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Solvable product-type system of difference equations with two dependent variables

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Abstract

It has been recently noticed that there is a finite number of two-dimensional classes of product-type systems of difference equations solvable in closed form. We present a new class of this type. A detailed analysis of the form of its solutions is given. Our results complement the previous ones on such systems and present one of the final steps in describing the forms of their solutions.

MSC: 39A20; 39A45

Keywords: system of difference equations; product-type system; solvable in closed form

1 Introduction

Many types of difference equations and systems have been studied so far. A part of the studies can be found in [1–24]. Some types of the systems essentially obtained by symmetrization of scalar ones were studied in [8–10], which was a motivation for further investigations in the field [6, 7, 11, 12, 14–24]. Historically, perhaps the first main problem of interest in the whole area was finding formulas for their solutions. For known methods for finding the formulas the reader can consult, for example, [1–5]. A note of ours from 2004 has influenced some investigation in this direction since that time (see, for example, [13, 15–24] and the references therein).

In the study of some classes of equations and systems, product-type ones appear as boundary cases. Finding formulas for positive solutions to the equations and systems in the boundary cases is a routine problem, so not of theoretical interest nowadays. It can be of practical interest only if another system or equation is reduced to such one. However, if all solutions are not positive, the problem is very complicated. The boundary cases of equations and systems have motivated us to study them for the case of non-positive initial values. In fact, the equations and systems on the complex domain have attracted our special attention. Our study started in [21], where a system with two dependent variables was investigated. The form of the system in [21] strikingly suggested the study of the solvability of the other systems of related forms (see, e.g., [16, 22]). Since the system, as well as a couple of other ones later studied (see, e.g., [22]), was of the form

$$z_n = z_{n-m_1}^{a_1} w_{n-m_2}^{a_2}, \qquad w_n = w_{n-m_3}^{a_3} z_{n-m_4}^{a_4}, \qquad n \in \mathbb{N}_0, \tag{1}$$

it naturally suggested the study of the solvability of this, as well as of some related systems. This motivated us to include some coefficients in (1) and study the solvability of such systems, which was for the first time done in [15], where we showed the solvability theoretically and gave some hints on how to deal with more concrete cases, that is, for some special values of parameters a, b, c and d. Later we realized that complete pictures of the form of the solutions of this type of systems could be given by studying all the quantities appearing there in detail. References [18] and [24] were the first ones which gave the complete pictures of the forms of the solutions to the systems studied therein. Later in [17] we devised another method which deals with the solvability problem, although technically somewhat complex. For some quite recent results on product-type systems see [19], [20] and [23].

To finish the project of studying the solvability of product-type systems with two dependent variables (see [15, 17–24] and the related references therein), we have to study a few more. Here we study the system

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta w_{n-2}^c z_{n-1}^d, \quad n \in \mathbb{N}_0, \tag{2}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C}$. In fact, we assume that $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, to avoid dealing with non-defined or trivial solutions. We will give a complete picture of the forms of the solutions to system (2) for all the values of the parameters and initial values.

2 Auxiliary results

Some classical auxiliary results that are employed in the section that follows are quoted in this one.

Lemma 1 (see, *e.g.*, [3, 25]) *Let*

$$R_k(s) = b_k \prod_{j=1}^{k} (s - s_j),$$

$s_j \neq s_t$, $j \neq t$, *and* $b_k \neq 0$. *Then*

$$\sum_{j=1}^{k} \frac{s_j^m}{R_k'(s_j)} = 0,$$

for each $m \in \{0, 1, \ldots, k-2\}$, *and*

$$\sum_{j=1}^{k} \frac{s_j^{k-1}}{R_k'(s_j)} = \frac{1}{b_k}.$$

Four more or less widely known formulas are listed in the following lemma (see, e.g., [3, 5]). A recurrent relation connecting this type of sums is given in [18].

Lemma 2 *Let*

$$s_n^{(m)}(z) = \sum_{j=1}^{n} j^m z^{j-1}, \quad n \in \mathbb{N},$$

$m \in \mathbb{N}_0$ *and* $z \in \mathbb{C}$.

Then

$$s_n^{(0)}(z) = \frac{1-z^n}{1-z},$$
$$s_n^{(1)}(z) = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2},$$
$$s_n^{(2)}(z) = \frac{1+z-(n+1)^2z^n + (2n^2+2n-1)z^{n+1} - n^2z^{n+2}}{(1-z)^3},$$
$$s_n^{(3)}(z) = \frac{n^3z^n(z-1)^3 - 3n^2z^n(z-1)^2 + 3nz^n(z^2-1) - (z^n-1)(z^2+4z+1)}{(1-z)^4},$$

for every $z \in \mathbb{C} \setminus \{1\}$ *and* $n \in \mathbb{N}$.

The following lemma describes the nature of the zeros of a polynomial of the fourth order in detail (see [26]).

Lemma 3 *Let*

$$P_4(t) = t^4 + bt^3 + ct^2 + dt + e,$$
$$\Delta_0 = c^2 - 3bd + 12e, \qquad \Delta_1 = 2c^3 - 9bcd + 27b^2e + 27d^2 - 72ce,$$
$$\Delta = \frac{1}{27}\left(4\Delta_0^3 - \Delta_1^2\right), \qquad P = 8c - 3b^2,$$
$$Q = b^3 + 8d - 4bc, \qquad D = 64e - 16c^2 + 16b^2c - 16bd - 3b^4.$$

(a) *If* $\Delta < 0$, *then two zeros of* P_4 *are real and different, and two are complex conjugate.*
(b) *If* $\Delta > 0$, *then all the zeros of* P_4 *are real or none is. More precisely,*

1° *if* $P < 0$ *and* $D < 0$, *then all four zeros of* P_4 *are real and different*;
2° *if* $P > 0$ *or* $D > 0$, *then there are two pairs of complex conjugate zeros of* P_4.

(c) *If* $\Delta = 0$, *then and only then* P_4 *has a multiple zero. The following cases can occur*:

1° *if* $P < 0$, $D < 0$ *and* $\Delta_0 \neq 0$, *then two zeros of* P_4 *are real and equal and two are real and simple*;
2° *if* $D > 0$ *or* $(P > 0$ *and* $(D \neq 0$ *or* $Q \neq 0))$, *then two zeros of* P_4 *are real and equal and two are complex conjugate*;
3° *if* $\Delta_0 = 0$ *and* $D \neq 0$, *there is a triple zero of* P_4 *and one simple, all real*;
4° *if* $D = 0$, *then*

4.1° *if* $P < 0$ *there are two double real zeros of* P_4;
4.2° *if* $P > 0$ *and* $Q = 0$ *there are two double complex conjugate zeros of* P_4;
4.3° *if* $\Delta_0 = 0$, *then all four zeros of* P_4 *are real and equal to* $-b/4$.

3 Main results

The main results in this paper are proved in this section.

Theorem 1 *Assume that* $b, c, d \in \mathbb{Z}$, $a = 0$, $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then*

(a) *if* $c + bd \neq 1$, *the general solution to* (2) *is given by*

$$z_{3m} = \alpha^{\frac{1-c-bd(c+bd)^{m-1}}{1-c-bd}} \beta^{b\frac{1-(c+bd)^m}{1-c-bd}} z_0^{bd(c+bd)^{m-1}} w_{-1}^{bc(c+bd)^{m-1}}, \tag{3}$$

$$z_{3m+1} = \alpha^{\frac{1-c-bd(c+bd)^m}{1-c-bd}} \beta^{b\frac{1-(c+bd)^m}{1-c-bd}} w_0^{b(c+bd)^m}, \tag{4}$$

$$z_{3m+2} = \alpha^{\frac{1-c-bd(c+bd)^m}{1-c-bd}} \beta^{b\frac{1-(c+bd)^{m+1}}{1-c-bd}} z_{-1}^{bd(c+bd)^m} w_{-2}^{bc(c+bd)^m}, \tag{5}$$

$$w_{3m} = \alpha^{d\frac{1-(c+bd)^m}{1-c-bd}} \beta^{\frac{1-(c+bd)^m}{1-c-bd}} w_0^{(c+bd)^m}, \tag{6}$$

$$w_{3m+1} = \alpha^{d\frac{1-(c+bd)^m}{1-c-bd}} \beta^{\frac{1-(c+bd)^{m+1}}{1-c-bd}} z_{-1}^{d(c+bd)^m} w_{-2}^{c(c+bd)^m}, \tag{7}$$

$$w_{3m+2} = \alpha^{d\frac{1-(c+bd)^m}{1-c-bd}} \beta^{\frac{1-(c+bd)^{m+1}}{1-c-bd}} z_0^{d(c+bd)^m} w_{-1}^{c(c+bd)^m}, \tag{8}$$

