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# A stochastic prey-predator model with time-dependent delays

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## Abstract

A stochastic predator-prey system with time-dependent delays is considered. Firstly, we show the existence of a global positive solution and stochastically ultimate boundedness. Secondly, the critical value between weak persistence and extinction of the prey is obtained and we also give the asymptotic pathwise estimation. Finally, we simulate the model to illustrate our results.

**MSC:** 92B05; 92D25; 93E03

**Keywords:** prey-predator system; time-dependent delays; stochastically ultimate bounded; persistence; extinction

## 1 Introduction

During the last decades, the classical predator-prey system has gained an important theoretical and practical significance, which has been studied extensively, and many excellent results concerning the permanence, extinction and global attractive of the predator-prey system have been obtained; see [1–5].

A classical two-species predator-prey system can be expressed as follows:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)[r - a_{11}x(t) - a_{12}y(t)], \\ \frac{dy(t)}{dt} = y(t)[-d + a_{21}x(t) - a_{22}y(t)], \end{cases} \quad (1)$$

where  $x(t)$  and  $y(t)$  stand for the population sizes of the prey and the predator, respectively, and  $r, d, a_{ij}$  ( $i, j = 1, 2$ ) are positive constants. For biological interpretation of each coefficient in system (1), we refer the reader to [1].

On the one hand, more realistic and interesting models of population interactions should take the effects of time delay into account [6–8], such as their maturation time. In consideration of the fact that time delays are not resistant to time fluctuations [9–11], system (1) becomes

$$\begin{cases} \frac{dx(t)}{dt} = x(t)[r - a_{11}x(t) - a_{12}x(t - \tau_1(t)) - a_{13}y(t - \tau_2(t))], \\ \frac{dy(t)}{dt} = y(t)[-d + a_{21}x(t - \tau_3(t)) - a_{22}y(t) - a_{23}y(t - \tau_4(t))], \end{cases}$$

where  $a_{13} > 0$ ,  $a_{23} > 0$  and  $\tau_i(t)$  ( $i = 1, 2, 3, 4$ ) is a nonnegative, bounded and differentiable function on  $[0, +\infty)$ , and satisfies

$$\bar{\tau} = \max_{i=1,2,3,4} \sup_{t \geq 0} \{\tau_i(t)\} \geq 0.$$

$\tau'_i(t)$  ( $i = 1, 2, 3, 4$ ) are continuous bounded functions on  $[0, +\infty)$  with  $\tau'_i(t) = \frac{d\tau_i(t)}{dt}$ .

On the other hand, population systems are inevitably subject to the environmental noises in the natural environment (see [12–15]). May [16] pointed out that, due to environmental fluctuations, the birth rate, carrying capacity, competition coefficients and other parameters involved in the system exhibit random fluctuation to a greater or lesser extent (see [17–20]). We suppose that the white noise affects all parameters. We usually estimate a value by an average value plus an error term. By the central limit theorem, the error term follows a normal distribution, thus, we can replace each growth rate by an average rate plus an error term,  $r \rightarrow r + \sigma_{11}\dot{B}_{11}(t)$ , and similarly,  $-d \rightarrow -d + \sigma_{21}\dot{B}_{21}(t)$ ,  $a_{ij} \rightarrow a_{ij} + \sigma_{i(j+1)}\dot{B}_{i(j+1)}(t)$ , where  $\sigma_{ij}^2$  denotes the intensity of the noise and  $\dot{B}_{i(j+1)}(t)$  is a standard white noise, as  $B_{ij}(t)$  is a standard Brownian motion defined on a complete probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions. Then this system will become the following stochastic autonomous predator-prey system with time-dependent delays:

$$\begin{cases} dx(t) = x(t)[r - a_{11}x(t) - a_{12}x(t - \tau_1(t)) - a_{13}y(t - \tau_2(t))]dt + \sigma_{11}x(t)dB_{11}(t) \\ \quad + \sigma_{12}x^2(t)dB_{12}(t) + \sigma_{13}x(t)x(t - \tau_1(t))dB_{13}(t) + \sigma_{14}x(t)y(t - \tau_2(t))dB_{14}(t), \\ dy(t) = y(t)[-d + a_{21}x(t - \tau_3(t)) - a_{22}y(t) - a_{23}y(t - \tau_4(t))]dt + \sigma_{21}y(t)dB_{21}(t) \\ \quad + \sigma_{22}y(t)x(t - \tau_3(t))dB_{22}(t) + \sigma_{23}y^2(t)dB_{23}(t) + \sigma_{24}y(t)y(t - \tau_4(t))dB_{24}(t), \end{cases} \quad (2)$$

with initial conditions  $x(\theta) = \varphi_1(\theta) > 0$ ,  $y(\theta) = \varphi_2(\theta) > 0$ ,  $\theta \in [-\bar{\tau}, 0]$ , where  $\varphi_i(\theta)$  is a continuous function on  $[-\bar{\tau}, 0]$ .

For simplicity, we introduce the following notations here:

$$R_+^n = \{(a_1, a_2, \dots, a_n) \in R^n : a_i > 0, i = 1, 2, \dots, n\},$$

$$\langle f(t) \rangle = \frac{1}{t} \int_0^t f(s) ds, \quad f^* = \limsup_{t \rightarrow +\infty} f(t), \quad f_* = \liminf_{t \rightarrow +\infty} f(t).$$

Throughout this paper, the delay functions satisfy the following condition  $\tau' < 1$ , where  $\tau' = \max_{i=1,2,3,4} \sup_{t \geq 0} \tau'_i(t)$ .

## 2 Positive and global solution

Since  $x(t)$  and  $y(t)$  in system (2) represent population sizes at time  $t$ , they must be non-negative, so, for further study, the first thing is to prove the solution of system (2) has a unique global positive solution.

**Lemma 2.1** *For any given initial value  $(\varphi_1(\theta), \varphi_2(\theta)) \in C([-\bar{\tau}, 0], R_+^2)$ , system (2) has a unique global positive solution  $(x(t), y(t))$  on  $t \geq -\bar{\tau}$  and the solution will remain in  $R_+^2$  with probability 1.*

*Proof* Since the coefficients of system (2) are locally Lipschitz continuous, for any given initial value  $(\varphi_1(\theta), \varphi_2(\theta)) \in C([-\bar{\tau}, 0], R_+^2)$ , there exists a local positive solution  $(x(t), y(t))$

on  $t \in [-\bar{\tau}, \tau_e)$ , where  $\tau_e$  denotes the explosion time. To verify that this solution is global, we only need to prove  $\tau_e = +\infty$  a.s. The proof is similar to [17] by defining functions

$$V_1(x, y) = \sqrt{x} - 1 - \ln \sqrt{x} + \sqrt{y} - 1 - \ln \sqrt{y}, \quad (3)$$

$$\begin{aligned} V_2(x, y) = & \frac{1 + \sigma_{13}^2}{4(1 - \tau')} \int_{t-\tau_1(t)}^t x^2(s - \tau_1(s)) ds + \frac{1 + \sigma_{22}^2}{4(1 - \tau')} \int_{t-\tau_3(t)}^t x^2(s - \tau_3(s)) ds \\ & + \frac{1 + \sigma_{14}^2}{4(1 - \tau')} \int_{t-\tau_2(t)}^t y^2(s - \tau_2(s)) ds + \frac{1 + \sigma_{24}^2}{4(1 - \tau')} \int_{t-\tau_4(t)}^t y^2(s - \tau_4(s)) ds. \end{aligned}$$

