Liu et al. *Advances in Difference Equations* (2017) 2017:258
DOI 10.1186/s13662-017-1295-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Threshold dynamical analysis on a class of age-structured tuberculosis model with immigration of population

Lili Liu[1*], Xinzhi Ren[2] and Zhen Jin[1]

*Correspondence: liulili03@126.com
[1]Complex Systems Research Center, Shanxi University, Taiyuan, Shan'xi 030006, China
Full list of author information is available at the end of the article

Abstract

Some studies show that latency and relapse, especially the age-dependent latency and relapse, may affect the transmission dynamics of tuberculosis model. Meanwhile, the immigration of infected individuals induces the loss of disease-free steady state and hence no basic reproduction number. In our work, a class of age-structured tuberculosis model with immigration is proposed, where the new individuals can immigrate into the susceptible, latent, infectious and removed compartments. We show that the endemic steady state is unique and globally asymptotically stable by using the Lyapunov functional. Numerical simulations are given to support our theoretical results.

Keywords: tuberculosis model; age-structured; immigration; global stability; Lyapunov functional

1 Introduction

Tuberculosis (TB), mainly caused by *Mycobacterium tuberculosis*, is a widespread infectious disease and has become a global public health issue. Despite various treatment strategies and beneficial policies on TB patients, the current global TB remains a leading cause of death from an infectious disease. According to reports, there were one death in five in England in the 17th century [1]. About 8.7×10^6 cases of TB globally is estimated in 2011, in which India has the largest total infected population, with an estimated 2.2 million new cases; China has the second largest TB epidemic, with more than 1.3×10^6 new cases every year [2].

Mathematical models have been a useful tool to understand and analyze the transmission dynamics of TB and other infectious diseases. In [3], a SEI type of TB model with a general contact rate is considered, and the global stability of equilibria is derived. In [4], a TB model with early and late latent stages is introduced to discuss effectiveness of treating TB patients at different stages. The reader can refer to more related mathematical models for TB; see [5–7]. It is well known that TB experiences a latent phase as well as a relapse phase which are the removed individuals who have been previously infected, but revert back to the infectious compartments due to the reactivation (see [8, 9]). Hence, in this paper, we focus on an SEIR-type model.

The level of infectiousness and the likelihood of progression may depend on the types of infectious diseases and individuals' status. Thus, the age-structured epidemic models are thought to be more practical to describe certain disease features. Taking into consideration the age-dependent latency and relapse, Liu [10] established and researched the following initial-boundary-value problem for a hybrid system of ordinary and partial differential equations:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda_s - \mu_s S(t) - \beta S(t)I(t), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})e(t,a) = -\sigma(a)e(t,a) - \mu_e e(t,a), \\ \frac{dI(t)}{dt} = \int_0^\infty \sigma(a)e(t,a)\,da - (\mu_i + k)I(t) + \int_0^\infty \gamma(b)r(t,b)\,db, \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial b})r(t,b) = -\gamma(b)r(t,b) - \mu_r r(t,b), \end{cases} \tag{1.1}$$

with the boundary conditions

$$e(t,0) = \beta S(t)I(t), \qquad r(t,0) = kI(t) \tag{1.2}$$

for $t \geq 0$ and the initial conditions

$$S(0) = S_0, \qquad e(0,a) = e_0(a), \qquad I(0) = I_0, \qquad r(0,b) = r_0(b) \tag{1.3}$$

for $a, b \geq 0$. This model can be used to describe TB transmission dynamics. The initial conditions satisfy $S_0, I_0 \in \mathbb{R}^+$ and $e_0(a), r_0(b) \in L^1_+(\mathbb{R}^+, \mathbb{R}^+)$, where $L^1_+(\mathbb{R}^+, \mathbb{R}^+)$ is the space of functions on $[0,\infty)$ that are nonnegative and Lebesgue integrable. Here, $S(t)$ and $I(t)$ are the densities of susceptible and infectious individuals at time t, and $e(t,a)$ and $r(t,b)$ represent the densities of latent and removed individuals at time t, age a and b, respectively. a and b are referred to the latent age and the time spending at the removed compartment. Λ_s denotes the recruitment of susceptible individuals, μ_j ($j = s, e, i, r$) denotes their per capita mortality rate and k denotes the recovery rate. $\sigma(a)$ and $\gamma(b)$ are the removal rate from latent and removed compartments, respectively. The authors showed the asymptotic smoothness of solutions and uniform persistence of system (1.1). Furthermore, they obtained the basic reproduction number $\Re_0$, which played as a threshold parameter and satisfied that if $\Re_0 < 1$, then the disease-free equilibrium of system (1.1) is locally and globally asymptotically stable, while if $\Re_0 > 1$, then the endemic equilibrium uniquely exists and it is locally and globally asymptotically stable. More related work is done to understand the dynamics of age-structured models; see [11–15]. Famous books on age-structured models are by Webb [16], Iannelli [17] and Smith *et al.* [18].