(b) *if* $c + bd = 1$, *the general solution to* (2) *is given by*

$$z_{3m} = \alpha^{1+bd(m-1)} \beta^{bm} z_0^{bd} w_{-1}^{bc}, \tag{9}$$

$$z_{3m+1} = \alpha^{1+bdm} \beta^{bm} w_0^b, \tag{10}$$

$$z_{3m+2} = \alpha^{1+bdm} \beta^{b(m+1)} z_{-1}^{bd} w_{-2}^{bc}, \tag{11}$$

$$w_{3m} = \alpha^{dm} \beta^m w_0, \tag{12}$$

$$w_{3m+1} = \alpha^{dm} \beta^{m+1} z_{-1}^d w_{-2}^c, \tag{13}$$

$$w_{3m+2} = \alpha^{dm} \beta^{m+1} z_0^d w_{-1}^c. \tag{14}$$

Proof Since $a = 0$, we have

$$z_{n+1} = \alpha w_n^b, \qquad w_{n+1} = \beta w_{n-2}^c z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{15}$$

From (15), we have

$$w_n = \beta \alpha^d w_{n-3}^{c+bd}, \quad n \geq 3, \tag{16}$$

which implies that

$$w_{3m+i} = \left(\alpha^d \beta\right)^{\sum_{j=0}^{m-1}(c+bd)^j} w_i^{(c+bd)^m}, \quad m \in \mathbb{N}, i = 0, 1, 2. \tag{17}$$

Hence,

$$w_{3m} = \alpha^{d\sum_{j=0}^{m-1}(c+bd)^j} \beta^{\sum_{j=0}^{m-1}(c+bd)^j} w_0^{(c+bd)^m}, \tag{18}$$

$$\begin{aligned} w_{3m+1} &= \left(\alpha^d \beta\right)^{\sum_{j=0}^{m-1}(c+bd)^j} \left(\beta w_{-2}^c z_{-1}^d\right)^{(c+bd)^m} \\ &= \alpha^{d\sum_{j=0}^{m-1}(c+bd)^j} \beta^{\sum_{j=0}^{m}(c+bd)^j} z_{-1}^{d(c+bd)^m} w_{-2}^{c(c+bd)^m}, \end{aligned} \tag{19}$$

$$\begin{aligned} w_{3m+2} &= \left(\alpha^d \beta\right)^{\sum_{j=0}^{m-1}(c+bd)^j} \left(\beta w_{-1}^c z_0^d\right)^{(c+bd)^m} \\ &= \alpha^{d\sum_{j=0}^{m-1}(c+bd)^j} \beta^{\sum_{j=0}^{m}(c+bd)^j} z_0^{d(c+bd)^m} w_{-1}^{c(c+bd)^m}. \end{aligned} \tag{20}$$

Using (18)-(20) in the first equality in (15), we get

$$z_{3m} = \alpha^{1+bd\sum_{j=0}^{m-2}(c+bd)^j}\beta^{b\sum_{j=0}^{m-1}(c+bd)^j}z_0^{bd(c+bd)^{m-1}}w_{-1}^{bc(c+bd)^{m-1}}, \tag{21}$$

$$z_{3m+1} = \alpha^{1+bd\sum_{j=0}^{m-1}(c+bd)^j}\beta^{b\sum_{j=0}^{m-1}(c+bd)^j}w_0^{b(c+bd)^m}, \tag{22}$$

$$z_{3m+2} = \alpha^{1+bd\sum_{j=0}^{m-1}(c+bd)^j}\beta^{b\sum_{j=0}^{m}(c+bd)^j}z_{-1}^{bd(c+bd)^m}w_{-2}^{bc(c+bd)^m}. \tag{23}$$

From (18)-(23) and some calculations, we easily get (3)-(14), as desired. □

Theorem 2 *Assume that $a,c,d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (2) *is solvable in closed form.*

Proof Since $b = 0$ system (2) becomes

$$z_{n+1} = \alpha z_n^a, \qquad w_{n+1} = \beta w_{n-2}^c z_{n-1}^d, \quad n \in \mathbb{N}_0, \tag{24}$$

which is system (2.11) in [17]. Hence, if $c \neq 0$ the theorem follows from Theorem 2.2 in [17], while the case $c = 0$ follows from equations (2.13) and (2.14) in [17], as well as the second equation in (24). □

The case $d = 0$ has been recently studied in [20], where, among others, the following theorem was proved.

Theorem 3 *Assume that $a,b,c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (2) *is solvable in closed form.*

Theorem 4 *Assume that $a,b,c,d \in \mathbb{Z}$, $abcd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (2) *is solvable in closed form.*

Proof From $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ and (2) we get $z_n w_n \neq 0$ for $n \in \mathbb{N}_0$. Hence,

$$w_n^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0, \tag{25}$$

$$w_{n+1}^b = \beta^b w_{n-2}^{bc} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{26}$$

and consequently

$$z_{n+2} = \alpha^{1-c}\beta^b z_{n+1}^a z_{n-1}^{bd+c} z_{n-2}^{-ac}, \tag{27}$$

for $n \geq 2$.

Note also that

$$\begin{aligned} &z_1 = \alpha z_0^a w_0^b, \qquad z_2 = \alpha\left(\alpha z_0^a w_0^b\right)^a\left(\beta w_{-2}^c z_{-1}^d\right)^b = \alpha^{1+a}\beta^b z_{-1}^{bd} z_0^{a^2} w_{-2}^{bc} w_0^{ab}, \\ &z_3 = \alpha z_2^a w_2^b = \alpha^{1+a+a^2}\beta^{b(1+a)} z_{-1}^{abd} z_0^{a^3+bd} w_{-2}^{abc} w_{-1}^{bc} w_0^{a^2 b}. \end{aligned} \tag{28}$$

Let $\delta = \alpha^{1-c}\beta^b$,

$$a_1 = a, \qquad b_1 = 0, \qquad c_1 = bd + c, \qquad d_1 = -ac, \qquad y_1 = 1, \tag{29}$$

then

$$z_{n+2}=\delta^{y_1}z_{n+1}^{a_1}z_n^{b_1}z_{n-1}^{c_1}z_{n-2}^{d_1},\quad n\geq 2, \tag{30}$$

and consequently

$$\begin{aligned} z_{n+2}&=\delta^{y_1}\left(\delta z_n^{a_1}z_{n-1}^{b_1}z_{n-2}^{c_1}z_{n-3}^{d_1}\right)^{a_1}z_n^{b_1}z_{n-1}^{c_1}z_{n-2}^{d_1}\\ &=\delta^{y_1+a_1}z_n^{a_1a_1+b_1}z_{n-1}^{b_1a_1+c_1}z_{n-2}^{c_1a_1+d_1}z_{n-3}^{d_1a_1}\\ &=\delta^{y_2}z_n^{a_2}z_{n-1}^{b_2}z_{n-2}^{c_2}z_{n-3}^{d_2}, \end{aligned}$$

for $n\geq 3$, where

$$\begin{aligned} &a_2:=a_1a_1+b_1,\qquad b_2:=b_1a_1+c_1,\qquad c_2:=c_1a_1+d_1,\\ &d_2:=d_1a_1,\qquad y_2:=y_1+a_1. \end{aligned}$$

