function $x(t)$ is a solution of the following equation of order $\alpha \in (n-1,n]$ $(n \geq 3)$:*

$$\left(D_a^\alpha x\right)(t) + H(t) = 0, \quad a < t < b \tag{2.1}$$

*with the boundary conditions* (1.11) *if and only if $x(t)$ satisfies the integral equation*

$$x(t) = \int_a^b g(t,s) H(s)\,ds,$$

*where*

$$g(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \times (b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} \times (b-s)^{\alpha-1}, & a \leq t \leq s \leq b \end{cases} \tag{2.2}$$

*is the Green's function of the boundary value problem* (2.1) *and* (1.11).

**Lemma 2.3** ([26]) *The Green's function* (2.2) *satisfies the following properties*:

(i) $g(t,s) \geq 0$ *for all* $a \leq s, t \leq b$;

(ii)

$$\max_{t\in[a,b]} g(t,s) = g\big(s^*,s\big) = \frac{(s-a)^{\alpha-1}(b-s)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-1}[1-(\frac{b-s}{b-a})^{\frac{\alpha-1}{\alpha-2}}]^{\alpha-2}},$$

*where* $s^* = \frac{s-a(\frac{b-s}{b-a})^{\frac{\alpha-1}{\alpha-2}}}{1-(\frac{b-s}{b-a})^{\frac{\alpha-1}{\alpha-2}}}$;

(iii)

$$\max_{s\in[a,b]} g\big(s^*,s\big) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \frac{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}{(1-z_\alpha^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}},$$

*where* $z_\alpha$ *is the unique zero of the nonlinear equation* $z^{\frac{2\alpha-3}{\alpha-2}} - 2z + 1 = 0$ *in the interval* $z_\alpha \in (0, (\frac{2\alpha-4}{2\alpha-3})^{\frac{\alpha-2}{\alpha-1}})$.

Similarly, we need the following lemmas corresponding to the boundary conditions (1.12).

**Lemma 2.4** *A function* $x(t)$ *is a solution of equation* (2.1) *with boundary conditions* (1.12) *if and only if* $x(t)$ *satisfies the integral equation*

$$x(t) = \int_a^b G(t,s)H(s)\,ds,$$

*where*

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}} \times (b-s)^{\alpha-2} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}} \times (b-s)^{\alpha-2}, & a \leq t \leq s \leq b \end{cases} \tag{2.3}$$

*is the Green's function of the boundary value problem* (2.1) *and* (1.12).

*Proof* By Lemma 2.1, the general solutions to the boundary value problem (2.1) and (1.11) in $[a,b]$ can be represented as

$$x(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}H(s)\,ds \tag{2.4}$$

for constants $c_i$ $(i = 1,2,\ldots,n)$.

By the boundary conditions

$$x(a) = x'(a) = x''(a) = \cdots = x^{(n-2)}(a) = 0,$$

we obtain $c_2 = c_3 = \cdots = c_n = 0$.

Hence

$$x(t) = c_1(t-a)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}H(s)\,ds.$$

Since

$$x'(t) = c_1(\alpha - 1)(t-a)^{\alpha-2} - \frac{\alpha - 1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-2} H(s)\, ds,$$

the boundary condition $x'(b) = 0$ implies

$$c_1(b-a)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} H(s)\, ds = 0.$$

This shows

$$c_1 = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-2}} \int_a^b (b-s)^{\alpha-2} H(s)\, ds.$$

Then

$$\begin{aligned} x(t) &= \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)(b-a)^{\alpha-2}} \int_a^b (b-s)^{\alpha-2} H(s)\, ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} H(s)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left[ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}} (b-s)^{\alpha-2} - (t-s)^{\alpha-1} \right] H(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_t^b \left[ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}} (b-s)^{\alpha-2} \right] H(s)\, ds \\ &= \int_a^b G(t,s) H(s)\, ds, \end{aligned}$$

which completes the proof. □

**Lemma 2.5** *The Green's function* (2.3) *satisfies the following properties*:

(i) $G(t,s) \geq 0$ *for all* $a \leq s,t \leq b$;

(ii) $G(t,s)$ *is non-decreasing about the first variable*;

(iii)

$$0 \leq G(a,s) \leq G(t,s) \leq G(b,s) = \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-2}(s-a), \quad (t,s) \in [a,b] \times [a,b].$$

**Lemma 2.6** ([21]) *Let* $\varphi(s) = \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-2}(s-a)$, $s \in (a,b)$. *Then*

$$\max\{\varphi(s) : a \leq s \leq b\} = \varphi(s^{**}) = \frac{1}{\Gamma(\alpha)}(\alpha-2)^{\alpha-2}\left(\frac{b-a}{\alpha-1}\right)^{\alpha-1},$$

*where* $s^{**} = \frac{b+(\alpha-2)a}{\alpha-1}$.

## 3 Main results

Throughout this section we shall denote $u^{\pm} = \max\{\pm u, 0\}$. At the beginning, we introduce the following lemmas.

**Lemma 3.1** *Let* $A > 0$, $B \geq 0$ *be real numbers. For* $z \geq 0$, *we have*

$$Az^n - Bz^\alpha \geq -(n-\alpha)\alpha^{\frac{\alpha}{n-\alpha}} n^{\frac{n}{\alpha-n}} A^{\frac{\alpha}{\alpha-n}} B^{\frac{n}{n-\alpha}} \tag{3.1}$$

*for any* $\alpha \in (0,n]$ ($n \geq 2$).