In the modern world, the worldwide transportation leads to tremendous movement of individuals and it is inevitable that infectious diseases may be introduced into a population from outside the population. Although most of the developed countries have screening policies for new immigrants, latent tuberculosis may take long time to become infectious and latent tuberculosis individuals may travel, thus, they are usually not detected by the TB screening. Furthermore, the removed tuberculosis patients who often have higher relapse rate may travel into another region; or the infectious individuals may also travel. In [19], McCluskey *et al.* proposed a TB model with immigration of both latent and infective compartments. Later, In [20], Guo *et al.* introduced a TB model with treatment and immigration into the latent compartment. It is well known that the model will no longer have the disease-free equilibrium and the basic reproduction number, and there will always have a unique endemic equilibrium which is globally asymptotically stable by using a global Lyapunov function. More related work can be found in [21–25] and references therein.

In this paper, we propose and investigate a TB model with immigration into the four compartments. We also incorporate into the continuous age-dependent in latent and removed compartments. It is a generalization of model proposed in [10]. According to mathematical analysis, we show that TB always exists in a region and the endemic equilibrium is unique and globally asymptotically stable. This paper is organized as follows. In the next section, we will formulate the model. In Section 3, we show the mathematical well-posedness of our model. In Section 4, we investigate the asymptotic smoothness of the semi-flow generated by our model and the existence of compact attractor. In Section 5, we show our dynamics results, including the existence and global stability of the unique endemic steady state. Some simulations and conclusion are provided in Section 6 and Section 7, respectively.

2 Model formulation

This section we devote to formulating our model.

Assume that there is constant recruitment into the susceptible and infectious compartments at rates Λ_s and Λ_i, respectively. The recruitment into the latent and removed compartments with age-in-class a and b take place at rates denoted $\Lambda_e(a)$ and $\Lambda_r(b)$, respectively. The transfer diagram is shown in Figure 1. The corresponding hybrid system of ordinary and partial differential equations is the following form:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda_s - \mu_s S(t) - \beta S(t)I(t), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})e(t,a) = \Lambda_e(a) - \sigma(a)e(t,a) - \mu_e e(t,a), \\ \frac{dI(t)}{dt} = \Lambda_i + \int_0^\infty \sigma(a)e(t,a)\,da - (\mu_i + k)I(t) + \int_0^\infty \gamma(b)r(t,b)\,db, \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial b})r(t,b) = \Lambda_r(b) - \gamma(b)r(t,b) - \mu_r r(t,b), \end{cases} \tag{2.1}$$

with the boundary conditions

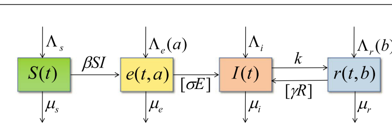
$$e(t,0) = \beta S(t)I(t), \qquad r(t,0) = kI(t), \quad t \geq 0,$$

and the initial conditions

$$S(0) = S_0, \qquad e(0,a) = e_0(a), \qquad I(0) = I_0, \qquad r(0,b) = r_0(b), \quad a,b \geq 0.$$

Note that system (2.1) considers the immigration of all compartments, which is different from system (1.1), where only considers the immigration of susceptible compartment. All

Figure 1 The schematic flow diagram of our model. Here, the terms $[\sigma E]$ and $[\gamma R]$ stand for $\int_0^\infty \sigma(a)e(t,a)\,da$ and $\int_0^\infty \gamma(b)r(t,b)\,db$.



parameters have the same biological meanings with system (1.1) and satisfy the following assumptions.

Assumption 2.1 Consider system (2.1), we assume that:

(A_1) All constant parameters $\Lambda_s, \Lambda_i, \mu_s, \mu_e, \mu_i, \mu_r, k > 0$.
(A_2) The functions $\sigma(a), \gamma(b) \in L^\infty(\mathbb{R}^+, \mathbb{R}^+)$, and denote $\sigma^{\inf}$, $\gamma^{\inf}$ and $\sigma^{\sup}$, $\gamma^{\sup}$ as the essential infimums and the essential supremums of σ and γ, respectively.
(A_3) $\sigma(a)$ and $\gamma(b)$ are Lipschitz continuous on $\mathbb{R}^+$ with Lipschitz coefficients M_σ and M_γ, respectively.
(A_4) $\int_0^\infty \sigma(a)\,da = 0$ and $\int_0^\infty \gamma(b)\,db = 0$.
(A_5) $\Lambda_e, \Lambda_r \in L^1(\mathbb{R}^+, \mathbb{R}^+)$, and denote $\bar{\Lambda}_e = \int_0^\infty \Lambda_e(a)\,da$, $\bar{\Lambda}_r = \int_0^\infty \Lambda_r(b)\,db$.
(A_6) $\bar{\Lambda}_e, \bar{\Lambda}_r \in \mathbb{R}^+$, and $\bar{\Lambda}_e + \bar{\Lambda}_r > 0$.