Assume

$$z_{n+2}=\delta^{y_k}z_{n+2-k}^{a_k}z_{n+1-k}^{b_k}z_{n-k}^{c_k}z_{n-k-1}^{d_k}, \tag{31}$$

for a $k\geq 2$ and every $n\geq k+1$, and

$$\begin{aligned} &a_k=a_1a_{k-1}+b_{k-1},\qquad b_k=b_1a_{k-1}+c_{k-1},\\ &c_k=c_1a_{k-1}+d_{k-1},\qquad d_k=d_1a_{k-1}, \end{aligned} \tag{32}$$

$$y_k=y_{k-1}+a_{k-1}. \tag{33}$$

Using (30) in (31), we get

$$\begin{aligned} z_{n+2}&=\delta^{y_k}\left(\delta z_{n+1-k}^{a_1}z_{n-k}^{b_1}z_{n-k-1}^{c_1}z_{n-k-2}^{d_1}\right)^{a_k}z_{n+1-k}^{b_k}z_{n-k}^{c_k}z_{n-k-1}^{d_k}\\ &=\delta^{y_k+a_k}z_{n+1-k}^{a_1a_k+b_k}z_{n-k}^{b_1a_k+c_k}z_{n-k-1}^{c_1a_k+d_k}z_{n-k-2}^{d_1a_k}\\ &=\delta^{y_{k+1}}z_{n+1-k}^{a_{k+1}}z_{n-k}^{b_{k+1}}z_{n-k-1}^{c_{k+1}}z_{n-k-2}^{d_{k+1}}, \end{aligned}$$

for $n\geq k+2$, where

$$\begin{aligned} &a_{k+1}=a_1a_k+b_k,\qquad b_{k+1}=b_1a_k+c_k,\qquad c_{k+1}=c_1a_k+d_k,\qquad d_{k+1}:=d_1a_k,\\ &y_{k+1}:=y_k+a_k. \end{aligned}$$

Hence, by induction we have proved that (31)-(33) hold.

From (31)-(33) and (28), we get

$$\begin{aligned} z_{n+2}&=\delta^{y_{n-1}}z_3^{a_{n-1}}z_2^{b_{n-1}}z_1^{c_{n-1}}z_0^{d_{n-1}}\\ &=\left(\alpha^{1-c}\beta^b\right)^{y_{n-1}}\left(\alpha^{1+a+a^2}\beta^{b(1+a)}z_{-1}^{abd}z_0^{a^3+bd}w_{-2}^{abc}w_{-1}^{bc}w_0^{a^2b}\right)^{a_{n-1}}\\ &\quad\times\left(\alpha^{1+a}\beta^bz_{-1}^{bd}z_0^{a^2}w_{-2}^{bc}w_0^{ab}\right)^{b_{n-1}}\left(\alpha z_0^aw_0^b\right)^{c_{n-1}}z_0^{d_{n-1}} \end{aligned}$$

$$
\begin{aligned}
&= \alpha^{(1-c)y_{n-1}+(1+a+a^2)a_{n-1}+(1+a)b_{n-1}+c_{n-1}} \beta^{by_{n-1}+b(1+a)a_{n-1}+bb_{n-1}} \\
&\quad \times z_{-1}^{abda_{n-1}+bdb_{n-1}} z_0^{(a^3+bd)a_{n-1}+a^2b_{n-1}+ac_{n-1}+d_{n-1}} w_{-2}^{abca_{n-1}+bcb_{n-1}} w_{-1}^{bca_{n-1}} \\
&\quad \times w_0^{a^2ba_{n-1}+abb_{n-1}+bc_{n-1}} \\
&= \alpha^{y_{n+2}-cy_{n-1}} \beta^{by_{n+1}} z_{-1}^{bda_n} z_0^{a_{n+2}-ca_{n-1}} w_{-2}^{bca_n} w_{-1}^{bca_{n-1}} w_0^{ba_{n+1}},
\end{aligned}
\tag{34}
$$

for $n \geq 2$.

From (32) one sees that a_k, b_k, c_k and d_k are solutions to

$$
\hat{x}_{k+4} = a_1\hat{x}_{k+3} + b_1\hat{x}_{k+2} + c_1\hat{x}_{k+1} + d_1\hat{x}_k, \quad k \in \mathbb{N}, \tag{35}
$$

and, along with (33) (for $k = 1, 0, -1, -2$), we also obtain

$$
a_{-3} = 0, \qquad a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_0 = 1, \tag{36}
$$

$$
y_{-3} = y_{-2} = y_{-1} = y_0 = 0, \qquad y_1 = 1, \tag{37}
$$

and

$$
y_k = \sum_{j=0}^{k-1} a_j. \tag{38}
$$

The solvability of (35) is well known, from which, along with (36), a formula for a_k is obtained. Using it in (38), a formula for y_k is obtained by Lemma 2. Hence, (27) is solvable.

We have

$$
z_{n-1}^d = \frac{w_{n+1}}{\beta w_{n-2}^c}, \quad n \in \mathbb{N}_0, \tag{39}
$$

$$
z_{n+1}^d = \alpha^d z_n^{ad} w_n^{bd}, \quad n \in \mathbb{N}_0, \tag{40}
$$

so that

$$
w_{n+3} = \alpha^d \beta^{1-a} w_{n+2}^a w_n^{bd+c} w_{n-1}^{-ac}, \quad n \in \mathbb{N}_0. \tag{41}
$$

We also have

$$
w_1 = \beta w_{-2}^c z_{-1}^d \quad \text{and} \quad w_2 = \beta w_{-1}^c z_0^d. \tag{42}
$$

As above we get

$$
w_{n+3} = \eta^{y_k} w_{n+3-k}^{a_k} w_{n+2-k}^{b_k} w_{n+1-k}^{c_k} w_{n-k}^{d_k}, \quad n \geq k-1, \tag{43}
$$

where $\eta = \alpha^d \beta^{1-a}$, a_k satisfies (35) and (36), and y_k is given by (38).

From (43) with $k = n+1$ and by using (42) we get

$$
\begin{aligned}
w_{n+3} &= \eta^{y_{n+1}} w_2^{a_{n+1}} w_1^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\
&= \left(\alpha^d \beta^{1-a}\right)^{y_{n+1}} \left(\beta w_{-1}^c z_0^d\right)^{a_{n+1}} \left(\beta w_{-2}^c z_{-1}^d\right)^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}}
\end{aligned}
$$

$$\begin{aligned}
&= \alpha^{dy_{n+1}}\beta^{(1-a)y_{n+1}+a_{n+1}+b_{n+1}}z_{-1}^{db_{n+1}}z_0^{da_{n+1}}w_{-2}^{cb_{n+1}}w_{-1}^{ca_{n+1}+d_{n+1}}w_0^{c_{n+1}}\\
&= \alpha^{dy_{n+1}}\beta^{y_{n+3}-ay_{n+2}}z_{-1}^{d(a_{n+2}-aa_{n+1})}z_0^{da_{n+1}}w_{-2}^{c(a_{n+2}-aa_{n+1})}\\
&\quad\times w_{-1}^{c(a_{n+1}-aa_n)}w_0^{a_{n+3}-aa_{n+2}},
\end{aligned}\tag{44}$$

for $n \in \mathbb{N}_0$.

As we have already seen, formulas for a_k and y_k can be found. Using them in (44) we show the solvability of (41). Some calculations show that (34) and (44) present a solution to (2), from which the result follows. □

Corollary 1 *Assume that* $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the general solution to* (2) *is given by* (34) *and* (44), *where* a_k *satisfies* (35) *and* (36), *and* y_k *is given by* (37) *and* (38).