Applying Itô's formula to  $V(x, y) = V_1(x, y) + V_2(x, y)$ , we have

$$\begin{aligned} dV(x, y) = & LV_1(x, y) dt + LV_2(x, y) dt + \frac{(\sqrt{x} - 1)\sigma_{11}}{2} dB_{11}(t) + \frac{\sigma_{12}(\sqrt{x} - 1)x}{2} dB_{12}(t) \\ & + \frac{\sigma_{13}(\sqrt{x} - 1)x(t - \tau_1(t))}{2} dB_{13}(t) + \frac{\sigma_{14}(\sqrt{x} - 1)y(t - \tau_2(t))}{2} dB_{14}(t) \\ & + \frac{\sigma_{21}(\sqrt{y} - 1)}{2} dB_{21}(t) + \frac{\sigma_{23}(\sqrt{y} - 1)y}{2} dB_{23}(t) \\ & + \frac{\sigma_{22}(\sqrt{y} - 1)y(t - \tau_3(t))}{2} dB_{22}(t) + \frac{\sigma_{24}(\sqrt{y} - 1)y(t - \tau_4(t))}{2} dB_{24}(t), \end{aligned}$$

where

$$\begin{aligned} LV_1(x, y) = & \frac{(\sqrt{x} - 1)}{2} [r - a_{11}x - a_{12}x(t - \tau_1(t)) - a_{13}y(t - \tau_2(t))] + \frac{(2 - \sqrt{x})\sigma_{11}^2}{8} \\ & + \frac{(2 - \sqrt{x})\sigma_{12}^2x^2}{8} + \frac{(2 - \sqrt{x})\sigma_{13}^2x(t - \tau_1(t))^2}{8} + \frac{(2 - \sqrt{x})\sigma_{14}^2y(t - \tau_2(t))^2}{8} \\ & + \frac{(\sqrt{y} - 1)}{2} [-d + a_{21}x(t - \tau_3(t)) - a_{22}y - a_{23}y(t - \tau_4(t))] + \frac{(2 - \sqrt{y})\sigma_{21}^2}{8} \\ & + \frac{(2 - \sqrt{y})\sigma_{22}^2x(t - \tau_3(t))^2}{8} + \frac{(2 - \sqrt{y})\sigma_{23}^2y^2}{8} + \frac{(2 - \sqrt{y})\sigma_{24}^2y(t - \tau_4(t))^2}{8} \\ \leq & 0.5r\sqrt{x} - 0.5a_{11}x^{1.5} + 0.5a_{11}x + 0.25(1 + \sigma_{13}^2)x^2(t - \tau_1(t)) \\ & + 0.25a_{12}^2(\sqrt{x} - 1)^2 + 0.25a_{13}^2(\sqrt{x} - 1)^2 + 0.25(1 + \sigma_{14}^2)y^2(t - \tau_2(t)) \\ & + 0.125(2 - \sqrt{x})\sigma_{11}^2 + 0.25\sigma_{12}^2x^2 - 0.125\sigma_{12}^2x^{2.5} + 0.5d + 0.25a_{21}^2(\sqrt{y} - 1) \\ & + 0.25(1 + \sigma_{22}^2)x^2(t - \tau_3(t)) - 0.5a_{22}y^{1.5} + 0.5a_{22} + 0.25a_{23}^2(\sqrt{y} - 1)^2 \\ & + 0.25(1 + \sigma_{24}^2)y^2(t - \tau_4(t)) + 0.125(2 - \sqrt{y})\sigma_{21}^2 + 0.25\sigma_{23}^2y^2 - 0.125\sigma_{23}^2y^{2.5}, \end{aligned}$$

because  $\tau' < 1$ , we have

$$\begin{aligned} LV_2(x, y) = & \frac{1 + \sigma_{13}^2}{4(1 - \tau')} x^2(t) - \frac{(1 - \tau'_1(t))(1 + \sigma_{13}^2)}{4(1 - \tau')} x^2(t - \tau_1(t)) + \frac{1 + \sigma_{22}^2}{4(1 - \tau')} x^2(t) \\ & - \frac{(1 - \tau'_3(t))(1 + \sigma_{22}^2)}{4(1 - \tau')} x^2(t - \tau_3(t)) - \frac{(1 - \tau'_2(t))(1 + \sigma_{22}^2)}{4(1 - \tau')} y^2(t - \tau_2(t)) \\ & + \frac{1 + \sigma_{22}^2}{4(1 - \tau')} y^2(t) + \frac{1 + \sigma_{24}^2}{4(1 - \tau')} y^2(t) - \frac{(1 - \tau'_4(t))(1 + \sigma_{24}^2)}{4(1 - \tau')} y^2(t - \tau_4(t)) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1+\sigma_{13}^2}{4(1-\tau')}x^2(t) - \frac{1+\sigma_{13}^2}{4}x^2(t-\tau_1(t)) + \frac{1+\sigma_{22}^2}{4(1-\tau')}x^2(t) - \frac{1+\sigma_{22}^2}{4}x^2(t-\tau_3(t)) \\
&\quad + \frac{1+\sigma_{14}^2}{4(1-\tau')}y^2(t) - \frac{1+\sigma_{14}^2}{4}y^2(t-\tau_2(t)) + \frac{1+\sigma_{24}^2}{4(1-\tau')}y^2(t) \\
&\quad - \frac{1+\sigma_{24}^2}{4}y^2(t-\tau_4(t)),
\end{aligned}$$

then

$$\begin{aligned}
LV(x, y) &= LV_1(x, y) + LV_2(x, y) \\
&= 0.5r\sqrt{x} - 0.5a_{11}x^{1.5} + 0.5a_{11}x + \frac{1+\sigma_{13}^2}{4(1-\tau')}x^2 + 0.25a_{12}^2(\sqrt{x}-1)^2 \\
&\quad + 0.25a_{13}^2(\sqrt{x}-1)^2 + \frac{1+\sigma_{14}^2}{4(1-\tau')}y^2 + 0.125(2-\sqrt{x})\sigma_{11}^2 + 0.25\sigma_{12}^2x^2 \\
&\quad - 0.125\sigma_{12}^2x^{2.5} + 0.5d + 0.25a_{21}^2(\sqrt{y}-1) + \frac{1+\sigma_{22}^2}{4(1-\tau')}x^2 - 0.5a_{22}y^{1.5} \\
&\quad + 0.5a_{22} + 0.25a_{23}^2(\sqrt{y}-1)^2 + \frac{1+\sigma_{24}^2}{4(1-\tau')}y^2(t) + 0.125(2-\sqrt{y})\sigma_{21}^2 \\
&\quad + 0.25\sigma_2^2y^2 - 0.125\sigma_2^2y^{2.5} \\
&\leq K.
\end{aligned}$$