Define the space of functions X as $X = \mathbb{R}^+ \times L^1_+(\mathbb{R}^+, \mathbb{R}^+) \times \mathbb{R}^+ \times L^1_+(\mathbb{R}^+, \mathbb{R}^+)$ with the norm

$$\|(x_1, x_2, x_3, x_4)\|_X = |x_1| + \int_0^\infty |x_2(a)|\,da + |x_3| + \int_0^\infty |x_4(b)|\,db.$$

Following the standard theory [16], it can be verified that system (2.1) with initial-boundary conditions has a unique nonnegative solution for all time. Thus, there exists a continuous semi-flow associated with system (2.1), that is, $\Phi : \mathbb{R}^+ \times X \to X$ takes the following form:

$$\Phi(t, x_0) = \big(S(t), e(t,\cdot), I(t), r(t,\cdot)\big), \quad t \geq 0, x_0 \in X,$$

with

$$\begin{aligned} \|\Phi(t,x_0)\|_X &= \big\|\big(S(t), e(t,\cdot), I(t), r(t,\cdot)\big)\big\|_X \\ &= |S(t)| + \int_0^\infty |e(t,a)|\,da + |I(t)| + \int_0^\infty |r(t,b)|\,db. \end{aligned} \tag{2.2}$$

3 The well-posedness of system (2.1)

This section is devoted to the positivity and boundedness of solutions.

Assume that function $f(a,s)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^+$, then one has

$$\int_0^t \int_0^a f(a,s)\,ds\,da + \int_t^\infty \int_{a-t}^a f(a,s)\,ds\,da = \int_0^\infty \int_s^{s+t} f(a,s)\,ds\,da, \tag{3.1}$$

which is achieved by changing the order of integration in two double integrals. This will be useful in the later proofs.

For simplification, we denote

$$\begin{aligned} &\varepsilon(a) = \sigma(a) + \mu_e, \qquad \rho_e(a) = \exp\left(-\int_0^a \varepsilon(s)\,ds\right), \qquad \theta_e = \int_0^\infty \sigma(a)\rho_e(a)\,da, \\ &\eta(b) = \gamma(b) + \mu_r, \qquad \rho_r(b) = \exp\left(-\int_0^b \eta(s)\,ds\right), \qquad \theta_r = \int_0^\infty \gamma(b)\rho_r(b)\,db. \end{aligned}$$

It follows from (A_3) of Assumption 2.1 that

$$0 \le \rho_e(a) \le e^{-\mu_e a} \le 1, \qquad 0 \le \rho_r(b) \le e^{-\mu_r b} \le 1 \tag{3.2}$$

and

$$\frac{d\rho_e(a)}{da} = -\varepsilon(a)\rho_e(a), \qquad \frac{d\rho_r(b)}{db} = -\eta(b)\rho_r(b). \tag{3.3}$$

Similar to [16], solutions of the PDE parts are

$$e(t,a) = \begin{cases} e(t-a,0)\rho_e(a) + \int_0^a \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds, & t > a \ge 0; \\ e_0(a-t)\frac{\rho_e(a)}{\rho_e(a-t)} + \int_{a-t}^a \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds, & a \ge t \ge 0, \end{cases} \tag{3.4}$$

and

$$r(t,b) = \begin{cases} r(t-b,0)\rho_r(b) + \int_0^b \Lambda_r(s)\frac{\rho_r(b)}{\rho_r(s)}\,ds, & t > b \ge 0; \\ r_0(b-t)\frac{\rho_r(b)}{\rho_r(b-t)} + \int_{b-t}^b \Lambda_r(s)\frac{\rho_r(b)}{\rho_r(s)}\,ds, & b \ge t \ge 0. \end{cases} \tag{3.5}$$

Finally, we define the state space for system (2.1) as

$$\Omega = \left\{ \bigl(S(t), e(t,\cdot), I(t), r(t,\cdot)\bigr) \in X : S(t) + \int_0^\infty e(t,a)\,da + I(t) + \int_0^\infty r(t,b)\,db \le \frac{\Lambda^*}{\mu^*} \right\},$$

where $\Lambda^* = \Lambda_s + \bar{\Lambda}_e + \Lambda_i + \bar{\Lambda}_r$, $\mu^* = \min\{\mu_j, j = s, e, i, r\}$. We can prove that Ω and Φ have the following properties.