*Proof* Let

$$F(z) = Az^n - Bz^\alpha, \quad z \geq 0.$$

It is clear that (3.1) is obvious when $z = 0$ or $B = 0$. By direct computation, we obtain $F'(z_0) = 0$ and $F''(z_0) > 0$. Hence $F$ attains its minimum at $z_0 = (\frac{\alpha B}{nA})^{\frac{1}{n-\alpha}}$ if $B \geq 0$ and

$$\begin{aligned} F_{\min} &= F\left(\left(\frac{\alpha B}{nA}\right)^{\frac{1}{n-\alpha}}\right) \\ &= A\left(\left(\frac{\alpha B}{nA}\right)^{\frac{1}{n-\alpha}}\right)^n - B\left(\left(\frac{\alpha B}{nA}\right)^{\frac{1}{n-\alpha}}\right)^\alpha \\ &= -(n-\alpha)\alpha^{\frac{\alpha}{n-\alpha}} n^{\frac{n}{\alpha-n}} A^{\frac{\alpha}{\alpha-n}} B^{\frac{n}{n-\alpha}}. \end{aligned}$$

So (3.1) holds. Note that inequality (3.1) is strict if $B > 0$. □

**Lemma 3.2** *If $A, B \in R^+$ and $C \in R$, then the function $f(x) = Ax^n - Bx + C$ has a minimal value point at $x_0 = (\frac{B}{nA})^{\frac{1}{n-1}}$ when $n$ is a positive even number.*

*Proof* Since

$$f'(x) = nAx^{n-1} - B = \left(\sqrt[n-1]{nA}x - \sqrt[n-1]{B}\right)g_{n-2}(x),$$

where the function $g_{n-2}(x)$ is a polynomial of degree $n-2$, we obtain $x_0 = (\frac{B}{nA})^{\frac{1}{n-1}}$ is a stagnation point. By direct computation, we get $f''(x) = nAx^{n-2} \geq 0$ and $\lim_{x\to\infty} f(x) = +\infty$. Thus the proof of the lemma is completed. □

Now we show and prove our results relating to the boundary conditions (1.11).

**Theorem 3.1** *Let $x(t)$ be a positive solution of the boundary value problem* (1.10)-(1.11) *in $(a,b)$. Then*

$$\begin{aligned} &\left(\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\left[\mu_0 p^+(s) + \gamma_0 q^+(s) + f^-(s)\right]ds\right) \\ &\quad\times\left(\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}} \\ &\quad> \left[\Gamma(\alpha)(b-a)^{\alpha-1}\right]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}, \end{aligned} \tag{3.2}$$

*where $\mu_0 = (n-\mu)\mu^{\frac{\mu}{n-\mu}} n^{\frac{n}{\mu-n}}$ and $\gamma_0 = (n-\gamma)\gamma^{\frac{\gamma}{n-\gamma}} n^{\frac{n}{\gamma-n}}$ and $z_\alpha$ is the same as in Lemma* 2.3.

*Proof* Set $x(t)$ to be a positive solution of the boundary value problem (1.10)-(1.11). From Lemma 2.2, $x(t)$ can be represented as

$$x(t) = \int_a^b g(t,s)\left[p(s)x^\mu(s) + q(s)x^\gamma(s) - f(s)\right]ds. \tag{3.3}$$

Let $x(c) = \max_{t\in(a,b)} x(t)$. It is clear from Lemma 2.3 that

$$0 \le g(t,s) \le g(s^*,s) = \frac{1}{\Gamma(\alpha)}\left[\frac{(b-s)(s-a)}{b-a}\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}. \tag{3.4}$$

Then, making use of (3.3)-(3.4), we have

$$\begin{aligned} x(c) &= \int_a^b g(c,s)\left[p(s)x^{\mu}(s) + q(s)x^{\gamma}(s) - f(s)\right] ds \\ &\le \int_a^b g(c,s)\left[p^+(s)x^{\mu}(s) + q^+(s)x^{\gamma}(s) + f^-(s)\right] ds \\ &\le \int_a^b g(s^*,s)\left[p^+(s)x^{\mu}(s) + q^+(s)x^{\gamma}(s) + f^-(s)\right] ds \\ &= \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}}\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} \\ &\quad \times \left[p^+(s)x^{\mu}(s) + q^+(s)x^{\gamma}(s) + f^-(s)\right] ds \\ &\le P_0 x^{\mu}(c) + Q_0 x^{\gamma}(c) + F_0, \end{aligned} \tag{3.5}$$

where

$$P_0 = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}}\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} p^+(s)\, ds,$$

$$Q_0 = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}}\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} q^+(s)\, ds,$$

and

$$F_0 = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}}\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} f^-(s)\, ds.$$

Besides, when $A = B = 1$, inequality (3.1) in Lemma 3.1 suggests that

$$x^{\mu}(c) < x^{n}(c) + \mu_0$$

and

$$x^{\gamma}(c) < x^{n}(c) + \gamma_0.$$

Combining these inequalities and inequality (3.5), we find that the following inequality

$$(P_0 + Q_0)x^{n}(c) - x(c) + \mu_0 P_0 + \gamma_0 Q_0 + F_0 > 0 \tag{3.6}$$

holds if and only if

$$(\mu_0 P_0 + \gamma_0 Q_0 + F_0)(P_0 + Q_0)^{\frac{1}{n-1}} > (n-1)n^{\frac{n}{1-n}},$$

which is the same as (3.2). The proof of Theorem 3.1 is finished. □

**Remark 1** From Lemma 2.3, we know that

$$\max_{s\in[a,b]} g\big(s^*,s\big) = \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \frac{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}{(1-z_\alpha^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}.$$

Hence, the following corollary is obvious.