There is a positive constant  $K$  satisfying the above inequality. By a similar proof to [17], we can obtain the desired assertion, and it is omitted in here.  $\square$

### 3 Stochastically ultimate boundedness

**Definition 3.1** (See [20]) The solution  $X(t) = (x(t), y(t))$  of system (2) is said to be stochastically ultimately bounded, if for any  $\epsilon \in (0, 1)$ , there is a positive constant  $\delta = \delta(\epsilon)$ , such that, for any initial value  $(x(\theta), y(\theta)) \in C([- \bar{\tau}, 0], R_+^2)$ , the solution  $X(t)$  to (2) has the property

$$\limsup_{t \rightarrow +\infty} P(|X(t)| = \sqrt{x^2(t) + y^2(t)} > \delta) < \epsilon.$$

**Lemma 3.2** Let  $\theta \in (0, 1)$ . Then there exists a positive constant  $K = K(\theta)$ , which is independent of the initial value  $(\phi_1(\theta), \phi_2(\theta)) \in C([- \bar{\tau}, 0], R_+^2)$ , such that the solution  $X = (x, y)$  of system (2) has

$$\limsup E|X|^\theta \leq K.$$

*Proof* Define  $V(x, y) = x^\theta + y^\theta$ . Applying Itô's formula to system (2), we obtain

$$\begin{aligned}
dV(x, y) &= LV(x, y)dt + \sigma_{11}\theta x^\theta dB_{11}(t) + \sigma_{12}\theta x^{\theta+1} dB_{12}(t) + \sigma_{13}\theta x^\theta x(t-\tau_1(t)) dB_{13}(t) \\
&\quad + \sigma_{14}\theta x^\theta y(t-\tau_2(t)) dB_{14}(t) + \sigma_{21}\theta y^\theta dB_{21}(t) + \sigma_{23}\theta y^{\theta+1} dB_{23}(t) \\
&\quad + \sigma_{22}\theta y^\theta x(t-\tau_3(t)) dB_{22}(t) + \sigma_{24}\theta y^\theta y(t-\tau_4(t)) dB_{24}(t),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
 LV(x, y) &= \theta x^\theta \left[ r - a_{11}x - a_{12}x(t - \tau_1(t)) - a_{13}y(t - \tau_2(t)) \right] + \frac{\theta(\theta - 1)\sigma_{11}^2}{2} x^\theta \\
 &\quad + \frac{\theta(\theta - 1)\sigma_{12}^2}{2} x^{\theta+2} + \frac{\theta(\theta - 1)\sigma_{13}^2}{2} x^\theta x^2(t - \tau_1(t)) + \frac{\theta(\theta - 1)\sigma_{14}^2}{2} x^\theta y^2(t - \tau_2(t)) \\
 &\quad + \theta y^\theta \left[ -d + a_{21}x(t - \tau_3(t)) - a_{22}y(t) - a_{23}y(t - \tau_4(t)) \right] dt + \frac{\theta(\theta - 1)\sigma_{21}^2}{2} y^\theta \\
 &\quad + \frac{\theta(\theta - 1)\sigma_{23}^2}{2} y^{\theta+2} + \frac{\theta(\theta - 1)\sigma_{22}^2}{2} y^\theta x^2(t - \tau_3(t)) + \frac{\theta(\theta - 1)\sigma_{24}^2}{2} y^\theta y^2(t - \tau_4(t)) \\
 &\leq r\theta x^\theta + \theta a_{21}y^\theta x(t - \tau_3(t)) + \frac{\theta(\theta - 1)\sigma_{12}^2}{2} x^{\theta+2} + \frac{\theta(\theta - 1)\sigma_{23}^2}{2} y^{\theta+2} \\
 &\leq r\theta x^\theta + \frac{(\theta a_{21})^2}{4} y^{2\theta} + \frac{\theta(\theta - 1)\sigma_{12}^2}{2} x^{\theta+2} + \frac{\theta(\theta - 1)\sigma_{23}^2}{2} y^{\theta+2} + x^2(t - \tau_3(t)) \\
 &= F(x, y) - V(x, y) + x^2(t - \tau_3(t)) - \frac{e^{\bar{\tau}}}{1 - \tau'} x^2(t),
 \end{aligned}$$

where  $F(x, y) = r\theta x^\theta + \frac{(\theta a_{21})^2}{4} y^{2\theta} + \frac{\theta(\theta - 1)\sigma_{12}^2}{2} x^{\theta+2} + \frac{\theta(\theta - 1)\sigma_{23}^2}{2} y^{\theta+2} + \frac{1 - \tau' + e^{\bar{\tau}}}{1 - \tau'} x^2 + y^2$ . Since  $\theta \in (0, 1)$ ,  $2\theta < \theta + 2$ , hence we easily see that  $F(x, y)$  is bounded in  $R_+^2$ , namely,

$$F(x, y) \leq M, \quad \forall X \in R_+^2.$$

Hence, we have

$$\begin{aligned}
 dV(x, y) &\leq \left[ M - V(x, y) + x^2(t - \tau_3(t)) - \frac{e^{\bar{\tau}}}{1 - \tau'} x^2(t) \right] dt \\
 &\quad + \sigma_{11}\theta x^\theta dB_{11}(t) + \sigma_{12}\theta x^{\theta+1} dB_{12}(t) + \sigma_{13}\theta x^\theta x(t - \tau_1(t)) dB_{13}(t) \\
 &\quad + \sigma_{14}\theta x^\theta y(t - \tau_2(t)) dB_{14}(t) + \sigma_{21}\theta y^\theta dB_{21}(t) + \sigma_{23}\theta y^{\theta+1} dB_{23}(t) \\
 &\quad + \sigma_{22}\theta y^\theta x(t - \tau_3(t)) dB_{22}(t) + \sigma_{24}\theta y^\theta y(t - \tau_4(t)) dB_{24}(t). \tag{5}
 \end{aligned}$$