Proposition 3.1 *Consider system* (2.1), *we have*

(i) *Ω is positively invariant for Φ, that is, $\Phi(t,x_0) \in \Omega$, for $\forall t \ge 0$ and $x_0 \in \Omega$;*
(ii) *Φ is point dissipative and Ω attracts all points in X.*

Proof From (2.2), we have

$$\frac{d}{dt}\bigl\|\Phi(t,x_0)\bigr\|_X = \frac{dS(t)}{dt} + \frac{d}{dt}\int_0^\infty e(t,a)\,da + \frac{dI(t)}{dt} + \frac{d}{dt}\int_0^\infty r(t,b)\,db. \tag{3.6}$$

Before calculating the derivative of semi-flow, we first show that

$$\begin{aligned} \int_0^\infty e(t,a)\,da &= \int_0^t e(t,a)\,da + \int_t^\infty e(t,a)\,da \\ &= \int_0^t e(t-a,0)\rho_e(a)\,da + \int_0^t \int_0^a \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da \\ &\quad + \int_t^\infty e_0(a-t)\frac{\rho_e(a)}{\rho_e(a-t)}\,da + \int_t^\infty \int_{a-t}^a \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da \\ &= \int_0^t e(t-a,0)\rho_e(a)\,da + \int_t^\infty e_0(a-t)\frac{\rho_e(a)}{\rho_e(a-t)}\,da \\ &\quad + \int_0^\infty \int_s^{s+t} \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da. \end{aligned} \tag{3.7}$$

Note that the last term is obtained by using (3.1). Furthermore, let $\tau = t - a$ and $\tau = a - t$ in the first and second integrals on the right-hand side of (3.7), respectively, one has

$$\begin{aligned}\int_0^\infty e(t,a)\,da &= \int_0^t e(\tau,0)\rho_e(t-\tau)\,d\tau + \int_t^\infty e_0(\tau)\frac{\rho_e(t+\tau)}{\rho_e(\tau)}\,d\tau \\ &\quad + \int_0^\infty \int_s^{s+t} \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da.\end{aligned}$$

Then we have

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)\,da &= \frac{d}{dt}\int_0^t e(\tau,0)\rho_e(t-\tau)\,d\tau + \frac{d}{dt}\int_t^\infty e_0(\tau)\frac{\rho_e(t+\tau)}{\rho_e(\tau)}\,d\tau \\ &\quad + \frac{d}{dt}\int_0^\infty \int_s^{s+t} \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da \\ &= e(t,0) + \int_0^t e(\tau,0)\frac{d}{dt}\rho_e(t-\tau)\,d\tau + \int_t^\infty e_0(\tau)\frac{\frac{d}{dt}\rho_e(t+\tau)}{\rho_e(\tau)}\,d\tau \\ &\quad + \int_0^\infty \Lambda_e(s)\frac{\rho_e(s+t)}{\rho_e(s)}\,ds \\ &= e(t,0) + \int_0^t e(t-a,0)\frac{d}{da}\rho_e(a)\,da + \int_t^\infty e_0(a-t)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(a-t)}\,da \\ &\quad + \int_0^\infty \Lambda_e(s)\frac{\rho_e(s+t)}{\rho_e(s)}\,ds. \end{aligned} \tag{3.8}$$

Here, we put our main attentions on dealing with the last term. Following (3.1), we have

$$\begin{aligned}&\int_0^t \int_0^a \Lambda_e(s)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(s)}\,ds\,da + \int_t^\infty \int_{a-t}^a \Lambda_e(s)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(s)}\,ds\,da \\ &\quad = \int_0^\infty \int_s^{s+t} \Lambda_e(s)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(s)}\,ds\,da.\end{aligned}$$

Furthermore, one has

$$\begin{aligned}\int_0^\infty \int_s^{s+t} \Lambda_e(s)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(s)}\,ds\,da &= \int_0^\infty \Lambda_e(s)\frac{\rho_e(s+t)-\rho_e(s)}{\rho_e(s)}\,ds \\ &= \int_0^\infty \Lambda_e(s)\frac{\rho_e(s+t)}{\rho_e(s)}\,ds - \int_0^\infty \Lambda_e(s)\,ds. \end{aligned} \tag{3.9}$$