Theorem 4 gives a general form of solutions to system (2) when $abcd \neq 0$, but does not present explicit formulas for sequences a_n and y_n involved in the solutions. Now we give some explicit formulas for them in more concrete cases, following some arguments related to the system in [19]. Since $ac \neq 0$, we can find the zeros of the characteristic polynomial associated to (35)

$$p_4(\lambda) = \lambda^4 - a\lambda^3 - (bd + c)\lambda + ac. \tag{45}$$

To do this, we consider the following equivalent equation with a parameter [25]:

$$\left(\lambda^2 - \frac{a}{2}\lambda + \frac{s}{2}\right)^2 - \left(\left(\frac{a^2}{4} + s\right)\lambda^2 - \left(\frac{as}{2} - (bd + c)\right)\lambda + \frac{s^2}{4} - ac\right) = 0. \tag{46}$$

The parameter is chosen so that $(as - 2(bd + c))^2 = (a^2 + 4s)(s^2 - 4ac)$, that is,

$$s^3 + a(bd - 3c)s - a^3c - (bd + c)^2 = 0. \tag{47}$$

We have

$$\left(\lambda^2 - \frac{a}{2}\lambda + \frac{s}{2}\right)^2 - \left(\frac{\sqrt{a^2 + 4s}}{2}\lambda - \frac{as - 2(bd + c)}{2\sqrt{a^2 + 4s}}\right)^2 = 0, \tag{48}$$

or equivalently

$$\lambda^2 - \left(\frac{a}{2} + \frac{\sqrt{a^2 + 4s}}{2}\right)\lambda + \frac{s}{2} + \frac{as - 2(bd + c)}{2\sqrt{a^2 + 4s}} = 0, \tag{49}$$

$$\lambda^2 - \left(\frac{a}{2} - \frac{\sqrt{a^2 + 4s}}{2}\right)\lambda + \frac{s}{2} - \frac{as - 2(bd + c)}{2\sqrt{a^2 + 4s}} = 0. \tag{50}$$

Let $p = a(bd - 3c)$, $q = -a^3c - (bd + c)^2$, and $s = u + v$. Assuming that $uv = -p/3$, from (47) we get $u^3 + v^3 = -q$. Hence, u^3 and v^3 are solutions to $z^2 + qz - p^3/27$. Thus

$$s = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{51}$$

or

$$s = \frac{1}{3\sqrt[3]{2}}\left(\sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\right), \tag{52}$$

by using the change of variables $p = -\Delta_0/3$ and $q = -\Delta_1/27$ in (51).

For s given in (52) we solve equations (49) and (50). So, the zeros of polynomial (45) are

$$\lambda_1 = \frac{a}{4} + \frac{\sqrt{a^2+4s}}{4} + \frac{1}{2}\sqrt{\frac{a^2}{2} - s - \frac{Q}{2\sqrt{a^2+4s}}}, \tag{53}$$

$$\lambda_2 = \frac{a}{4} + \frac{\sqrt{a^2+4s}}{4} - \frac{1}{2}\sqrt{\frac{a^2}{2} - s - \frac{Q}{2\sqrt{a^2+4s}}}, \tag{54}$$

$$\lambda_3 = \frac{a}{4} - \frac{\sqrt{a^2+4s}}{4} + \frac{1}{2}\sqrt{\frac{a^2}{2} - s + \frac{Q}{2\sqrt{a^2+4s}}}, \tag{55}$$

$$\lambda_4 = \frac{a}{4} - \frac{\sqrt{a^2+4s}}{4} - \frac{1}{2}\sqrt{\frac{a^2}{2} - s + \frac{Q}{2\sqrt{a^2+4s}}}, \tag{56}$$

where

$$\Delta_0 := 3a(3c - bd), \tag{57}$$

$$\Delta_1 := 27\left(a^3c + (bd+c)^2\right), \tag{58}$$

$$Q := -a^3 - 8bd - 8c. \tag{59}$$

By Lemma 3, the nature of λ_j, $j = \overline{1,4}$, depends also on

$$\Delta = \frac{1}{27}\left(4\Delta_0^3 - \Delta_1^2\right), \tag{60}$$

$$P = -3a^2, \tag{61}$$

$$D = a\left(48c - 16bd - 3a^3\right). \tag{62}$$

Zeros of p_4 are mutually different and different from 1. If $a = 1$, $c = 2$ and $bd = 3$, polynomial (45) becomes

$$p_4(\lambda) = \lambda^4 - \lambda^3 - 5\lambda + 2 = (\lambda - 2)\left(\lambda^3 + \lambda^2 + 2\lambda - 1\right).$$

Since in this case $\Delta < 0$, all the zeros of the polynomial are different. Since $p_4(1) \neq 0$, 1 is not a zero of the polynomial. In fact, there are many polynomials of the form in (45) such that $\Delta < 0$. For example, they are those for which holds $3ac < abd$, that is, $\Delta_0 < 0$.

Since $\lambda_j \neq \lambda_i$, $i \neq j$,

$$a_n = \gamma_1\lambda_1^n + \gamma_2\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{63}$$

where γ_i, $i = \overline{1,4}$ are constants, is the general solution to (35).

Equalities (36), along with Lemma 1 applied to polynomial (45), yield

$$\begin{aligned} a_n &= \sum_{j=1}^{4} \frac{\lambda_j^{n+3}}{p_4'(\lambda_j)} \\ &= \frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &\quad + \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}, \end{aligned} \tag{64}$$

for $n \geq -3$, from which, along with (38) and the fact that $\lambda_i \neq 1$, $i = \overline{1,4}$, is obtained:

$$y_n = \sum_{j=0}^{n-1} \sum_{i=1}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \sum_{i=1}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \quad n \in \mathbb{N}. \tag{65}$$

Moreover, (65) holds for $n \geq -3$.

Zeros of p_4 are different and one of them is 1. In this case it must be $p_4(1) = 1 - a - bd - c + ac = 0$. Hence,

$$(a-1)(c-1) = bd, \tag{66}$$

which implies

$$\begin{aligned} p_4(\lambda) &= \lambda^4 - a\lambda^3 - (ac - a + 1)\lambda + ac \\ &= (\lambda - 1)\bigl(\lambda^3 - (a-1)\lambda^2 - (a-1)\lambda - ac\bigr). \end{aligned} \tag{67}$$

Let $\lambda_1 = 1$. To find the other zeros of p_4, we have to solve the equation

$$\lambda^3 - (a-1)\lambda^2 - (a-1)\lambda - ac = 0.$$

By using the change of variables $\lambda = t + \frac{a-1}{3}$ and some simple calculations, we get

$$t^3 + \tilde{p}t + \tilde{q} = 0,$$

where

$$\tilde{p} = \frac{(1-a)(a+2)}{3} \quad \text{and} \quad \tilde{q} = -\left(\frac{2(a-1)^3}{27} + \frac{(a-1)^2}{3} + ac\right).$$

Using the standard arguments, as those in getting (51), we obtain

$$\lambda_j = \frac{a-1}{3} + \varepsilon^{j-2}\sqrt[3]{-\frac{\tilde{q}}{2} - \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}} + \bar{\varepsilon}^{j-2}\sqrt[3]{-\frac{\tilde{q}}{2} + \sqrt{\frac{\tilde{q}^2}{4} + \frac{\tilde{p}^3}{27}}}, \quad j = \overline{2,4}, \tag{68}$$

where ε is such that $\varepsilon^3 = 1$, $\varepsilon \neq 1$.

For example, if $a = 3$ and $c = 2$, then $bd = 2 \neq 0$, $\Delta \neq 0$ and

$$p_4(\lambda) = \lambda^4 - 3\lambda^3 - 4\lambda + 6 = (\lambda - 1)\bigl(\lambda^3 - 2\lambda^2 - 2\lambda - 6\bigr),$$

so by Lemma 3, the polynomial has four different zeros, and one of them is 1.

Equality (64) holds with, say, $\lambda_1 = 1$. Further, we have

$$y_n = \sum_{j=0}^{n-1} \frac{1}{p_4'(1)} + \sum_{j=0}^{n-1} \sum_{i=2}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \frac{n}{3-2a-ac} + \sum_{i=2}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \tag{69}$$

for $n \in \mathbb{N}$. It is easily shown that (69) also holds for $n = -j$, $j = \overline{0,3}$.

This analysis, along with Corollary 1, implies the following result.

Corollary 2 *Assume that $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$, $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta \neq 0$. Then the following statements are true*:

(a) *If $(a-1)(c-1) \neq bd$, then the general solution to* (2) *is given by* (34) *and* (44), *where $(a_n)_{n\geq -3}$ is given by* (64), *$(y_n)_{n\geq -3}$ is given by* (65), *while λ_j, $j = \overline{1,4}$, are given by* (53)-(56).