**Corollary 3.2** *Let* $x(t)$ *be a positive solution of the boundary value problem* (1.10)-(1.11) *in* $(a,b)$. *Then*

$$\begin{aligned}&\left(\int_a^b \big[p^+(s)+q^+(s)\big]\,ds\right)^{\frac{1}{n-1}} \left(\int_a^b \big[\mu_0 p^+(s)+\gamma_0 q^+(s)+f^-(s)\big]\,ds\right)\\ &\quad > (n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},\end{aligned}$$

*where the constants* $\mu_0$, $\gamma_0$ *and* $z_\alpha$ *are the same as in Theorem* 3.1.

The following conclusions are given for equations (1.13) and (1.14).

**Corollary 3.3** *Let* $x(t)$ *be a positive solution of the boundary value problem* (1.13) *and* (1.11) *in* $(a,b)$. *Then*

(i) *Hartman type inequality*:

$$\begin{aligned}&\left(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\big[\gamma_0 q^+(s)+f^-(s)\big]\,ds\right)\\ &\qquad\times\left(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} q^+(s)\,ds\right)^{\frac{1}{n-1}}\\ &\quad > \big[\Gamma(\alpha)(b-a)^{\alpha-1}\big]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};\end{aligned}$$

(ii) *Lyapunov type inequality*:

$$\begin{aligned}&\left(\int_a^b q^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \big[\gamma_0 q^+(s)+f^-(s)\big]\,ds\right)\\ &\quad > (n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},\end{aligned}$$

*where the constants* $\gamma_0$ *and* $z_\alpha$ *are the same as in Theorem* 3.1.

**Corollary 3.4** *Let* $x(t)$ *be a positive solution of the boundary value problem* (1.14) *and* (1.11) *in* $(a,b)$. *Then*

(i) *Hartman type inequality*:

$$\begin{aligned}&\left(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\big[\mu_0 p^+(s)+f^-(s)\big]\,ds\right)\\ &\qquad\times\left(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} p^+(s)\,ds\right)^{\frac{1}{n-1}}\\ &\quad > \big[\Gamma(\alpha)(b-a)^{\alpha-1}\big]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};\end{aligned}$$

(ii) *Lyapunov type inequality*:

$$\left(\int_a^b p^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \left[\mu_0 p^+(s)+f^-(s)\right]ds\right)$$
$$> (n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},$$

*where the constants* $\mu_0$ *and* $z_\alpha$ *are the same as in Theorem* 3.1.

Next, the results relating to boundary conditions (1.12) will be introduced and proved.

**Theorem 3.5** (Hartman type inequality) *Let* $x(t)$ *be a positive solution of equation* (1.10) *satisfying the boundary conditions* (1.12) *in* $(a,b)$. *Then*

$$\begin{aligned}&\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[\mu_0 p^+(s)+\gamma_0 q^+(s)+f^-(s)\right]ds\right)\\ &\quad\times\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\\ &\quad> \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}},\end{aligned}\tag{3.7}$$

*where* $\mu_0$ *and* $\gamma_0$ *are the same as in Theorem* 3.1.

*Proof* Set $x(t)$ to be a positive solution of equation (1.10) with (1.12). From Lemma 2.4, $x(t)$ can be represented as

$$x(t)=\int_a^b G(t,s)\left[p(s)x^\mu(s)+q(s)x^\gamma(s)-f(s)\right]ds.\tag{3.8}$$

Let $x(c)=\max_{t\in(a,b)}x(t)$. It is clear from Lemma 2.5 that

$$0\le G(a,s)\le G(t,s)\le G(b,s)=\frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-2}(s-a).\tag{3.9}$$

Then, making use of (3.9)-(3.10), we have

$$\begin{aligned}x(c)&=\int_a^b G(c,s)\left[p(s)x^\mu(s)+q(s)x^\gamma(s)-f(s)\right]ds\\ &\le\int_a^b G(c,s)\left[p^+(s)x^\mu(s)+q^+(s)x^\gamma(s)+f^-(s)\right]ds\\ &\le\int_a^b G(b,s)\left[p^+(s)x^\mu(s)+q^+(s)x^\gamma(s)+f^-(s)\right]ds\\ &=\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-2}(s-a)\left[p^+(s)x^\mu(s)+q^+(s)x^\gamma(s)+f^-(s)\right]ds\\ &\le P_1x^\mu(c)+Q_1x^\gamma(c)+F_1,\end{aligned}\tag{3.10}$$

where

$$P_1 = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2}(s-a)p^+(s)\,ds,$$
$$Q_1 = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2}(s-a)q^+(s)\,ds,$$

and

$$F_1 = \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-2}(s-a)f^-(s)\,ds.$$

Besides, when $A = B = 1$, inequality (3.1) in Lemma 3.1 suggests that

$$x^{\mu}(c) < x^n(c) + \mu_0,$$

and

$$x^{\gamma}(c) < x^n(c) + \gamma_0.$$

Combining these inequalities and inequality (3.10), we find that the following inequality

$$(P_1 + Q_1)x^n(c) - x(c) + \mu_0 P_1 + \gamma_0 Q_1 + F_1 > 0 \tag{3.11}$$

holds if and only if

$$(\mu_0 P_1 + \gamma_0 Q_1 + F_1)(P_1 + Q_1)^{\frac{1}{n-1}} > (n-1)n^{\frac{n}{1-n}},$$

which is the same as (3.7). Hence the proof of Theorem 3.5 is completed. □

**Remark 2** From Lemmas 2.5 and 2.6, it is easy to see that

$$\max_{s\in[a,b]} G(b,s) = \frac{1}{\Gamma(\alpha)}(\alpha-1)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}.$$

Thus, Lyapunov type inequality of the boundary value problem (1.10) and (1.12) can be obtained.