By virtue of (5), we use Itô's formula again for  $e^t V(x, y)$ . We have

$$\begin{aligned}
 de^t V(x, y) &= e^t [V(x, y) dt + dV(x, y)] \\
 &\leq e^t \left[ M + x^2(t - \tau_3(t)) - \frac{e^{\bar{\tau}}}{1 - \tau'} x^2(t) \right] dt \\
 &\quad + \sigma_{11}\theta e^t x^\theta dB_{11}(t) + \sigma_{12}\theta e^t x^{\theta+1} dB_{12}(t) + \sigma_{13}\theta e^t x^\theta x(t - \tau_1(t)) dB_{13}(t) \\
 &\quad + \sigma_{14}\theta e^t x^\theta y(t - \tau_2(t)) dB_{14}(t) + \sigma_{21}\theta e^t y^\theta dB_{21}(t) + \sigma_{23}\theta e^t y^{\theta+1} dB_{23}(t) \\
 &\quad + \sigma_{22}\theta e^t y^\theta x(t - \tau_3(t)) dB_{22}(t) + \sigma_{24}\theta e^t y^\theta y(t - \tau_4(t)) dB_{24}(t),
 \end{aligned}$$

then

$$\begin{aligned}
 e^t EV(x, y) &\leq V(x(0), y(0)) + Me^t - E \int_0^t \frac{e^{\bar{\tau}}}{1 - \tau'} e^s x^2(s) ds + E \int_0^t e^s x^2(s - \tau_3(s)) ds \\
 &\leq V(x(0), y(0)) + Me^t - E \int_0^t \frac{e^{\bar{\tau}}}{1 - \tau'} e^s x^2(s) ds
 \end{aligned}$$

$$\begin{aligned}
& + E \left[ \frac{e^{\bar{\tau}}}{1-\tau'} \int_0^t e^{(s-\tau_3(s))} x^2(s-\tau_3(s)) d(s-\tau_3(s)) \right] \\
& \leq V(x(0), y(0)) + Me^t + \frac{e^{\bar{\tau}}}{1-\tau'} \int_{-\tau_3(0)}^0 e^s x^2(s) ds.
\end{aligned}$$

We easily infer that

$$\limsup_{t \rightarrow +\infty} EV(x, y) \leq M.$$

On the other hand,

$$|X|^\theta = (x^2 + y^2)^{\frac{\theta}{2}} \leq 2^{\frac{\theta}{2}} \max(x^\theta, y^\theta) \leq 2^{\frac{\theta}{2}} V(x, y).$$

Hence,

$$\limsup_{t \rightarrow +\infty} E|X|^\theta \leq 2^{\frac{\theta}{2}} M := K.$$

The proof is completed.  $\square$

**Theorem 3.3** *Let  $\theta \in (0, 1)$ , and system (2) is stochastically ultimate bounded.*

*Proof* From Lemma 3.2, we know there is a positive constant  $K$ , such that  $\limsup_{t \rightarrow +\infty} E|X|^\theta \leq K$ . For any  $\epsilon > 0$ , let  $\delta = K^2/\epsilon^2$ . By Chebyshev's inequality, we have

$$P\{|X(t)| > \delta\} \leq \frac{\sqrt{|X(t)|}}{\sqrt{\delta}}.$$

Then

$$\limsup_{t \rightarrow +\infty} P\{|X(t)| > \delta\} \leq \epsilon. \quad \square$$

#### 4 Asymptotic pathwise estimation

Lemma 2.1 and Theorem 3.3 show that the global positive solution of system (2) is stochastically ultimate bounded. This nice property motivates us to further study the system. In this section, we will study the pathwise properties of system (2).

**Theorem 4.1** *For any given initial value satisfying  $(\phi_1(\theta), \phi_2(\theta)) \in C([-\bar{\tau}, 0], R_+^2)$ , the solutions  $(x(t), y(t))$  of system (2) have the properties*

$$\limsup_{t \rightarrow +\infty} \frac{\ln x(t)}{\ln t} \leq 1, \quad \limsup_{t \rightarrow +\infty} \frac{\ln y(t)}{\ln t} \leq 1.$$

*Proof* Applying Itô's formula to system (2), we have

$$\begin{aligned}
de^t \ln V(x, y) &= e^t \ln(x + y) dt + e^t d \ln(x + y) \\
&= e^t \ln(x + y) + \frac{e^t x}{x + y} [r - a_{11}x - a_{12}x(t - \tau_1(t)) - a_{13}y(t - \tau_2(t))] dt
\end{aligned}$$

$$\begin{aligned}
& -0.5 \frac{e^t x^2}{(x+y)^2} [\sigma_{11}^2 + \sigma_{12}^2 x^2 + \sigma_{13}^2 x^2 (t - \tau_1(t)) + \sigma_{14}^2 y^2 (t - \tau_2(t))] dt \\
& + \frac{e^t x}{x+y} [\sigma_{11} dB_{11}(t) + \sigma_{12} x dB_{12}(t) + \sigma_{13} x (t - \tau_1(t)) dB_{13}(t) \\
& + \sigma_{14} y (t - \tau_2(t)) dB_{14}(t)] \\
& + \frac{e^t y}{x+y} [-d + a_{21} x (t - \tau_3(t)) - a_{22} y (t) - a_{23} y (t - \tau_4(t))] dt \\
& -0.5 \frac{e^t y^2}{(x+y)^2} [\sigma_{21}^2 + \sigma_{23}^2 y^2 + \sigma_{22}^2 x^2 (t - \tau_3(t)) + \sigma_{24}^2 y^2 (t - \tau_4(t))] dt \\
& + \frac{e^t y}{x+y} [\sigma_{21} dB_{21}(t) + \sigma_{23} y dB_{23}(t) + \sigma_{22} x (t - \tau_3(t)) dB_{22}(t) \\
& + \sigma_{24} y (t - \tau_4(t)) dB_{24}(t)],
\end{aligned}$$

thus

$$\begin{aligned}
e^t \ln(x(t) + y(t)) &= \ln(x(0) + y(0)) + \int_0^t e^s \ln(x(s) + y(s)) ds \\
&+ \int_0^t \frac{e^s x(s)}{x(s) + y(s)} [r - a_{11} x(s) - a_{12} x(s - \tau_1(s)) - a_{13} y(s - \tau_2(s))] ds \\
&- 0.5 \int_0^t \frac{e^s x^2(s)}{(x(s) + y(s))^2} [\sigma_{11}^2 + \sigma_{12}^2 x^2(s) + \sigma_{13}^2 x^2(s - \tau_1(s)) \\
&+ \sigma_{14}^2 y^2(s - \tau_2(s))] ds \\
&+ \int_0^t \frac{e^s y(s)}{x(s) + y(s)} [-d + a_{21} x(s - \tau_3(s)) - a_{22} y(s) - a_{23} y(s - \tau_4(s))] ds \\
&- \int_0^t 0.5 \frac{e^s y^2(s)}{(x(s) + y(s))^2} [\sigma_{21}^2 + \sigma_{23}^2 y^2(s) + \sigma_{22}^2 x^2(s - \tau_3(s)) \\
&+ \sigma_{24}^2 y^2(s - \tau_4(s))] ds \\
&+ M_1(t) + M_2(t) + M_3(t) + M_4(t) \\
&+ M_5(t) + M_6(t) + M_7(t) + M_8(t),
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
M_1(t) &= \int_0^t \sigma_{11} \frac{e^s x(s)}{x(s) + y(s)} dB_{11}(s), & M_2(t) &= \int_0^t \sigma_{12} \frac{e^s x^2(s)}{x(s) + y(s)} dB_{12}(s), \\
M_3(t) &= \int_0^t \sigma_{13} \frac{e^s x(s)x(s - \tau_1(s))}{x(s) + y(s)} dB_{13}(s), & M_4(t) &= \int_0^t \sigma_{14} \frac{e^s x(s)y(s - \tau_2(s))}{x(s) + y(s)} dB_{14}(s), \\
M_5(t) &= \int_0^t \sigma_{21} \frac{e^s y(s)}{x(s) + y(s)} dB_{21}(s), & M_6(t) &= \int_0^t \sigma_{22} \frac{e^s y(s)x(s - \tau_3(s))}{x(s) + y(s)} dB_{22}(s), \\
M_7(t) &= \int_0^t \sigma_{23} \frac{e^s y^2(s)}{x(s) + y(s)} dB_{23}(s), & M_8(t) &= \int_0^t \sigma_{24} \frac{e^s y(s)y(s - \tau_4(s))}{x(s) + y(s)} dB_{24}(s).
\end{aligned}$$