(b) *If $(a-1)(c-1) = bd$ and $3 - 2a \neq ac$, then the general solution to* (2) *is given by* (34) *and* (44), *where $(a_n)_{n\geq -3}$ is given by* (64) *with $\lambda_1 = 1$, $(y_n)_{n\geq -3}$ is given by* (69), *$\lambda_1 = 1$, while λ_j, $j = \overline{2,4}$, are given by* (68).

1 *is the only double zero of* p_4. Polynomial p_4 has a double zero equal to 1 if (66) holds and

$$p_4'(1) = 3 - 2a - ac = 0, \tag{70}$$

that is, if and only if

$$c = \frac{3}{a} - 2. \tag{71}$$

Then we have

$$p_4(\lambda) = \lambda^4 - a\lambda^3 + (3a-4)\lambda + 3 - 2a = (\lambda - 1)^2\big(\lambda^2 + (2-a)\lambda + 3 - 2a\big),$$

and consequently

$$\lambda_{1,2} = 1, \qquad \lambda_{3,4} = \frac{a - 2 \pm \sqrt{a^2 + 4a - 8}}{2}. \tag{72}$$

From (71) we must have $a = 3$ and $c = -1$, or $a = 1$ and $c = 1$, or $a = -1$ and $c = -5$, or $a = -3$ and $c = -3$.

If $a = c = 1$, then

$$p_4(\lambda) = \lambda^4 - \lambda^3 - \lambda + 1 = (\lambda - 1)^2\big(\lambda^2 + \lambda + 1\big),$$

and consequently

$$\lambda_{1,2} = 1, \qquad \lambda_{3,4} = \frac{-1 \pm i\sqrt{3}}{2}.$$

Since (66) holds we see that this case is not possible when $abcd \neq 0$.

If $a = 3$, $c = -1$, then

$$p_4(\lambda) = \lambda^4 - 3\lambda^3 + 5\lambda - 3 = (\lambda - 1)^2\big(\lambda^2 - \lambda - 3\big),$$

and consequently

$$\lambda_{1,2} = 1, \qquad \lambda_{3,4} = \frac{1 \pm \sqrt{13}}{2}. \tag{73}$$

If $a = c = -3$, then

$$p_4(\lambda) = \lambda^4 + 3\lambda^3 - 13\lambda + 9 = (\lambda - 1)^2\big(\lambda^2 + 5\lambda + 9\big),$$

and consequently

$$\lambda_{1,2} = 1, \qquad \lambda_{3,4} = \frac{-5 \pm i\sqrt{11}}{2}. \tag{74}$$

If $a = -1$ and $c = -5$, then

$$p_4(\lambda) = \lambda^4 + \lambda^3 - 7\lambda + 5 = (\lambda - 1)^2\big(\lambda^2 + 3\lambda + 5\big),$$

and consequently

$$\lambda_{1,2} = 1, \qquad \lambda_{3,4} = \frac{-3 \pm i\sqrt{11}}{2}. \tag{75}$$

In these four cases, we have [19]

$$\begin{aligned} a_n = {} & \frac{n(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ & + \frac{\lambda_3^{n+3}}{(\lambda_3-1)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-1)^2(\lambda_4-\lambda_3)} \end{aligned} \tag{76}$$

and

$$\begin{aligned} y_n = {} & \sum_{j=0}^{n-1}\Bigg(\frac{j(1-\lambda_3)(1-\lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ & + \frac{\lambda_3^{j+3}}{(\lambda_3-1)^2(\lambda_3-\lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4-1)^2(\lambda_4-\lambda_3)}\Bigg) \\ = {} & \frac{(n-1)n}{2(1-\lambda_3)(1-\lambda_4)} + \frac{n(3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1)}{(1-\lambda_3)^2(1-\lambda_4)^2} \\ & + \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3-1)^3(\lambda_3-\lambda_4)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4-1)^3(\lambda_4-\lambda_3)}. \end{aligned} \tag{77}$$

p_4 has only one double zero different from 1. Assume that $\lambda = m \notin \{0,1\}$ is a double zero of p_4. Then we have

$$m^4 - am^3 - (bd + c)m + ac = 0 \quad \text{and} \quad 4m^3 - 3am^2 - bd - c = 0. \tag{78}$$

If m is not a triple zero, then it must be $12m^2 - 6am \neq 0$, that is, $a \neq 2m$.

From (78), we get

$$\begin{aligned} p_4(\lambda) &= \lambda^4 - a\lambda^3 + \left(3am^2 - 4m^3\right)\lambda + 3m^4 - 2am^3 \\ &= (\lambda - m)^2\left(\lambda^2 + (2m - a)\lambda + m(3m - 2a)\right), \end{aligned} \tag{79}$$

and consequently

$$\lambda_{1,2} = m, \qquad \lambda_{3,4} = \frac{a - 2m \pm \sqrt{-8m^2 + 4am + a^2}}{2}. \tag{80}$$

Hence, if we additionally assume that $2a \neq 3m$, $3am^2 - 4m^3 \in \mathbb{Z}$, $3m^4 - 2am^3 \in \mathbb{Z}$, we get a family of polynomials of the form in (45) which have double zeros different from 1. For example, if $a = m \in \mathbb{Z} \setminus \{0, 1\}$, then from (79) it follows that

$$p_4(\lambda) = (\lambda - a)^2\left(\lambda^2 + a\lambda + a^2\right).$$

Since, in the case $\lambda_1 = \lambda_2$, $\lambda_i \neq \lambda_j$, $2 \leq i, j \leq 4$, we have

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{81}$$

where γ_i and $i = \overline{1,4}$ are constants, and the solution satisfying (36) is

$$\begin{aligned} a_n = {} & \frac{\lambda_2^{n+2}((n+3)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2(2\lambda_2 - \lambda_3 - \lambda_4))}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2} \\ & + \frac{\lambda_3^{n+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)}. \end{aligned} \tag{82}$$

From (38) and (82) and by Lemma 2, we get

$$\begin{aligned} y_n = {} & \sum_{j=0}^{n-1}\left(\frac{\lambda_2^{j+2}((j+3)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2(2\lambda_2 - \lambda_3 - \lambda_4))}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2}\right. \\ & \left. + \frac{\lambda_3^{j+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)}\right) \\ = {} & \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(1 - \lambda_2)^2} + \frac{(\lambda_2^4 - 2\lambda_2^3\lambda_3 - 2\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_3\lambda_4)(\lambda_2^n - 1)}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2(\lambda_2 - 1)} \\ & + \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)(\lambda_3 - 1)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)(\lambda_4 - 1)}. \end{aligned} \tag{83}$$

Corollary 3 *Assume that $a, b, c, d \in \mathbb{Z}$, $abcd \neq 0$ and $\alpha, \beta, z_{-1}, z_0, w_{-2}, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true*:

(a) *If only one of the zeros of p_4 is double and different from* 1, *say λ_1 and λ_2, then the general solution to* (2) *is given by* (34) *and* (44), *where $(a_n)_{n\geq-3}$ is given by* (82), *$(y_n)_{n\geq-3}$ is given by* (83), *while λ_j, $j = \overline{1,4}$, are given by* (80), *where $m \notin \{0, 1\}$, $2m \neq a \neq 3m$ and $3am^2 - 4m^3, 3m^4 - 2am^3 \in \mathbb{Z}$.*

(b) *If* 1 *is a unique double zero of polynomial p_4, say $\lambda_1 = \lambda_2 = 1$, then the general solution to* (2) *is given by* (34) *and* (44), *where $(a_n)_{n\geq-3}$ is given by* (76), *$(y_n)_{n\geq-3}$ is given by*

(77), *while* λ_j, $j = \overline{1,4}$, *are given by* (73) *when* $a = 3$, $c = -1$, *by* (74) *when* $a = c = -3$, *and by* (75) *when* $a = -1$, $c = -5$.

p_4 *has two pairs of different double zeros.* In this case it must be $D = 0$, which implies that $a = 0$ or $16bd = 48c - 3a^3$. The case $a = 0$ is impossible due to the condition $abcd \neq 0$. In the other case we have $\Delta = 0$ if and only if

$$4\left(3a\left(3c + \frac{3a^3 - 48c}{16}\right)\right)^3 = \left(27\left(a^3c + \left(4c - \frac{3a^3}{16}\right)^2\right)\right)^2,$$

that is,

$$2^3a^6 = \pm\left(3^2a^6 - 2^7a^3c + 2^{12}c^2\right). \tag{84}$$

From (84) we have

$$17a^6 - 2^7a^3c + 2^{12}c^2 = 0,$$

from which it follows that $a^3/c = \frac{2^6}{17}(1 \pm 4i)$, which is impossible due to the rationality of a^3/c, or is

$$a^6 - 2^7a^3c + 2^{12}c^2 = \left(a^3 - 2^6c\right)^2 = 0,$$

which implies $c = a^3/2^6$.