Applying (3.9) to replace the last term of (3.8), one gets

$$\begin{aligned}\frac{d}{dt}\int_0^\infty e(t,a)\,da &= e(t,0) + \int_0^t e(t-a,0)\frac{d}{da}\rho_e(a)\,da + \int_t^\infty e_0(a-t)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(a-t)}\,da \\ &\quad + \int_0^t \int_0^a \Lambda_e(s)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(s)}\,ds\,da \\ &\quad + \int_t^\infty \int_{a-t}^a \Lambda_e(s)\frac{\frac{d}{da}\rho_e(a)}{\rho_e(s)}\,ds\,da + \int_0^\infty \Lambda_e(s)\,da \\ &= e(t,0) - \int_0^\infty \varepsilon(a)e(t,a)\,da + \bar{\Lambda}_e. \end{aligned} \tag{3.10}$$

Similarly, one can obtain

$$\frac{d}{dt}\int_0^\infty r(t,b)\,db = r(t,0) - \int_0^\infty \eta(b)r(t,b)\,db + \bar{\Lambda}_r. \tag{3.11}$$

Combining (3.10)-(3.11) and the first and third equations of (2.1) yields

$$\begin{aligned}
&\frac{d}{dt}\left(S(t) + \int_0^\infty e(t,a)\,da + I(t) + \int_0^\infty r(t,b)\,db\right)\\
&\quad = (\Lambda_s + \bar{\Lambda}_e + \Lambda_i + \bar{\Lambda}_r) - \left(\mu_s S(t) + \mu_e\int_0^\infty e(t,a)\,da + \mu_i I(t) + \mu_r\int_0^\infty r(t,b)\,db\right)\\
&\quad \le \Lambda^* - \mu^*\left(S(t) + \int_0^\infty e(t,a)\,da + I(t) + \int_0^\infty r(t,b)\,db\right),
\end{aligned}$$

which implies

$$\left\|\Phi(t,x_0)\right\|_X \le \frac{\Lambda^*}{\mu^*} - e^{-\mu^* t}\left(\frac{\Lambda^*}{\mu^*} - \|x_0\|_X\right), \quad \text{for } t \ge 0. \tag{3.12}$$

Obviously, $\Phi(t,x_0) \in \Omega$ holds true for any solution of (2.1) satisfying $x_0 \in \Omega$ and $t \ge 0$. Thus, Ω is positively invariant for semi-flow $\{\Phi(t)\}_{t\ge 0}$.

Moreover, it follows from (3.12) that $\limsup_{t\to\infty}\|\Phi(t,x_0)\|_X \le \Lambda^*/\mu^*$ for any $x_0 \in X$. Therefore, Φ is point dissipative and Ω attracts all points in X. This completes the proof. □

Combining Assumption 2.1 and Proposition 3.1, we have the following two propositions.

Proposition 3.2 *If $x_0 \in X$ and $\|x_0\|_X \le M$ for some constant $M \ge \Lambda^*/\mu^*$, then the following statements hold true for $t \ge 0$:*

(i) $0 \le S(t), \int_0^\infty e(t,a)\,da, I(t), \int_0^\infty r(t,b)\,db \le M$;

(ii) $e(t,0) \le \beta M^2, r(t,0) \le kM$.

Proposition 3.3 *Let $B \subset X$ be bounded. Then*

(i) *$\Phi(\mathbb{R}^+,B)$ is bounded;*

(ii) *Φ is eventually bounded on B;*

(iii) *if $M \ge \Lambda^*/\mu^*$ is a bound for B, then M is also a bound for $\Phi(\mathbb{R}^+,B)$;*

(iv) *given any $L \ge \Lambda^*/\mu^*$, there exists $T = T(B,M)$ such that L is a bound for $\Phi(t,B)$ whenever $t \ge T$.*

Now, we give the following proposition on the asymptotic lower bounds for system (2.1).

Proposition 3.4 *If $x_0 \in X$, then*

$$\liminf_{t\to\infty} S(t) \ge m_s, \qquad \liminf_{t\to\infty} I(t) \ge m_i,$$

$$\liminf_{t\to\infty} e(t,0) \ge m_e, \qquad \liminf_{t\to\infty} r(t,0) \ge m_r,$$

where m_j, $j = s,i,e,r$ are defined below.