Let  $M(t) = \sum_{i=1}^8 M_i(t)$ . Then  $M(t)$  is a local martingale, and the quadratic form of  $M(t)$  is

$$\begin{aligned} \langle M(t), M(t) \rangle &= 0.5 \int_0^t \frac{e^{2s} x^2(s)}{(x(s) + y(s))^2} [\sigma_{11}^2 + \sigma_{12}^2 x^2(s) + \sigma_{13}^2 x^2(s - \tau_1(s)) \\ &\quad + \sigma_{14}^2 y^2(s - \tau_2(s))] ds \\ &\quad + 0.5 \int_0^t \frac{e^{2s} y^2(s)}{(x(s) + y(s))^2} [\sigma_{21}^2 + \sigma_{23}^2 y^2(s) + \sigma_{22}^2 x^2(s - \tau_3(s)) \\ &\quad + \sigma_{24}^2 y^2(s - \tau_4(s))] ds. \end{aligned}$$

According to the exponential martingale inequality (see [17]), for any positive constants  $T$ ,  $\alpha$ , and  $\beta$  we have

$$P \left\{ \sup_{0 \leq t \leq T} \left[ M(t) - \frac{\alpha}{2} \langle M(t), M(t) \rangle \right] > \beta \right\} \leq e^{-\alpha\beta}. \quad (7)$$

We choose  $T = \gamma k$ ,  $\alpha = e^{-\gamma k}$ ,  $\beta = \rho e^{\gamma k} \ln k$ , and then

$$P \left\{ \sup_{0 \leq t \leq \gamma k} \left[ M(t) - \frac{e^{-\gamma k}}{2} \langle M(t), M(t) \rangle \right] > \rho e^{\gamma k} \ln k \right\} \leq k^{-\rho},$$

where  $\rho > 1$  and  $\gamma > 0$ . By virtue of the Borel-Cantelli lemma [13], for almost all  $\omega \in \Omega$ , there exists a  $k_0(\omega)$ , such that, for all  $k \geq k_0(\omega)$ ,

$$M(t) \leq \frac{e^{-\gamma k}}{2} \langle M(t), M(t) \rangle + \rho e^{\gamma k} \ln k, \quad 0 \leq t \leq \gamma k.$$

Substituting the above inequality into equation (6), we have

$$\begin{aligned} e^t \ln(x(t) + y(t)) &\leq \ln(x(0) + y(0)) + \int_0^t e^s \ln(x(s) + y(s)) ds + \rho e^{\gamma k} \ln k \\ &\quad + \int_0^t \frac{e^s x(s)}{x(s) + y(s)} [r - a_{11} x(s)] ds \\ &\quad + \int_0^t \frac{e^s y(s)}{x(s) + y(s)} [a_{21} x(s - \tau_3(s)) - a_{22} y(s)] ds \\ &\quad - 0.5 \int_0^t \frac{e^s (1 - e^{s-\gamma k}) x^2(s)}{(x(s) + y(s))^2} [\sigma_{11}^2 + \sigma_{12}^2 x^2(s) + \sigma_{13}^2 x^2(s - \tau_1(s)) \\ &\quad + \sigma_{14}^2 y^2(s - \tau_2(s))] ds \\ &\quad - 0.5 \int_0^t \frac{e^s (1 - e^{s-\gamma k}) y^2(s)}{(x(s) + y(s))^2} [\sigma_{21}^2 + \sigma_{23}^2 y^2(s) + \sigma_{22}^2 x^2(s - \tau_3(s)) \\ &\quad + \sigma_{24}^2 y^2(s - \tau_4(s))] ds. \end{aligned} \quad (8)$$

Because  $0 \leq t \leq \gamma k$ ,  $k \geq K_0(\omega)$ , we know  $s \leq \gamma k$ . Note that

$$\begin{aligned} \int_0^t e^s x(s - \tau_3(s)) ds &\leq \int_0^t \frac{e^{s-\tau_3(s)} e^{\tau_3(s)} x(s - \tau_3(s))}{1 - \tau'} d(s - \tau_3(s)) \leq \frac{e^{\bar{\tau}}}{1 - \tau'} \int_{-\tau_3(0)}^{t-\tau_3(t)} e^s x(s) ds \\ &\leq \frac{e^{\bar{\tau}}}{1 - \tau'} \int_{-\tau_3(0)}^0 e^s x(s) ds + \frac{e^{\bar{\tau}}}{1 - \tau'} \int_0^t e^s x(s) ds. \end{aligned}$$



Then we can rewrite (8)

$$\begin{aligned}
 e^t \ln(x(t) + y(t)) &\leq \ln(x(0) + y(0)) + \frac{a_{21}e^{\bar{\tau}}}{1-\tau'} \int_{-\tau_3(0)}^0 e^s x(s) ds \\
 &\quad + \int_0^t e^s [\ln(x(s) + y(s)) + r] ds + \frac{a_{21}e^{\bar{\tau}}}{1-\tau'} \int_0^t e^s x(s) ds \\
 &\quad - \int_0^t \frac{\min\{\sigma_{12}^2, \sigma_{23}^2\} e^s (1 - e^{s-\gamma k})(x^4(s) + y^4(s))}{2(x(s) + y(s))^2} ds + \rho e^{\gamma k} \ln k \\
 &\leq \ln(x(0) + y(0)) + \frac{a_{21}e^{\bar{\tau}}}{1-\tau'} \int_{-\tau_3(0)}^0 e^s x(s) ds + \rho e^{\gamma k} \ln k \\
 &\quad + \int_0^t e^s \left[ \ln(x + y) + r + \frac{a_{21}e^{\bar{\tau}}}{1-\tau'} [x + y] \right. \\
 &\quad \left. - \frac{\min\{\sigma_{12}^2, \sigma_{23}^2\} (1 - e^{s-\gamma k})}{16} (x + y)^2 \right] ds. \tag{9}
 \end{aligned}$$

Obviously, there is a positive constant  $K_1$  such that

$$\ln(x + y) + r + \frac{a_{21}e^{\bar{\tau}}}{1-\tau'} [x + y] - \frac{\min\{\sigma_{12}^2, \sigma_{23}^2\} (1 - e^{t-\gamma k})}{16} (x + y)^2 \leq K_1.$$