Assume $c = a^3/2^6$. Then

$$p_4(\lambda) = \lambda^4 - a\lambda^3 + \frac{a^3}{8}\lambda + \frac{a^4}{2^6} = \left(\lambda^2 - \frac{a\lambda}{2} - \frac{a^2}{8}\right)^2$$

(for more details see [19], p.14).

Hence

$$\lambda_{1,2} = \frac{a}{4}(1 + \sqrt{3}), \qquad \lambda_{3,4} = \frac{a}{4}(1 - \sqrt{3}), \tag{85}$$

are two double zeros of p_4, for each $a \neq 0$.

Since

$$a(1 \pm \sqrt{3})/4 \neq 1,$$

for every $a \in \mathbb{Z}$, p_4 cannot have two pairs of double zeros such that one of them is equal to 1.

The general solution to (35) in this case is of the following form:

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_1^n + (\gamma_3 + \gamma_4 n)\lambda_3^n, \quad n \in \mathbb{N}, \tag{86}$$

for some constants γ_i, $i = \overline{1,4}$. The solution with initial conditions (36) is

$$\begin{aligned} a_n = {} & \frac{\lambda_2^{n+2}(n(\lambda_2 - \lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2 - \lambda_4)^4} \\ & + \frac{\lambda_4^{n+2}(n(\lambda_4 - \lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4 - \lambda_2)^4}. \end{aligned} \tag{87}$$

From (38), (87) and Lemma 2, we get

$$\begin{aligned} y_n &= \sum_{j=0}^{n-1}\left(\frac{\lambda_2^{j+2}(j(\lambda_2-\lambda_4)^2+\lambda_2^2-4\lambda_2\lambda_4+3\lambda_4^2)}{(\lambda_2-\lambda_4)^4}\right. \\ &\quad \left.+\frac{\lambda_4^{j+2}(j(\lambda_4-\lambda_2)^2+\lambda_4^2-4\lambda_2\lambda_4+3\lambda_2^2)}{(\lambda_4-\lambda_2)^4}\right) \\ &= \frac{\lambda_2^3-n\lambda_2^{n+2}+(n-1)\lambda_2^{n+3}}{(\lambda_2-\lambda_4)^2(1-\lambda_2)^2}+\frac{(\lambda_2^4-4\lambda_2^3\lambda_4+3\lambda_2^2\lambda_4^2)(\lambda_2^n-1)}{(\lambda_2-\lambda_4)^4(\lambda_2-1)} \\ &\quad +\frac{\lambda_4^3-n\lambda_4^{n+2}+(n-1)\lambda_4^{n+3}}{(\lambda_4-\lambda_2)^2(1-\lambda_4)^2}+\frac{(\lambda_4^4-4\lambda_2\lambda_4^3+3\lambda_2^2\lambda_4^2)(\lambda_4^n-1)}{(\lambda_4-\lambda_2)^4(\lambda_4-1)}. \end{aligned} \tag{88}$$

Corollary 4 *Assume that $a,b,c,d\in\mathbb{Z}$, $abcd\neq 0$ and $\alpha,\beta,z_{-1},z_0,w_{-2},w_{-1},w_0\in\mathbb{C}\setminus\{0\}$. Then the following statements are true*:

(a) *If polynomial p_4 has two pairs of double zeros both different from* 1, *then the general solution to* (2) *is given by* (34) *and* (44), *where $(a_n)_{n\geq -3}$ is given by* (87), *$(y_n)_{n\geq -3}$ is given by* (88), *while λ_j, $j=\overline{1,4}$, are given by* (85).

(b) *The characteristic polynomial* (45) *cannot have two pairs of double zeros such that one of them is equal to* 1.

Triple zero case. In this case we must have $\Delta=\Delta_0=0$ or equivalently $\Delta_0=\Delta_1=0$, that is,

$$a=0 \quad \text{or} \quad bd=3c.$$

Since $abcd\neq 0$, the case $a=0$ is not possible. If $c=bd/3$, then

$$\Delta_1=27c\bigl(a^3+16c\bigr).$$

Since the case $c=0$ is excluded, we must have $c=-a^3/16$.

We have

$$p_4(\lambda)=\lambda^4-a\lambda^3+\frac{a^3}{4}\lambda-\frac{a^4}{16}=\left(\lambda-\frac{a}{2}\right)^3\left(\lambda+\frac{a}{2}\right)$$

(see, for example, [19], p.15), and consequently

$$\lambda_j=\frac{a}{2},\quad j=\overline{1,3},\qquad \lambda_4=-\frac{a}{2}. \tag{89}$$

Thus, for every $a\neq 0$, $a/2$ is a triple zero of p_4, and p_4 cannot have a zero of the fourth order.

Hence

$$a_n=\bigl(\gamma_1+\gamma_2 n+\gamma_3 n^2\bigr)\lambda_1^n+\gamma_4\lambda_4^n,\quad n\in\mathbb{N}, \tag{90}$$

where γ_i and $i=\overline{1,4}$ are constants, is the general solution to (35) in this case.

Further, by using the initial conditions in (36), we obtain

$$\gamma_1 = 1 - \frac{\lambda_4^3}{(\lambda_4-\lambda_1)^3}, \qquad \gamma_2 = \frac{\lambda_1(3\lambda_1-5\lambda_4)}{2(\lambda_4-\lambda_1)^2},$$
$$\gamma_3 = \frac{\lambda_1}{2(\lambda_1-\lambda_4)}, \qquad \gamma_4 = \frac{\lambda_4^3}{(\lambda_4-\lambda_1)^3}.$$

Thus

$$a_n = 1 - \frac{\lambda_4^3}{(\lambda_4-1)^3} + \frac{3-5\lambda_4}{2(\lambda_4-1)^2}n + \frac{1}{2(1-\lambda_4)}n^2 + \frac{\lambda_4^{n+3}}{(\lambda_4-1)^3}, \quad n \ge -3, \tag{91}$$

when $\lambda_1 = 1$, while if $\lambda_1 \ne 1$, then

$$a_n = \left(1 - \frac{\lambda_4^3}{(\lambda_4-\lambda_1)^3} + \frac{\lambda_1(3\lambda_1-5\lambda_4)}{2(\lambda_4-\lambda_1)^2}n + \frac{\lambda_1}{2(\lambda_1-\lambda_4)}n^2\right)\lambda_1^n + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)^3}, \tag{92}$$

for $n \ge -3$.