**Corollary 3.6** (Lyapunov type inequality) *Let $x(t)$ be a positive solution of the boundary value problem* (1.10) *and* (1.12) *in $(a,b)$. Then*

$$\left(\int_a^b \left[p^+(s) + q^+(s)\right] ds\right)^{\frac{1}{n-1}} \left(\int_a^b \left[\mu_0 p^+(s) + \gamma_0 q^+(s) + f^-(s)\right] ds\right)$$
$$> \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\left[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\right]^{\frac{n}{1-n}},$$

*where the constants $\mu_0$ and $\gamma_0$ are the same as in Theorem* 3.1.

As before, the following conclusions are given for two equations (1.13) and (1.14).

**Corollary 3.7** *Let $x(t)$ be a positive solution of the boundary value problem* (1.12)-(1.13) *in* $(a,b)$. *Then*

(i) *Hartman type inequality*:

$$\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[\gamma_0 q^+(s)+f^-(s)\right]ds\right)\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]q^+(s)\,ds\right)^{\frac{1}{n-1}} > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};$$

(ii) *Lyapunov type inequality*:

$$\left(\int_a^b \left[\gamma_0 q^+(s)+f^-(s)\right]ds\right)\left(\int_a^b q^+(s)\,ds\right)^{\frac{1}{n-1}} > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\left[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\right]^{\frac{n}{1-n}},$$

*where the constant $\gamma_0$ is the same as in Theorem* 3.1.

**Corollary 3.8** *Let $x(t)$ be a positive solution of the boundary value problem* (1.14) *and* (1.12) *in* $(a,b)$. *Then*

(i) *Hartman type inequality*:

$$\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[\mu_0 p^+(s)+f^-(s)\right]ds\right)\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]p^+(s)\,ds\right)^{\frac{1}{n-1}} > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};$$

(ii) *Lyapunov type inequality*:

$$\left(\int_a^b p^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \left[\mu_0 p^+(s)+f^-(s)\right]ds\right) > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\left[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\right]^{\frac{n}{1-n}},$$

*where the constant $\mu_0$ is the same as in Theorem* 3.1.

When $n$ ($n \geq 3$) is a positive even integer, the above theorems are valid for all invariant solutions of equations (1.10), (1.13) and (1.14) under the boundary conditions (1.11) or (1.12). Now the results corresponding to the boundary conditions (1.11) are presented when $n$ ($n \geq 3$) is a positive even integer.

**Theorem 3.9** *Let $x(t)$ be a negative solution of the boundary value problem* (1.10)-(1.11) *in* $(a,b)$. *Then*

$$\begin{aligned}&\left(\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\left[\mu_0 p^+(s)+\gamma_0 q^+(s)+f^+(s)\right]ds\right)\\ &\quad\times\left(\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\\ &\quad> \left[\Gamma(\alpha)(b-a)^{\alpha-1}\right]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}},\end{aligned} \tag{3.12}$$

*where $\mu_0$ and $\gamma_0$ are the same as in Theorem* 3.1.

*Proof* If $x(t)$ is a negative solution of equation (1.10), then $-x(t)$ is a positive solution of

$$\big(D_a^\alpha x\big)(t) + p(t)\big|x(t)\big|^{\mu-1}x(t) + q(t)\big|x(t)\big|^{\gamma-1}x(t) = -f(t). \tag{3.13}$$

Then, similar to the proof of Theorem 3.1, we know that if equation (3.13) has a positive solution, then

$$\begin{aligned} x(c) &= \int_a^b g(c,s)\big[p(s)x^\mu(s) + q(s)x^\gamma(s) + f(s)\big]\,ds \\ &\le \int_a^b g(c,s)\big[p^+(s)x^\mu(s) + q^+(s)x^\gamma(s) + f^+(s)\big]\,ds \\ &\le \int_a^b g\big(s^*,s\big)\big[p^+(s)x^\mu(s) + q^+(s)x^\gamma(s) + f^+(s)\big]\,ds \\ &= \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}} \int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\bigg[1-\bigg(\frac{b-s}{b-a}\bigg)^{\frac{\alpha-1}{\alpha-2}}\bigg]^{2-\alpha} \\ &\quad \times \big[p^+(s)x^\mu(s) + q^+(s)x^\gamma(s) + f^+(s)\big]\,ds \\ &\le P_0 x^\mu(c) + Q_0 x^\gamma(c) + F_2, \end{aligned} \tag{3.14}$$

where $P_0$ and $Q_0$ are defined in Theorem 3.1, and

$$F_2 = \frac{1}{\Gamma(\alpha)(b-a)^{\alpha-1}} \int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\bigg[1-\bigg(\frac{b-s}{b-a}\bigg)^{\frac{\alpha-1}{\alpha-2}}\bigg]^{2-\alpha} f^+(s)\,ds.$$

Repeating the same steps as in Theorem 3.1, we know that

$$(P_0 + Q_0)x^n(c) - x(c) + \mu_0 P_0 + \gamma_0 Q_0 + F_2 > 0, \quad n = 2k\ (k \in N_+) \tag{3.15}$$

holds if and only if the minimum of the function $f(x) = (P_0 + Q_0)x^n - x + \mu_0 P_0 + \gamma_0 Q_0 + F_2$ satisfies $f(x)_{\min} \ge 0$. From Lemma 3.2, we know that

$$f_{\min} = f\bigg(\frac{1}{(n(P_0 + Q_0))^{\frac{1}{n-1}}}\bigg) = (P_0 + Q_0)^{\frac{1}{1-n}} n^{\frac{n}{1-n}}(1-n) + \mu_0 P_0 + \gamma_0 Q_0 + F_2.$$

Hence it is necessary that $(P_0 + Q_0)^{\frac{1}{1-n}} n^{\frac{n}{1-n}}(1-n) + \mu_0 P_0 + \gamma_0 Q_0 + F_1 > 0$ holds. By direct computation, inequality (3.12) is obvious. □

From Theorems 3.1 and 3.9, we obtain Theorem 3.10 since $|f(s)| \ge \max\{f^+(s), f^-(s)\}$.