Proof It follows from the first equation of system (2.1) that

$$S' = \Lambda_s - \mu_s S - \beta SI \geq \Lambda_s - (\mu_s + \beta M)S,$$

which implies that

$$\liminf_{t\to\infty} S(t) \geq \frac{\Lambda_s}{\mu_s + \beta M} \triangleq m_s.$$

Similarly, one has

$$\liminf_{t\to\infty} I(t) \geq \frac{\Lambda_i + (\sigma^{\inf} + \gamma^{\inf})M}{\mu_i + k} \triangleq m_i.$$

Following (1.2), one can see that

$$\liminf_{t\to\infty} e(t,0) = \beta SI \geq \beta m_s m_i \triangleq m_e, \qquad \liminf_{t\to\infty} r(t,0) = kI \geq k m_i \triangleq m_r.$$

The proof is completed. □

4 Asymptotic smoothness

In order to obtain global properties of the semi-flow $\{\Phi(t)\}_{t\geq 0}$, it is important to prove that the semi-flow is asymptotically smooth.

We introduce two definitions and a useful lemma.

Definition 4.1 A function $Y : \mathbb{R} \to X$ is called a total trajectory of Φ if Y satisfies $\Phi_s(Y(t)) = Y(t+s)$ for all $t \in \mathbb{R}$ and $s \geq 0$.

Definition 4.2 A non-empty invariant compact set A is called the compact attractor of a class $\mathcal{B}$ of sets if $\operatorname{dist}(\Phi(t,B),A) \to 0$ for each $B \in \mathcal{B}$, where

$$\operatorname{dist}\big(\Phi(t,B),A\big) = \sup_{x\in\Phi_t(B)} \inf_{y\in A} \|x-y\|_X.$$

Remark 4.1 For any point $y_0 \in A$, it follows from Definitions 4.1 and 4.2, and [18], Theorem 1.40, that there exists a total trajectory $Y(\cdot)$ with $y(0) = y_0$ and $y(t) \in A$ for all $t \in \mathbb{R}$.

Lemma 4.1 [13] *Let $D \subseteq \mathbb{R}$. For $j = 1,2$, suppose $f_j : D \to \mathbb{R}$ is a bounded Lipshitz continuous function with bound K_j and Lipschitz coefficient M_j. Then the product function $f_1 f_2$ is Lipschitz with coefficient $K_1M_2 + K_2M_1$.*

In order to prove the asymptotic smoothness of the semi-flow, we will apply the following result, which is a special case of [18], Theorem 2.46.

Lemma 4.2 *If the following two conditions hold for any bounded closed set $B \subset X$, then the semi-flow $\Phi(t,x_0) = \Phi_1(t,x_0) + \Phi_2(t,x_0) : \mathbb{R}^+ \times X \to X$ is asymptotically smooth in X.*

(i) $\lim_{t\to\infty} \operatorname{diam} \Phi_1(t,B) = 0$;

(ii) *for $t \geq 0$, $\Phi_2(t,B)$ has compact closure.*

The following result is used to verify (ii) of Lemma 4.2, which is based on Theorem B.2 in [18].

Lemma 4.3 *A set $K \subset L^1(\mathbb{R}^+,\mathbb{R}^+)$ has compact closure if and only if the following conditions hold*:

(i) $\sup_{f\in K}\int_0^\infty f(a)\,da < \infty$;
(ii) $\lim_{h\to\infty}\int_h^\infty f(a)\,da \to 0$ *uniformly in* $f \in K$;
(iii) $\lim_{h\to 0^+}\int_0^\infty |f(a+h)-f(a)|\,da \to 0$ *uniformly in* $f \in K$;
(iv) $\lim_{h\to 0^+}\int_0^h f(a)\,da \to 0$ *uniformly in* $f \in K$.

Based on Lemmas 4.2 and 4.3, we have the following theorem.

Theorem 4.1 *The semi-flow* $\{\Phi(t)\}_{t\geq 0}$ *is asymptotically smooth*.

Proof We first decompose $\Phi:\mathbb{R}^+\times X\to X$ into the following two operators $\Phi_1(t,X)$ and $\Phi_2(t,X)$. Let $\Phi_1(t,X) := (0, y_2(t,\cdot), 0, y_4(t,\cdot))$ and $\Phi_2(t,X) := (S(t), \tilde{y}_2(t,\cdot), I(t), \tilde{y}_4(t,\cdot))$, where

$$y_2(t,a) := \begin{cases} \int_0^a \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds, & t > a \geq 0; \\ e(t,a), & a \geq t \geq 0 \end{cases} \quad \text{and} \quad y_4(t,b) := \begin{cases} \int_0^b \Lambda_r(s)\frac{\rho_r(b)}{\rho_r(s)}\,ds, & t > b \geq 0; \\ r(t,b), & b \geq t \geq 0, \end{cases} \tag{4.1}$$

$$\tilde{y}_2(t,a) := \begin{cases} e(t-a,0)\rho_e(a), & t > a \geq 0; \\ 0, & a \geq t \geq 0 \end{cases} \quad \text{and} \quad \tilde{y}_4(t,b) := \begin{cases} r(t-b)\rho_r(b), & t > b \geq 0; \\ 0, & b \geq t \geq 0. \end{cases} \tag{4.2}$$

Then we have $\Phi(t,X) = \Phi_1(t,X) + \Phi_2(t,X)$ for all $t \geq 0$.