Let  $C = \ln(x(0) + y(0)) + \frac{a_{21}e^{\bar{\tau}}}{1-\tau'} \int_{-\tau_3(0)}^0 e^s x(s) ds$ . Then, for all  $0 \leq t \leq \gamma k$ ,  $k \geq K_0(\omega)$ ,

$$e^t \ln(x(t) + y(t)) \leq C + \int_0^t K_1 e^s ds + \rho e^{\gamma k} \ln k.$$

If  $\gamma(k-1) \leq t \leq \gamma k$  and  $k \geq k_0(\omega)$ , we obtain

$$\frac{\ln(x(t) + y(t))}{\ln t} \leq \frac{C}{e^t \ln t} + \frac{K_1 e^t - K_1}{e^t \ln t} + \frac{\rho e^{\gamma k} \ln k}{e^t \ln t}.$$

We infer

$$\limsup_{t \rightarrow +\infty} \frac{\ln(x(t) + y(t))}{\ln t} \leq \rho e^{\gamma}. \tag{10}$$

Letting  $\rho \rightarrow 1$ ,  $\gamma \rightarrow 0$ , by (10) we have

$$\limsup_{t \rightarrow +\infty} \frac{\ln(x(t))}{\ln t} \leq 1, \quad \limsup_{t \rightarrow +\infty} \frac{\ln(y(t))}{\ln t} \leq 1, \quad \text{a.s.}$$

## 5 Persistence and extinction

It is critical to discuss the persistence and extinction for an ecological population system. In this section, we will mainly investigate the persistence in mean and extinction of system (2).

### Theorem 5.1

- (1) If  $r < 0.5\sigma_{11}^2$ , populations  $x$  and  $y$  of system (2) will go extinct a.s., namely,
- $$\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow +\infty} y(t) = 0, \text{ a.s.}$$

- (2) If  $r > 0.5\sigma_{11}^2$ , population  $x$  is weakly persistent a.s., namely,  $x^* > 0$ .
- (a) If  $r > 0.5\sigma_{11}^2$  and  $\frac{a_{21}(r-0.5\sigma_{11}^2)}{(1-\tau')a_{11}} - (d + 0.5\sigma_{21}^2) < 0$ , population  $y$  of system (2) will go extinct a.s., namely,  $\lim_{t \rightarrow +\infty} y(t) = 0$ , a.s.
- (b) If  $r > 0.5\sigma_{11}^2$  and  $\frac{a_{21}(r-0.5\sigma_{11}^2)}{(1-\tau')a_{11}} - (d + 0.5\sigma_{21}^2) > 0$ , then  $\langle y(t) \rangle^* \leq \frac{\frac{a_{21}(r-0.5\sigma_{11}^2)}{(1-\tau')a_{11}} - (d+0.5\sigma_{21}^2)}{a_{22}}$ , a.s.
- (3) If  $r = 0.5\sigma_{11}^2$ , then the population  $x$  is nonpersistent in the mean a.s., namely,  $\langle x \rangle^* = 0$ .

*Proof* Applying Itô's formula to system (2), we have

$$\begin{aligned} d \ln x &= [r - a_{11}x - a_{12}x(t - \tau_1(t)) - a_{13}y(t - \tau_2(t))] dt \\ &\quad - 0.5[\sigma_{11}^2 + \sigma_{12}^2x^2 + \sigma_{13}^2x^2(t - \tau_1(t)) + \sigma_{14}^2y^2(t - \tau_2(t))] dt \\ &\quad + \sigma_{11} dB_{11}(t) + \sigma_{12}x dB_{12}(t) + \sigma_{13}x(t - \tau_1(t)) dB_{13}(t) \\ &\quad + \sigma_{14}y(t - \tau_2(t)) dB_{14}(t), \\ d \ln y &= [-d + a_{21}x(t - \tau_3(t)) - a_{22}y(t) - a_{23}y(t - \tau_4(t))] dt \\ &\quad - 0.5[\sigma_{21}^2 + \sigma_{23}^2y^2 + \sigma_{22}^2x^2(t - \tau_3(t)) + \sigma_{24}^2y^2(t - \tau_4(t))] dt \\ &\quad + \sigma_{21} dB_{21}(t) + \sigma_{23}y dB_{23}(t) + \sigma_{22}x(t - \tau_3(t)) dB_{22}(t) \\ &\quad + \sigma_{24}y(t - \tau_4(t)) dB_{24}(t). \end{aligned}$$

Integrating both sides from 0 to  $t$ , we get

$$\begin{aligned} \ln x(t)/x(0) &= \int_0^t r - a_{11}x(s) - a_{12}x(s - \tau_1(s)) - a_{13}y(s - \tau_2(s)) ds \\ &\quad - 0.5 \int_0^t \sigma_{11}^2 + \sigma_{12}^2x^2(s) + \sigma_{13}^2x^2(s - \tau_1(s)) + \sigma_{14}^2y^2(s - \tau_2(s)) ds \\ &\quad + \int_0^t \sigma_{11} dB_{11}(s) + \int_0^t \sigma_{12}x(s) dB_{12}(s) + \int_0^t \sigma_{13}x(s - \tau_1(s)) dB_{13}(s) \\ &\quad + \int_0^t \sigma_{14}y(s - \tau_2(s)) dB_{14}(s), \end{aligned} \quad (11)$$

$$\begin{aligned} \ln y(t)/y(0) &= \int_0^t -d + a_{21}x(s - \tau_3(s)) - a_{22}y(s) - a_{23}y(s - \tau_4(s)) ds \\ &\quad - 0.5 \int_0^t \sigma_{21}^2 + \sigma_{23}^2y^2(s) + \sigma_{22}^2x^2(s - \tau_3(s)) + \sigma_{24}^2y^2(s - \tau_4(s)) ds \\ &\quad + \int_0^t \sigma_{21} dB_{21}(s) + \int_0^t \sigma_{23}y(s) dB_{23}(s) + \int_0^t \sigma_{22}x(s - \tau_3(s)) dB_{22}(s) \\ &\quad + \int_0^t \sigma_{24}y(s - \tau_4(s)) dB_{24}(s), \end{aligned} \quad (12)$$

where

$$\begin{aligned} N_1(t) &= \int_0^t \sigma_{11} dB_{11}(s), & N_2(t) &= \int_0^t \sigma_{12}x(s) dB_{12}(s), \\ N_3(t) &= \int_0^t \sigma_{13}x(s - \tau_1(s)) dB_{13}(s), & N_4(t) &= \int_0^t \sigma_{14}y(s - \tau_2(s)) dB_{14}(s), \end{aligned}$$

$$\begin{aligned} N_5(t) &= \int_0^t \sigma_{21} dB_{21}(s), & N_6(t) &= \int_0^t \sigma_{22} x(s - \tau_3(s)) dB_{22}(s), \\ N_7(t) &= \int_0^t \sigma_{23} y(s) dB_{23}(s), & N_8(t) &= \int_0^t \sigma_{24} y(s - \tau_4(s)) dB_{24}(s). \end{aligned}$$