**Theorem 3.10** *Let $x(t)$ be a nontrivial solution of the boundary value problem* (1.10)-(1.11). *If $x(t) \neq 0$ in $(a,b)$, then*

$$\begin{aligned} &\bigg(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\bigg[1-\bigg(\frac{b-s}{b-a}\bigg)^{\frac{\alpha-1}{\alpha-2}}\bigg]^{2-\alpha}\big[\mu_0 p^+(s) + \gamma_0 q^+(s) + \big|f(s)\big|\big]\,ds\bigg) \\ &\quad \times \bigg(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\bigg[1-\bigg(\frac{b-s}{b-a}\bigg)^{\frac{\alpha-1}{\alpha-2}}\bigg]^{2-\alpha}\big[p^+(s) + q^+(s)\big]\,ds\bigg)^{\frac{1}{n-1}} \\ &> \big[\Gamma(\alpha)(b-a)^{\alpha-1}\big]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}, \end{aligned}$$

*where $\mu_0$ and $\gamma_0$ are the same as in Theorem* 3.1.

**Corollary 3.11** (Disconjugacy) *If*

$$\begin{aligned}&\left(\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\left[\mu_0 p^+(s)+\gamma_0 q^+(s)+\left|f(s)\right|\right]ds\right)\\&\quad\times\left(\int_a^b \left[(b-s)(s-a)\right]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\\&\quad\le\left[\Gamma(\alpha)(b-a)^{\alpha-1}\right]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}},\end{aligned}$$

*holds, then equation* (1.10) *is disconjugate in* $[a,b]$, *where* $\mu_0$ *and* $\gamma_0$ *are the same as in Theorem* 3.1.

Similarly, Lyapunov type inequality can be easily obtained according to Theorem 3.9.

**Corollary 3.12** *Let* $x(t)$ *be a negative solution of equation* (1.10) *satisfying the boundary conditions* (1.11) *in* $(a,b)$. *Then*

$$\begin{aligned}&\left(\int_a^b \left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\left(\int_a^b \left[\mu_0 p^+(s)+\gamma_0 q^+(s)+f^+(s)\right]ds\right)\\&\quad>(n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},\end{aligned}$$

*where* $\mu_0$, $\gamma_0$ *and* $z_\alpha$ *are the same as in Theorem* 3.1.

Based on Corollaries 3.2 and 3.12, Corollary 3.13 is obvious.

**Corollary 3.13** *Let* $x(t)$ *be a nontrivial solution of the boundary value problem* (1.10)-(1.11). *If* $x(t)\neq 0$ *in* $(a,b)$, *then*

$$\begin{aligned}&\left(\int_a^b \left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\left(\int_a^b \left[\mu_0 p^+(s)+\gamma_0 q^+(s)+\left|f(s)\right|\right]ds\right)\\&\quad>(n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},\end{aligned}$$

*where* $\mu_0$, $\gamma_0$ *and* $z_\alpha$ *are the same as in Theorem* 3.1.

**Corollary 3.14** (Disconjugacy) *If*

$$\begin{aligned}&\left(\int_a^b \left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\left(\int_a^b \left[\mu_0 p^+(s)+\gamma_0 q^+(s)+\left|f(s)\right|\right]ds\right)\\&\quad\le(n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}}\end{aligned}$$

*holds, then equation* (1.10) *is disconjugate in* $[a,b]$, *where* $\mu_0$, $\gamma_0$ *and* $z_0$ *are the same as in Theorem* 3.1.

As before, we show our results relating to equations (1.13) and (1.14) under the boundary conditions (1.11) when $n$ ($n\geq 3$) is a positive even integer.

**Corollary 3.15** *Let $x(t)$ be a nontrivial solution of the boundary value problem* (1.13) *and* (1.11). *If $x(t) \neq 0$ in $(a,b)$, then*

(i) *Hartman type inequality*:

$$\begin{aligned}&\left(\int_a^b \bigl[(b-s)(s-a)\bigr]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\bigl[\gamma_0 q^+(s)+\bigl|f(s)\bigr|\bigr]\,ds\right)\\&\quad\times\left(\int_a^b \bigl[(b-s)(s-a)\bigr]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} q^+(s)\,ds\right)^{\frac{1}{n-1}}\\&\quad>\bigl[\Gamma(\alpha)(b-a)^{\alpha-1}\bigr]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};\end{aligned}$$

(ii) *Lyapunov type inequality*:

$$\begin{aligned}&\left(\int_a^b \bigl[\gamma_0 q^+(s)+\bigl|f(s)\bigr|\bigr]\,ds\right)\left(\int_a^b q^+(s)\,ds\right)^{\frac{1}{n-1}}\\&\quad>(n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z_\alpha^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},\end{aligned}$$

*where the constants $\gamma_0$ and $z_\alpha$ are the same as in Theorem* 3.1.