First, we show that $\lim_{t\to\infty}\operatorname{diam}\Phi_1(t,B) = 0$. For $j = 1,2$, let $x_0^j = (S_0, e_0^j(\cdot), I_0, r_0^j(\cdot)) \in B$ be two initial conditions and $\Phi(t,x_0^j) = (S^j, e^j(t,\cdot), I^j, r^j(t,\cdot))$ be their corresponding solutions. Thus, one has

$$y_2^1(t,a) - y_2^2(t,a) = \begin{cases} 0, & t > a \geq 0; \\ (e_0^1(a-t) - e_0^2(a-t))\frac{\rho_e(a)}{\rho_e(a-t)}, & a \geq t \geq 0. \end{cases}$$

Then

$$\begin{aligned} \|y_2^1(t,\cdot) - y_2^2(t,\cdot)\|_1 &= \int_t^\infty |e_0^1(a-t) - e_0^2(a-t)|\frac{\rho_e(a)}{\rho_e(a-t)}\,da \\ &= \int_0^\infty |e_0^1(s) - e_0^2(s)|\frac{\rho_e(t+s)}{\rho_e(s)}\,ds \\ &= \int_0^\infty |e_0^1(s) - e_0^2(s)|e^{\int_s^{t+s}\varepsilon(\tau)\,d\tau}\,ds \\ &\leq e^{-\mu_e t}(\|e_0^1\|_1 + \|e_0^2\|_1) \\ &\leq 2Me^{-\mu_e t}, \end{aligned}$$

where $\|\cdot\|_1$ denotes the standard norm on L^1. Similarly, $\|y_4^1(t,\cdot)-y_4^2(t,\cdot)\| \leq 2Me^{-\mu_r t}$. Consequently, the distance between $\Phi_1(t,x_0^1)$ and $\Phi_2(t,x_0^2)$ satisfies

$$\left\|\Phi_1\left(t,x_0^1\right) - \Phi_2\left(t,x_0^2\right)\right\|_X \leq 2M\left(e^{-\mu_e t} + e^{-\mu_r t}\right),$$

which implies that $\Phi_1(t,B) \leq 2M(e^{-\mu_e t} + e^{-\mu_r t})$, due to the arbitrariness of $x_0^j \in B$, $j = 1,2$. Thus, $\lim_{t\to\infty} \operatorname{diam} \Phi_1(t,B) = 0$.

Now, we show that $\Phi_2(t,B)$ has compact closure. According to Proposition 3.3, $S(t)$ and $I(t)$ remain in the compact set $[0,\Lambda^*/\mu^*] \subset [0,M]$, where $M \geq \Lambda^*/\mu^*$ is a bound for B. Thus, it is only to show that $\tilde{y}_2(t,a)$ and $\tilde{y}_4(t,b)$ satisfy conditions (i)-(iv) in Lemma 4.3.

Now, following (3.4) and (4.2), we have

$$\begin{aligned} 0 \leq \tilde{y}_2(t,a) &= \begin{cases} \beta e(t-a,0)\rho_e(a), & t > a \geq 0, \\ 0, & a \geq t \geq 0 \end{cases} \\ &\leq \beta M^2 e^{-\mu_e a}, \end{aligned}$$

which implies that conditions (i), (ii) and (iv) in Lemma 4.3 are satisfied. It suffices to verify that (iii) in Lemma 4.3 holds true. For sufficiently small $h \in (0,t)$, we have

$$\begin{aligned} &\int_0^\infty \left|\tilde{y}_2(t,a+h) - \tilde{y}_2(t,a)\right| da \\ &\quad = \int_0^{t-h} \left|e(t-a-h,0)\rho_e(a+h) - e(t-a,0)\rho_e(a)\right| da + \int_{t-h}^t \left|0 - e(t-a,0)\rho_e(a)\right| da \\ &\quad \leq H_1 + H_2 + \int_{t-h}^t \left|e(t-a,0)\rho_e(a)\right| da, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} H_1 &= \int_0^{t-h} e(t-a-h,0)\left|\rho_e(a+h) - \rho_e(a)\right| da, \\ H_2 &= \int_0^{t-h} \rho_e(a)\left|e(t-a-h,0) - e(t-a,0)\right| da. \end{aligned}$$