Noting that  $N_i(t)$  is a local martingale, we have

$$\begin{aligned} \langle N_2(t), N_2(t) \rangle &= \int_0^t \sigma_{21}^2 x^2(s) ds, & \langle N_3(t), N_3(t) \rangle &= \int_0^t \sigma_{23}^2 x^2(s - \tau_1(s)) ds, \\ \langle N_4(t), N_4(t) \rangle &= \int_0^t \sigma_{14}^2 y^2(s - \tau_2(s)) ds, & \langle N_6(t), N_6(t) \rangle &= \int_0^t \sigma_{22}^2 x^2(s - \tau_3(s)) ds, \\ \langle N_7(t), N_7(t) \rangle &= \int_0^t \sigma_{23}^2 y^2(s) ds, & \langle N_8(t), N_8(t) \rangle &= \int_0^t \sigma_{24}^2 y^2(s - \tau_4(s)) ds. \end{aligned}$$

By the exponential martingale inequality (7) (choose  $T = k$ ,  $\alpha = 1$ ,  $\beta = 2 \ln k$ ), we have

$$P \left\{ \sup_{0 \leq t \leq \gamma k} \left[ N_i(t) - \frac{1}{2} \langle N_i(t), N_i(t) \rangle \right] > 2 \ln k \right\} \leq 1/k^2, \quad i = 2, 3, 4, 6, 7, 8.$$

By virtue of the Borel-Cantelli lemma, for almost all  $\omega \in \Omega$ , there exists a  $k_0(\omega)$  such that, for all  $k \geq k_0(\omega)$ ,

$$N_i(t) \leq \frac{1}{2} \langle N_i(t), N_i(t) \rangle + 2 \ln k, \quad 0 \leq t \leq k, k \geq k_0(\omega).$$

Substituting above inequalities into (11) and (12), we have

$$\begin{aligned} \ln x(t)/x(0) &\leq \int_0^t r - 0.5\sigma_{11}^2 - a_{11}x(s) - a_{12}x(s - \tau_1(s)) - a_{13}y(s - \tau_2(s)) ds \\ &\quad + N_1(t) + 6 \ln k \\ &\leq \int_0^t r - 0.5\sigma_{11}^2 ds + N_1(t) + 6 \ln k, \end{aligned} \tag{13}$$

$$\begin{aligned} \ln y(t)/y(0) &\leq \int_0^t -d - 0.5\sigma_{21}^2 + a_{21}x(s - \tau_3(s)) - a_{22}y(s) - a_{23}y(s - \tau_4(s)) ds \\ &\quad + N_5(t) + 6 \ln k \\ &\leq \int_0^t -d - 0.5\sigma_{21}^2 + a_{21}x(s - \tau_3(s)) ds + N_5(t) + 6 \ln k, \end{aligned} \tag{14}$$

for all  $0 \leq t \leq k$ ,  $k \geq k_0$ . When  $0 \leq k - 1 \leq t \leq k$ , by (13)

$$\frac{\ln x(t)/x(0)}{t} \leq r - 0.5\sigma_{11}^2 + \frac{N_1(t)}{t} + \frac{6 \ln k}{k - 1}.$$

(1) If  $r - 0.5\sigma_{11}^2 < 0$ , have  $\lim_{t \rightarrow +\infty} x(t) = 0$ . In combination with (14) we easily obtain  $\lim_{t \rightarrow +\infty} y(t) = 0$  a.s.

(2) If  $r - 0.5\sigma_{11}^2 > 0$ , suppose this assertion is not true, that is to say  $P(S) > 0$ , where  $S = \{x^* = 0\}$ . Then for  $\omega \in S$ , we have  $\lim_{t \rightarrow +\infty} x(t, \omega) = 0$ , hence,

$$\begin{aligned} \lim_{t \rightarrow +\infty} \langle x(t, \omega) \rangle &= \lim_{t \rightarrow +\infty} \langle x^2(t, \omega) \rangle = \lim_{t \rightarrow +\infty} \langle x(t - \tau_1(t), \omega) \rangle = \lim_{t \rightarrow +\infty} \langle x^2(t - \tau_1(t), \omega) \rangle \\ &= \lim_{t \rightarrow +\infty} \langle y(t - \tau_2(t), \omega) \rangle = \lim_{t \rightarrow +\infty} \langle y^2(t - \tau_2(t), \omega) \rangle = 0. \end{aligned} \quad (15)$$

Then, according to the law of large numbers for local martingales,  $\frac{N_i(t)}{t} = 0$ . By virtue of (12) and (15) we have

$$\left[ \frac{\ln x(t)/x(0)}{t} \right]^* = r - 0.5\sigma_{11}^2 > 0. \quad (16)$$

By Theorem 4.1 we obtain

$$\left[ \frac{\ln x(t)/x(0)}{t} \right]^* \leq \left[ \frac{\ln x(t)/x(0)}{\ln t} \right]^* \left[ \frac{\ln t}{t} \right]^* \leq 0.$$

Obviously, this is a contradiction, namely  $x^* > 0$ .

For all  $0 \leq t \leq k$ ,  $k \geq k_0$ . Then,  $\forall \epsilon > 0$ , there exists a  $T > 0$ , when  $0 < k - 1 \leq T \leq t \leq k$  and  $[\frac{6 \ln k}{t}]^* < \epsilon$ .

It follows from (13) that

$$\frac{\ln x(t)/x(0)}{t} \leq r - 0.5\sigma_{11}^2 + \epsilon - a_{11} \langle x(t) \rangle + \frac{N_1(t)}{t}, \quad t \geq T. \quad (17)$$

If  $r - 0.5\sigma_{11}^2 > 0$ , we have  $\langle x(t) \rangle^* \leq \frac{r - 0.5\sigma_{11}^2}{a_{11}}$ .

In the same way, by (14)

$$\begin{aligned} \frac{\ln y(t)/y(0)}{t} &\leq -d - 0.5\sigma_{21}^2 + t^{-1} \int_0^t a_{21} x(s - \tau_3(s)) ds - a_{22} \langle y(t) \rangle + \frac{N_5(t)}{t} + \frac{6 \ln k}{t} \\ &\leq -d - 0.5\sigma_{21}^2 + \frac{a_{21}}{t(1 - \tau')} \left[ \int_{-\tau(0)}^0 x(s) ds + \int_0^t x(s) ds \right] - a_{22} \langle y(t) \rangle \\ &\quad + \frac{N_5(t)}{t} + \frac{6 \ln k}{t}, \end{aligned}$$

for all  $0 \leq t \leq k$ ,  $k \geq k_0$ .  $\forall \epsilon > 0$ , there exists a  $T > 0$ , when  $0 < k - 1 \leq T \leq t \leq k$ ,  $[\frac{6 \ln k}{t}]^* < \epsilon/2$  and  $\frac{a_{21}}{t(1 - \tau')} \int_{-\tau(0)}^0 x(s) ds < \epsilon/2$ .