**Corollary 3.16** *Let $x(t)$ be a nontrivial solution of the boundary value problem* (1.14) *and* (1.11). *If $x(t) \neq 0$ in $(a,b)$, then*

(i) *Hartman type inequality*:

$$\begin{aligned}&\left(\int_a^b \bigl[(b-s)(s-a)\bigr]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\bigl[\mu_0 p^+(s)+\bigl|f(s)\bigr|\bigr]\,ds\right)\\&\quad\times\left(\int_a^b \bigl[(b-s)(s-a)\bigr]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha} p^+(s)\,ds\right)^{\frac{1}{n-1}}\\&\quad>\bigl[\Gamma(\alpha)(b-a)^{\alpha-1}\bigr]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};\end{aligned}$$

(ii) *Lyapunov type inequality*:

$$\begin{aligned}&\left(\int_a^b p^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \bigl[\mu_0 p^+(s)+\bigl|f(s)\bigr|\bigr]\,ds\right)\\&\quad>(n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z_\alpha^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},\end{aligned}$$

*where the constants $\mu_0$ and $z_\alpha$ are the same as in Theorem* 3.1.

Next we present the following conclusions corresponding to the boundary conditions (1.12).

**Theorem 3.17** (Hartman type inequality) *Let $x(t)$ be a negative solution of equation* (1.10) *satisfying the boundary conditions* (1.12) *in* $(a,b)$*. Then*

$$\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[\mu_0 p^+(s)+\gamma_0 q^+(s)+f^+(s)\right]ds\right) \times \left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}} > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}},$$

*where $\mu_0$ and $\gamma_0$ are the same as in Theorem* 3.1.

*Proof* Similar to the proof of Theorem 3.9. □

Based on Theorems 3.5 and 3.17, we get Theorem 3.18.

**Theorem 3.18** (Hartman type inequality) *Let $x(t)$ be a nontrivial solution of the boundary value problem* (1.10) *and* (1.12)*. If $x(t)\neq 0$ in $(a,b)$, then*

$$\left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[\mu_0 p^+(s)+\gamma_0 q^+(s)+\left|f(s)\right|\right]ds\right) \times \left(\int_a^b \left[(b-s)^{\alpha-2}(s-a)\right]\left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}} > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}},$$

*where $\mu_0$ and $\gamma_0$ are the same as in Theorem* 3.1.

As before, we obtain Corollary 3.19.

**Corollary 3.19** (Lyapunov type inequality) *Let $x(t)$ be a negative solution of equation* (1.10) *satisfying the boundary conditions* (1.12) *in* $(a,b)$*. Then*

$$\left(\int_a^b \left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\left(\int_a^b \left[\mu_0 p^+(s)+\gamma_0 q^+(s)+f^+(s)\right]ds\right) > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\left[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\right]^{\frac{n}{1-n}},$$

*where $\mu_0$ and $\gamma_0$ are the same as in Theorem* 3.1.

Based on Corollaries 3.6 and 3.19, we get Corollary 3.20.

**Corollary 3.20** (Lyapunov type inequality) *Let $x(t)$ be a nontrivial solution of the boundary value problem* (1.10) *and* (1.12)*. If $x(t)\neq 0$ in $(a,b)$, then*

$$\left(\int_a^b \left[p^+(s)+q^+(s)\right]ds\right)^{\frac{1}{n-1}}\left(\int_a^b \left[\mu_0 p^+(s)+\gamma_0 q^+(s)+\left|f(s)\right|\right]ds\right) > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\left[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\right]^{\frac{n}{1-n}},$$

*where $\mu_0$ and $\gamma_0$ are the same as in Theorem* 3.1.

Similarly, equations (1.13) and (1.14) also admit the above conclusions.

**Corollary 3.21** *Let $x(t)$ be a nontrivial solution of the boundary value problem* (1.12)-(1.13). *If $x(t) \neq 0$ in $(a,b)$, then*

(i) *Hartman type inequality*:

$$\left(\int_a^b \big[(b-s)^{\alpha-2}(s-a)\big]\big[\gamma_0 q^+(s) + \big|f(s)\big|\big]\,ds\right)\left(\int_a^b \big[(b-s)^{\alpha-2}(s-a)\big]q^+(s)\,ds\right)^{\frac{1}{n-1}}$$
$$> \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};$$

(ii) *Lyapunov type inequality*:

$$\left(\int_a^b q^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \big[\gamma_0 q^+(s) + \big|f(s)\big|\big]\,ds\right)$$
$$> \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\big[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\big]^{\frac{n}{1-n}},$$

*where the constant $\gamma_0$ is the same as in Theorem* 3.1.

**Corollary 3.22** *Let $x(t)$ be a nontrivial solution of the boundary value problem* (1.14) *and* (1.12). *If $x(t) \neq 0$ in $(a,b)$, then*

(i) *Hartman type inequality*:

$$\left(\int_a^b \big[(b-s)^{\alpha-2}(s-a)\big]\big[\mu_0 p^+(s) + \big|f(s)\big|\big]\,ds\right)\left(\int_a^b \big[(b-s)^{\alpha-2}(s-a)\big]p^+(s)\,ds\right)^{\frac{1}{n-1}}$$
$$> \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};$$

(ii) *Lyapunov type inequality*:

$$\left(\int_a^b p^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \big[\mu_0 p^+(s) + \big|f(s)\big|\big]\,ds\right)$$
$$> \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\big[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\big]^{\frac{n}{1-n}},$$

*where the constant $\mu_0$ is the same as in Theorem* 3.1.

When $\gamma \to 1^-$ (or $\mu \to 1^+$), equation (1.13) (or equation (1.14)) reduces to the forced Riemann-Liouville linear fractional differential equation of order $\alpha \in (n-1,n]$

$$D_a^\alpha x(t) + v(t)x(t) = f(t), \tag{3.16}$$

where $v(t) = q(t)$ (or $v(t) = p(t)$). Since

$$\lim_{\mu\to 1^+} \mu_0 = \lim_{\gamma\to 1^-} \gamma_0 = (n-1)n^{\frac{n}{1-n}},$$

we can also get Lyapunov type inequalities from the above conclusions. Here we take the following corollaries for instance.