From (38), (91) and Lemma 2, it follows that

$$y_n = \left(1 - \frac{\lambda_4^3}{(\lambda_4-1)^3}\right)n + \frac{(3-5\lambda_4)(n-1)n}{4(\lambda_4-1)^2} + \frac{(n-1)n(2n-1)}{12(1-\lambda_4)} + \frac{\lambda_4^3(\lambda_4^n-1)}{(\lambda_4-1)^4}. \tag{93}$$

From (38), (92) and Lemma 2, it follows that

$$\begin{aligned} y_n = {} & \left(1 - \frac{\lambda_4^3}{(\lambda_4-\lambda_1)^3}\right)\frac{\lambda_1^n-1}{\lambda_1-1} + \frac{\lambda_1^2(3\lambda_1-5\lambda_4)(1-n\lambda_1^{n-1}+(n-1)\lambda_1^n)}{2(\lambda_4-\lambda_1)^2(1-\lambda_1)^2} \\ & + \frac{\lambda_1^2(1+\lambda_1-n^2\lambda_1^{n-1}+(2n^2-2n-1)\lambda_1^n-(n-1)^2\lambda_1^{n+1})}{2(\lambda_1-\lambda_4)(1-\lambda_1)^3} \\ & + \frac{\lambda_4^3(\lambda_4^n-1)}{(\lambda_4-\lambda_1)^3(\lambda_4-1)}, \end{aligned} \tag{94}$$

for $n \in \mathbb{N}$.

Corollary 5 *Assume that $a,b,c,d \in \mathbb{Z}$, $abcd \ne 0$ and $\alpha,\beta,z_{-1},z_0,w_{-2},w_{-1},w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true*:

(a) *If polynomial* (45) *has a triple zero different from* 1, *then the general solution to system* (2) *is given by* (34) *and* (44), *where $(a_n)_{n\ge-3}$ is given by* (92), *$(y_n)_{n\ge-3}$ is given by* (94), *while λ_j, $j = \overline{1,4}$, are given by* (89).

(b) *If polynomial* (45) *has a triple zero equal to* 1, *say λ_1, λ_2 and λ_3, then the general solution to system* (2) *is given by* (34) *and* (44), *where $(a_n)_{n\ge-3}$ is given by* (91), *$(y_n)_{n\ge-3}$ is given by* (93), *while λ_j, $j = \overline{1,4}$, are given by* (89) *with $a = 2$.*

Theorem 5 *Assume that $a,b,d \in \mathbb{Z}$, $abd \ne 0$, $c = 0$, $\alpha,\beta,z_{-1},z_0,w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (2) *is solvable in closed form.*

Proof We modify our method in [18, 24]. We have

$$z_{n+1} = \alpha z_n^a w_n^b, \qquad w_{n+1} = \beta z_{n-1}^d, \quad n \in \mathbb{N}_0, \tag{95}$$

and consequently

$$z_{n+1} = \alpha\beta^b z_n^a z_{n-2}^{bd}, \quad n \in \mathbb{N}. \tag{96}$$

Let $\delta = \alpha\beta^b$,

$$a_1 = a, \qquad b_1 = 0, \qquad c_1 = bd, \qquad y_1 = 1. \tag{97}$$

Then clearly

$$z_{n+1} = \delta^{y_1} z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1}, \quad n \in \mathbb{N}. \tag{98}$$

Hence,

$$\begin{aligned} z_{n+1} &= \delta^{y_1}\left(\delta z_{n-1}^{a_1} z_{n-2}^{b_1} z_{n-3}^{c_1}\right)^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} \\ &= \delta^{y_1+a_1} z_{n-1}^{a_1a_1+b_1} z_{n-2}^{b_1a_1+c_1} z_{n-3}^{c_1a_1} \\ &= \delta^{y_2} z_{n-1}^{a_2} z_{n-2}^{b_2} z_{n-3}^{c_2}, \end{aligned}$$

for $n \geq 2$, where

$$a_2 := a_1a_1 + b_1, \qquad b_2 := b_1a_1 + c_1, \qquad c_2 := c_1a_1, \qquad y_2 := y_1 + a_1.$$

Assume

$$z_{n+1} = \delta^{y_k} z_{n+1-k}^{a_k} z_{n-k}^{b_k} z_{n-k-1}^{c_k}, \tag{99}$$

for a $k \geq 2$ and every $n \geq k$, and

$$a_k = a_1a_{k-1} + b_{k-1}, \qquad b_k = b_1a_{k-1} + c_{k-1}, \qquad c_k = c_1a_{k-1}, \tag{100}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{101}$$

Further, by (98), it follows that

$$\begin{aligned} z_{n+1} &= \delta^{y_k}\left(\delta z_{n-k}^{a_1} z_{n-k-1}^{b_1} z_{n-k-2}^{c_1}\right)^{a_k} z_{n-k}^{b_k} z_{n-k-1}^{c_k} \\ &= \delta^{y_k+a_k} z_{n-k}^{a_1a_k+b_k} z_{n-k-1}^{b_1a_k+c_k} z_{n-k-2}^{c_1a_k} \\ &= \delta^{y_{k+1}} z_{n-k}^{a_{k+1}} z_{n-k-1}^{b_{k+1}} z_{n-k-2}^{c_{k+1}}, \end{aligned}$$

for $n \geq k+1$, where

$$\begin{aligned} &a_{k+1} := a_1a_k + b_k, \qquad b_{k+1} := b_1a_k + c_k, \\ &c_{k+1} := c_1a_k, \qquad y_{k+1} := y_k + a_k. \end{aligned}$$

Hence, by induction we see that (99)-(101) hold.

Setting $k = n$ in (99), and employing (28), we get

$$\begin{aligned} z_{n+1} &= \delta^{y_n} z_1^{a_n} z_0^{b_n} z_{-1}^{c_n} \\ &= \left(\alpha\beta^b\right)^{y_n} \left(\alpha z_0^a w_0^b\right)^{a_n} z_0^{b_n} z_{-1}^{c_n} \\ &= \alpha^{y_n+a_n} \beta^{by_n} z_{-1}^{c_n} z_0^{aa_n+b_n} w_0^{ba_n} \\ &= \alpha^{y_{n+1}} \beta^{by_n} z_{-1}^{bda_{n-1}} z_0^{a_{n+1}} w_0^{ba_n}, \end{aligned} \tag{102}$$

for $n \geq 2$.

From (100) we see that a_k, b_k and c_k are solutions to

$$\tilde{x}_{k+3} = a_1\tilde{x}_{k+2} + b_1\tilde{x}_{k+1} + c_1\tilde{x}_k, \quad k \in \mathbb{N}, \tag{103}$$

and that along with (100) and (101) (for $k = 0, -1, -2$), we obtain

$$a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_0 = 1, \tag{104}$$

$$y_{-2} = y_{-1} = y_0 = 0, \qquad y_1 = 1, \tag{105}$$

and

$$y_k = \sum_{j=0}^{k-1} a_j. \tag{106}$$

The solvability of (103) is well known, from which along with (104) is obtained a formula for a_k, which along with (106) and Lemma 2 yields a formula for y_k. Hence, (96) is solvable.

Using (102), in the second equation in (95), is obtained:

$$w_n = \alpha^{dy_{n-2}} \beta^{bdy_{n-3}+1} z_{-1}^{bd^2 a_{n-4}} z_0^{da_{n-2}} w_0^{bda_{n-3}}, \quad n \geq 5. \tag{107}$$

It is shown that equations (102) and (107) are solutions to system (2), so it is solvable, as claimed. □

Theorem 5 gives a general form of solutions to system (2) in the case $c = 0$, $abd \neq 0$, but it does not present explicit formulas for a_n and y_n involved in the solutions. We give some explicit formulas for them, following also some arguments in [19]. Since $bd \neq 0$, we find the zeros of the characteristic polynomial associated to (103)

$$p_3(\lambda) = \lambda^3 - a\lambda^2 - bd. \tag{108}$$

For $\lambda = s + \frac{a}{3}$, the equation $p_3(\lambda) = 0$ becomes

$$s^3 - \frac{a^2}{3}s - \frac{2a^3 + 27bd}{27} = 0. \tag{109}$$

We know that

$$s_j = \frac{1}{3\sqrt[3]{2}}\left(\varepsilon^j \sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \overline{\varepsilon}^j \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\right), \tag{110}$$

$j=\overline{0,2}$, where

$$\Delta_0 = a^2 =: -3p, \qquad \Delta_1 = 2a^3 + 27bd =: -27q, \tag{111}$$

$\varepsilon^3 = 1$ and $\varepsilon \neq 1$, are the zeros of (109).