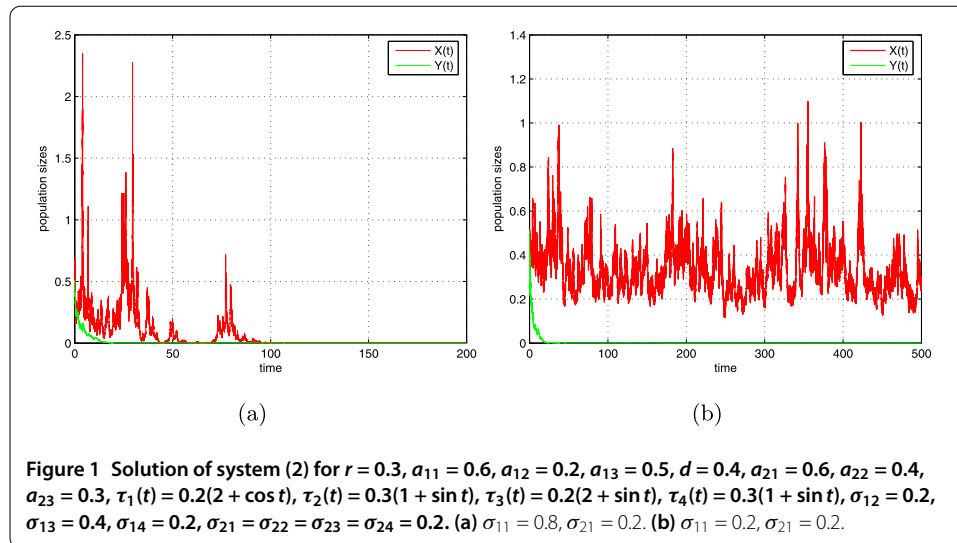
We have

$$\frac{\ln y(t)/y(0)}{t} \leq \frac{a_{21}(r - 0.5\sigma_{11}^2)}{a_{11}(1 - \tau')} - (d + 0.5\sigma_{21}^2) - a_{22} \langle y(t) \rangle + \frac{N_5(t)}{t}.$$

(a) If  $\frac{a_{21}(r - 0.5\sigma_{11}^2)}{a_{11}(1 - \tau')} - (d + 0.5\sigma_{21}^2) < 0$ , we have  $\lim_{t \rightarrow +\infty} y(t) = 0$ , a.s. (b) If  $\frac{a_{21}(r - 0.5\sigma_{11}^2)}{a_{11}(1 - \tau')} - (d + 0.5\sigma_{21}^2) > 0$ , we have  $\langle y(t) \rangle^* \leq \frac{\frac{a_{21}(r - 0.5\sigma_{11}^2)}{a_{11}(1 - \tau')} - (d + 0.5\sigma_{21}^2)}{a_{22}}$ , a.s.

(3) If  $r - 0.5\sigma_{11}^2 = 0$ , by (17) we have  $\langle x(t) \rangle^* \leq \frac{\epsilon}{a_{11}}$ . Since  $\epsilon$  is arbitrary, we have  $\langle x(t) \rangle^* = 0$ , a.s.

The proof of Theorem 5.1 is completed.  $\square$



**Remark 5.1** The definition of weak persistence is not a very appropriate one for stochastic models. Many authors have introduced some more appropriate definitions of permanence for stochastic population models, for example, stochastic persistence in probability (see [21, 22]) or a new definition of stochastic permanence (see [23]). We will continue to study them in future work.

## 6 Numerical simulations

Now let us use Milstein's method (see [24]) to support our results. See Figure 1. We choose  $r = 0.3$ ,  $a_{11} = 0.6$ ,  $a_{12} = 0.2$ ,  $a_{13} = 0.5$ ,  $d = 0.4$ ,  $a_{21} = 0.6$ ,  $a_{22} = 0.4$ ,  $a_{23} = 0.3$ ,  $\tau_1(t) = 0.2 \times (2 + \cos t)$ ,  $\tau_2(t) = 0.3(1 + \sin t)$ ,  $\tau_3(t) = 0.2(2 + \sin t)$ ,  $\tau_4(t) = 0.3(1 + \sin t)$ ,  $\sigma_{12} = 0.2$ ,  $\sigma_{13} = 0.4$ ,  $\sigma_{14} = 0.2$ ,  $\sigma_{21} = \sigma_{22} = \sigma_{23} = \sigma_{24} = 0.2$ . The difference between the conditions of Figure 1(a) and Figure 1(b) are the values of  $\sigma_{11}$  and  $\sigma_{21}$ . In Figure 1(a), we choose  $\sigma_{11} = 0.8$ ,  $\sigma_{21} = 0.2$ . Then  $r - 0.5\sigma_{11}^2 < 0$ , hence, by Theorem 5.1(1), we see that populations  $x$  and  $y$  go extinct; see Figure 1(a). In Figure 1(b), we choose  $\sigma_{11} = 0.2$ ,  $\sigma_{21} = 0.2$ . Then, by computation, we obtain  $r - 0.5\sigma_{11}^2 > 0$  and  $\frac{a_{21}(r - 0.5\sigma_{11}^2)}{a_{11}(1 - \tau)} - (d + 0.5\sigma_{21}^2) < 0$ , hence, by Theorem 5.1, we see that population  $x$  is weakly persistent and population  $y$  goes extinct; see Figure 1(b).

## 7 Conclusions

In this paper, we consider a stochastic predator-prey system with time-dependent delays. We first show that system (2) has a global positive solution and is stochastically ultimate bounded. And then we also discuss the asymptotic properties for the moments as well as the sample paths of the solution. Finally, we obtain the critical value between weak persistence and extinction of the prey and simulate the model to support our results.

There are still many interesting and challenging questions that need to be studied. We outline some open problems and topics for further research.

- (1) The stability of the positive equilibrium state is one of the most interesting topics in the study of population models. Considering the influence of environment noise, the stochastic models do not keep the positive equilibrium state of the corresponding deterministic systems. However, in recent years, many authors have studied the stability in distribution of stochastic population models (see [25–27]). Hence, the

study about the stability in distribution of system (2) is necessary. We will continue to investigate this in our future work.

- (2) In the natural environment, the populations are inevitably subject to the environmental noises, and when the intensity of the noise is sufficiently large, the population will go extinct (see Figure 1(a)). Hence, we can extend classical deterministic systems with time dependence. For example, in [9, 10], the authors considered a population model:

$$\dot{x}(t) = \sum_{k=1}^m \alpha_k(t)x(h_k(t)) - \beta(t)x^2(t), \quad t \geq 0,$$

where  $m \geq 1$  is an integer, and functions  $h_k : [0, \infty) \rightarrow R$  are continuous, such that  $t - \tau \leq h_k(t) \leq t$ ,  $\tau = \text{constant}$ . They considered the effects of environmental noise, and studied the stochastic permanence, the stability in distribution and other dynamic properties of stochastic population models.

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#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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