**Corollary 3.23** *Let $x(t)$ be a nontrivial solution of the boundary value problem* (3.16) *and* (1.11) *when $n$ ($n \geq 3$) is a positive even integer. If $x(t) \neq 0$ in $(a,b)$, then*

(i) *Hartman type inequality*:

$$\begin{aligned}&\left(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}\big[(n-1)n^{\frac{n}{1-n}}v^+(s)+\big|f(s)\big|\big]\,ds\right)\\&\quad\times\left(\int_a^b \big[(b-s)(s-a)\big]^{\alpha-1}\left[1-\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{2-\alpha}v^+(s)\,ds\right)^{\frac{1}{n-1}}\\&\quad>\big[\Gamma(\alpha)(b-a)^{\alpha-1}\big]^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};\end{aligned} \tag{3.17}$$

(ii) *Lyapunov type inequality*:

$$\begin{aligned}&\left(\int_a^b v^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \big[(n-1)n^{\frac{n}{1-n}}v^+(s)+\big|f(s)\big|\big]\,ds\right)\\&\quad>(n-1)n^{\frac{n}{1-n}}\left[\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\right]^{\frac{n}{n-1}}\left[\frac{(1-z^{\frac{\alpha-1}{\alpha-2}})^{\alpha-2}}{z_\alpha^{\alpha-1}(1-z_\alpha)^{\alpha-1}}\right]^{\frac{n}{n-1}},\end{aligned} \tag{3.18}$$

*where* $\mu_0$, $\gamma_0$ *and* $z_\alpha$ *are the same as in Theorem* 3.1.

**Remark 3** When $f(t)\equiv 0$ and $\mu\to 1^+$ (or $\gamma\to 1^-$), inequalities (3.17) and (3.18) coincide with [26, Corollaries 5.3 and 5.4]. The authors of [26] obtained Lyapunov type inequalities by means of norms rather than the property of functions in this paper.

**Corollary 3.24** *Let* $x(t)$ *be a nontrivial solution of the boundary value problem* (3.16) *and* (1.12) *when* $n$ ($n\geq 3$) *is a positive even integer. If* $x(t)\neq 0$ *in* $(a,b)$, *then*

(i) *Hartman type inequality*:

$$\begin{aligned}&\left(\int_a^b \big[(b-s)^{\alpha-2}(s-a)\big]\big[(n-1)n^{\frac{n}{1-n}}v^+(s)+\big|f(s)\big|\big]\,ds\right)\\&\quad\times\left(\int_a^b \big[(b-s)^{\alpha-2}(s-a)\big]v^+(s)\,ds\right)^{\frac{1}{n-1}}\\&\quad>\Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}};\end{aligned}$$

(ii) *Lyapunov type inequality*:

$$\begin{aligned}&\left(\int_a^b v^+(s)\,ds\right)^{\frac{1}{n-1}}\left(\int_a^b \big[(n-1)n^{\frac{n}{1-n}}v^+(s)+\big|f(s)\big|\big]\,ds\right)\\&\quad>\Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}\big[(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}(\alpha-1)^{1-\alpha}\big]^{\frac{n}{1-n}},\end{aligned}$$

*where* $\mu_0$ *and* $\gamma_0$ *are the same as in Theorem* 3.1.

Now we present an application of the obtained results to eigenvalue problems.

**Example 3.25** Consider the following eigenvalue problem:

$$\begin{cases}D_a^\alpha x(t)+\lambda x(t)=0, & a<t<b, n-1<\alpha\leq n;\\ x(a)=x'(a)=x''(a)=\cdots=x^{(n-2)}(a)=x'(b)=0,\end{cases} \tag{3.19}$$

where $n$ ($n \geq 3$) is a positive even integer. If $\lambda$ is an eigenvalue of the boundary value problem (3.19), then

$$|\lambda| > \frac{\Gamma(\alpha+1)(\alpha-1)}{(b-a)^{\alpha}}. \tag{3.20}$$

*Proof* Suppose that $\lambda$ is an eigenvalue of the boundary value problem (3.19), then (3.19) has at least one nontrivial continuous solution in $(a,b)$. From Corollary 3.24, we obtain that

$$\begin{aligned} &\left(\int_a^b \bigl[(b-s)^{\alpha-2}(s-a)\bigr](n-1)n^{\frac{n}{1-n}}|\lambda|\,ds\right) \\ &\quad\times \left(\int_a^b \bigl[(b-s)^{\alpha-2}(s-a)\bigr]|\lambda|\,ds\right)^{\frac{1}{n-1}} > \Gamma(\alpha)^{\frac{n}{n-1}}(n-1)n^{\frac{n}{1-n}}. \end{aligned} \tag{3.21}$$

From [21], we know that

$$\int_a^b (b-s)^{\alpha-2}(s-a)\,ds = \frac{(b-a)^{\alpha}}{\alpha(\alpha-1)}. \tag{3.22}$$

Substituting (3.22) into inequality (3.21), it is easy to see that

$$|\lambda| > \frac{\Gamma(\alpha+1)(\alpha-1)}{(b-a)^{\alpha}}. \qquad \square$$

## 4 Concluding remarks

We conclude this paper with the following remarks. The results obtained in this paper for equation (1.10) under the boundary conditions (1.11) or (1.12) can be easily generalized to the Riemann-Liouville fractional forced differential equations of order $\alpha \in (n-1,n]$ ($n \geq 3$) with no sign restrictions on coefficients

$$\bigl(D_a^{\alpha}x\bigr)(t) \pm p(t)\bigl|x(t)\bigr|^{\mu-1}x(t) \mp q(t)\bigl|x(t)\bigr|^{\gamma-1}x(t) = f(t)$$

or more universally equation of the form

$$\bigl(D_a^{\alpha}x\bigr)(t) + \sum_{k=1}^{n} q_k(t)\bigl|x(t)\bigr|^{\sigma_k-1}x(t) = f(t),$$

where

$$0 < \sigma_1 < \cdots < \sigma_m < 1 < \sigma_{m+1} < \cdots < n$$

and the functions $q_k$ ($k = 1,\ldots,n$) and the forcing term $f$ have no sign restrictions. When $n$ ($n \geq 3$) is a positive even integer, we also have corresponding results similar to Corollary 3.24. The reader can easily obtain the formulae of these results.