Hence, the zeros of p_3 are

$$\lambda_j = \frac{a}{3} + \frac{1}{3\sqrt[3]{2}}\left(\varepsilon^j \sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \overline{\varepsilon}^j \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\right), \quad j=\overline{0,2}. \tag{112}$$

Zeros of p_3 are mutually different and different from 1. In the case $\Delta_1^2 - 4\Delta_0^3 \neq 0$, we get $bd(4a^3 + 27bd) \neq 0$. If $0 \neq bd \neq -4a^3/27$, then the zeros of (108) are mutually different. If also $a + bd \neq 1$, then they are different from 1. The case $a = bd = k \in \mathbb{N}$ is such one.

Zeros of p_3 are different and one of them is 1. In this case we have $a + bd = 1$. Hence

$$p_3(\lambda) = \lambda^3 - a\lambda^2 + a - 1 = (\lambda - 1)\big(\lambda^2 - (a-1)\lambda - (a-1)\big),$$

and consequently

$$\lambda_1 = 1, \qquad \lambda_{2,3} = \frac{a - 1 \pm \sqrt{a^2 + 2a - 3}}{2}. \tag{113}$$

Since $p_3'(1) = 3 - 2a \neq 0$, when $a \in \mathbb{Z}$, the polynomial in (108) cannot have the unity as a double zero.

It is well known that the general solution to (103) in this case is

$$a_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \quad n \in \mathbb{N}, \tag{114}$$

where α_j, $j=\overline{1,3}$, are constants, which due to $c_1 = bd \neq 0$ can be prolonged for every non-positive index.

From (114) and by Lemma 1 with $R_3(s) = \prod_{j=1}^3 (s - \lambda_j)$, we get

$$a_n = \frac{\lambda_1^{n+2}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_2^{n+2}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_3^{n+2}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}, \tag{115}$$

for $n \geq -2$ (see, for example, [18]).

From (106) and (115), it follows that

$$y_n = \sum_{i=0}^{n-1}\sum_{j=1}^{3} \frac{\lambda_j^{i+2}}{p_3'(\lambda_j)}, \tag{116}$$

for $n \in \mathbb{N}$.

Equation (116) shows that

$$\begin{aligned} y_n = {} & \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - 1)} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - 1)} \\ & + \frac{\lambda_3^2(\lambda_3^n - 1)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - 1)}, \quad n \in \mathbb{N}, \end{aligned} \tag{117}$$

when $\lambda_j \neq 1$, $j=\overline{1,3}$.

If one of the zeros of p_3 is 1, say, λ_3, then

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_1 - \lambda_2)(\lambda_1 - 1)^2} + \frac{\lambda_2^2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2} + \frac{n}{(\lambda_1 - 1)(\lambda_2 - 1)}, \tag{118}$$

for $n \in \mathbb{N}$. Moreover, equations (117) and (118) hold for $n \geq -2$.

Corollary 6 *Assume that $a, b, c, d \in \mathbb{Z}$, $abd \neq 0$, $c = 0$, $\alpha, \beta, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta_1^2 \neq 4\Delta_0^3$. Then the following statements are true*:

(a) *If $a + bd \neq 1$, then the general solution to* (2) *is given by* (102) *and* (107), *where $(a_n)_{n\geq-2}$ is given by* (115), *$(y_n)_{n\geq-2}$ is given by* (117), *while λ_j, $j = \overline{1,3}$, are given by* (112).

(b) *If $a + bd = 1$, then p_3 has a unique zero equal to* 1, *say λ_3, and the general solution to* (2) *is given by formulas* (102) *and* (107), *where $(a_n)_{n\geq-2}$ is given by* (115) *with $\lambda_3 = 1$, $(y_n)_{n\geq-2}$ is given by* (118), *while λ_j, $j = \overline{1,3}$, are given by* (113).

p_3 has a double zero. Since it must be $\Delta_1^2 = 4\Delta_0^3$, we have $bd = -4a^3/27$, so that

$$p_3(\lambda) = \lambda^3 - a\lambda^2 + \frac{4}{27}a^3.$$

The following condition must also be satisfied: $p_3'(\lambda) = 0$. Hence, $\lambda_1 = -a/3$, $\lambda_{2,3} = 2a/3$, and consequently

$$p_3(\lambda) = \left(\lambda - \frac{2a}{3}\right)^2 \left(\lambda + \frac{a}{3}\right).$$

Due to $bd \in \mathbb{Z}$, we get $a = 3\hat{a}$, for some $\hat{a} \in \mathbb{Z}$. Now note that $2a/3 \neq 1$, for every $a \in \mathbb{Z}$, so that 1 cannot be a double zero of p_3.

Hence

$$a_n = \hat{\alpha}_1\lambda_1^n + (\hat{\alpha}_2 + \hat{\alpha}_3 n)\lambda_2^n, \quad n \in \mathbb{N}, \tag{119}$$

where $\hat{\alpha}_i$, $i = \overline{1,3}$, are constants. Using initial conditions (104) we obtain

$$a_n = \frac{\lambda_1^{n+2} + (\lambda_2 - 2\lambda_1 + n(\lambda_2 - \lambda_1))\lambda_2^{n+1}}{(\lambda_2 - \lambda_1)^2}, \tag{120}$$

for $n \geq -2$.

From (106) and (120), it follows that

$$y_n = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+2} + (\lambda_2 - 2\lambda_1 + j(\lambda_2 - \lambda_1))\lambda_2^{j+1}}{(\lambda_2 - \lambda_1)^2}, \tag{121}$$

for $n \in \mathbb{N}$.

Equation (121) along with Lemma 2 yields

$$y_n = \frac{\lambda_1^2(\lambda_1^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_1 - 1)} + \frac{(\lambda_2 - 2\lambda_1)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - \lambda_1)^2(\lambda_2 - 1)} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - \lambda_1)(\lambda_2 - 1)^2}, \tag{122}$$

for $n \in \mathbb{N}$.

On the other hand, if $\lambda_1 = 1 \neq \lambda_2 = \lambda_3$ ($a = -3$), then we get

$$y_n = \frac{n}{(\lambda_2 - 1)^2} + \frac{(\lambda_2 - 2)\lambda_2(\lambda_2^n - 1)}{(\lambda_2 - 1)^3} + \frac{\lambda_2^2(1 - n\lambda_2^{n-1} + (n-1)\lambda_2^n)}{(\lambda_2 - 1)^3}, \tag{123}$$

for $n \in \mathbb{N}$. Moreover, (122) and (123) hold for $n \geq -2$.

Corollary 7 *Assume that* $a, b, c, d \in \mathbb{Z}$, $abd \neq 0$, $c = 0$, $\alpha, \beta, z_{-1}, z_0, w_0 \in \mathbb{C} \setminus \{0\}$ *and* $\Delta_1^2 = 4\Delta_0^3$. *Then the following statements are true*:

(a) *If* $a + bd \neq 1$, *then the general solution to* (2) *is given by* (102) *and* (107), *where* $(a_n)_{n\geq -2}$ *is given by* (120), $(y_n)_{n\geq -2}$ *is given by* (122), *where* $\lambda_1 = -a/3$ *and* $\lambda_{2,3} = 2a/3$.

(b) *If only one of the zeros of the polynomial* (108) *is equal to* 1, *say*, λ_1, *then the general solution to system* (2) *is given by* (102) *and* (107), *where* $(a_n)_{n\geq -2}$ *is given by* (120) *with* $\lambda_1 = 1$, *while* $(y_n)_{n\geq -2}$ *is given by* (123).

(c) *It is not possible that two zeros of polynomial* (108) *are equal to one*.

Case when all the zeros of p_3 *are equal*. We have $p_3(\lambda) = p_3'(\lambda) = p_3''(\lambda) = 0$. So, $p_3''(\lambda) = 0$ would imply $\lambda = a/3$. From $p_3'(\lambda) = 3\lambda^2 - 2a\lambda$, we see that $a/3$ is a unique zero of p_3 if $a = 0$, which contradicts the assumption $abd \neq 0$. Hence, the case is not possible.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author has contributed solely to the writing of this paper. He read and approved the manuscript.

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