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**Competing interests**
The first and second authors announce to have no competing interests.

**Authors' contributions**
The first and second authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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**References**

1. Liapunov, AM: Probleme général de la stabilité du mouvement. Ann. Fac. Sci. Univ. Toulouse **2**, 27-247 (1907)
2. Wintner, A: On the nonexistence of conjugate points. Am. J. Math. **73**, 368-380 (1951)
3. Hartman, P: Ordinary Differential Equations. Wiley, New York (1964). 2nd edn. Birkhäuser, Boston (1982)
4. Zheng, Z, Zhang, W: Characterization of eigenvalues in spectral gap for singular differential operators. Abstr. Appl. Anal. **2012**, Article ID 271657 (2012)
5. Qi, J, Zheng, Z, Sun, H: Classification of Sturm-Liouville differential equations with complex coefficients and operator realizations. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **467**, 1835-1850 (2011)
6. Zheng, Z, Qi, J, Chen, S: Eigenvalues below the lower bound of minimal operators of singular Hamiltonian expressions. Comput. Math. Appl. **56**(11), 2825-2833 (2008)
7. Zheng, Z: Invariance of deficiency indices under perturbation for discrete Hamiltonian systems. J. Differ. Equ. Appl. **19**(8), 1243-1250 (2013)
8. Zheng, Z, Wang, X, Han, H: Oscillation criteria for forced second order differential equations with mixed nonlinearities. Appl. Math. Lett. **22**(7), 1096-1101 (2009)
9. Cai, J, Zheng, Z: A singular Sturm-Liouville problem with limit circle. Discrete Dyn. Nat. Soc. **2017**, Article ID 9673846 (2017)
10. Shao, J, Meng, F: Generalized variational principles on oscillation for nonlinear nonhomogeneous differential equations. Abstr. Appl. Anal. **2011**, Article ID 972656 (2011)
11. Meng, F, Huang, Y: Interval oscillation criteria for a forced second-order nonlinear differential equations with damping. Appl. Math. Comput. **218**, 1857-1861 (2011)
12. Shao, J, Meng, F, Pang, X: Generalized variational oscillation principles for second order differential equations with mixed nonlinearities. Discrete Dyn. Nat. Soc. **2011**, Article ID 972656 (2011)
13. Kong, X, Zheng, Z: Two-point oscillation for a class of second-order damped linear differential equations. Abstr. Appl. Anal. **2011**, Article ID 420589 (2011)
14. Tang, X-H, Zhang, M: Lyapunov inequalities and stability for linear Hamiltonian systems. J. Differ. Equ. **252**, 358-381 (2012)
15. Ferreira, RAC: A Lyapunov-type inequality for a fractional boundary value problem. Fract. Calc. Appl. Anal. **16**, 978-984 (2013)
16. Ferreira, RAC: On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function. J. Math. Anal. Appl. **412**, 1058-1063 (2014)
17. Jleli, M, Samet, B: Lyapunov-type inequality for fractional boundary-value problems. Electron. J. Differ. Equ. **2015**, Article ID 88 (2015)
18. Jleli, M, Ragoub, L, Samet, B: A Lyapunov-type inequality for a fractional differential equation under a Robin boundary condition. J. Funct. Spaces Appl. **2015**, Article ID 468536 (2015)
19. O'Regan, D, Samet, B: Lyapunov-type inequalities for a class of fractional differential equations. J. Inequal. Appl. **2015**, Article ID 247 (2015)
20. Rong, J, Bai, C: Lyapunov-type inequality for a fractional differential equation with fractional boundary value conditions. Electron. J. Qual. Theory Differ. Equ. **2015**, Article ID 82 (2015)
21. Cabrera, I, Sadarangani, K, Samet, B: Hartman-Winter-type inequalities for a class of nonlocal fractional boundary value problems. Math. Methods Appl. Sci. **40**, 129-136 (2017)
22. Agarwal, RP, Özbekler, A: Lyapunov type inequalities for mixed nonlinear Riemann-Liouville fractional differential equations with a forcing term. J. Comput. Appl. Math. **314**, 69-78 (2017)
23. Agarwal, RP, Özbekler, A: Lyapunov type inequalities for a fractional differential equation with mixed boundary equations. Math. Inequal. Appl. **18**(2), 443-451 (2015)
24. Agarwal, RP, Özbekler, A: Disconjugacy via Lyapunov and Vallee-Poussin type inequalities for forced differential equations. Appl. Math. Comput. **265**, 456-468 (2015)
25. Chidouh, A, Torres, DFM: A generalized Lyapunov inequality for a fractional boundary value problem. Appl. Math. Comput. **312**, 192-197 (2017)
26. Jleli, M, Nieto, JJ, Samet, B: Lyapunov-type inequalities for a higher order fractional differential equation with fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2017**, Article ID 16 (2017)