Recall that $0 \leq \rho_e(a) \leq e^{-\mu_e a} \leq 1$ and $\rho_e(a)$ is non-increasing function with respect to a, we have

$$\begin{aligned} \int_0^{t-h} \left|\rho_e(a+h) - \rho_e(a)\right| da &= \int_0^{t-h} \rho_e(a)\,da - \int_0^{t-h} \rho_e(a+h)\,da \\ &= \int_0^{t-h} \rho_e(a)\,da - \int_h^{t-h} \rho_e(a)\,da - \int_{t-h}^t \rho_e(a)\,da \\ &= \int_0^h \rho_e(a)\,da - \int_{t-h}^t \rho_e(a)\,da \leq h. \end{aligned} \tag{4.4}$$

Hence, from (ii) of Proposition 3.2, we yield $H_1 \leq \beta M^2 h$. It follows from (i) of Proposition 3.2 that $|dS(t)/dt|$ and $|dI(t)/dt|$ are bounded by $M_S = \Lambda_s + \mu_s M + \beta M^2$ and $M_I = \sigma^{\sup} M + \gamma^{\sup} M + (\mu_i + k)M$, respectively. Thus, $S(\cdot)$ and $I(\cdot)$ are Lipschitz continuous on

$[0,\infty)$ with coefficients M_S and M_I. Following Lemma 4.1, one sees that $S(\cdot)I(\cdot)$ is Lipschitz continuous on $[0,\infty)$ with coefficient $M_{SI}=M(M_S+M_I)$. Thus,

$$H_2 \le \int_0^{t-h} \beta M_{SI} e^{-\mu_e a}\, da \le \frac{\beta M_{SI} h}{\mu_e}.$$

Consequently, we have

$$\int_0^\infty \left|\tilde{y}_2(t,a+h)-\tilde{y}_2(t,a)\right| da \le \left(2\beta M^2 + \frac{\beta M_{SI}}{\mu_e}\right)h.$$

This shows that $\int_0^\infty |\tilde{y}_2(t,a+h)-\tilde{y}_2(t,a)|\, da \to 0$ as $h\to 0$. Then $\tilde{y}_2(t,a)$ satisfies condition (iii) in Lemma 4.3, which implies that there exists a precompact subset $B_e \subset L^1_+(\mathbb{R}^+,\mathbb{R}^+)$ such that $\tilde{y}_2(t,a)$ remains in B_e. Similarly, $\tilde{y}_4(t,b)$ remains in a precompact subset $B_r \subset L^1_+(\mathbb{R}^+,\mathbb{R}^+)$. Thus, $\Phi_2(t,B)\subseteq [0,M]\times B_e\times[0,M]\times B_r$. Applying Lemma 4.3, we can conclude that $\Phi_2(t,B)$ has compact closure. Thus, two conditions of Lemma 4.2 are satisfied and $\{\Phi(t)\}_{t\ge 0}$ is asymptotically smooth. This completes the proof. □

Propositions 3.1 and 3.3 and Theorem 4.1 show that Φ is point dissipative, eventually bounded on bounded sets, and asymptotically smooth. Thus, following [18], Theorem 2.33, we have the following proposition on the existence of a global attractor.

Theorem 4.2 *The semigroup $\{\Phi(t)\}_{t\ge 0}$ has a global attractor $\mathcal{A}$ contained in X, which attracts the bounded sets of X.*

5 Dynamical results

This section is devoted to the existence and global stability of the steady state.

5.1 Existence of steady state

In this subsection, we consider the existence of steady state for system (2.1). Clearly, system (2.1) has no disease-free steady state. The steady state $(S^*, e^*(\cdot), I^*, r^*(\cdot))$ of system (2.1) satisfies the equalities

$$\begin{cases} 0 = \Lambda_s - \mu_s S^* - \beta S^* I^*, \\ \frac{de^*(a)}{da} = \Lambda_e - \varepsilon(a)e^*(a), \\ 0 = \Lambda_i + \int_0^\infty \sigma(a)e^*(a)\,da - (\mu_i+k)I^* + \int_0^\infty \gamma(b)r^*(b)\,db, \\ \frac{dr^*(b)}{db} = \Lambda_r - \eta(b)r^*(b), \end{cases} \tag{5.1}$$

with the boundary conditions

$$e^*(0) = \beta S^* I^*, \qquad r^*(0) = kI^*. \tag{5.2}$$

It follows from the first equation of (5.1) that we have

$$I^* = \frac{\Lambda_s - \mu_s S^*}{\beta S^*}. \tag{5.3}$$

From the second equation of (5.1) and boundary conditions (5.2), one has

$$\begin{aligned} e^*(a) &= e^*(0)\rho_e(a) + \int_0^\infty \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds \\ &= \bigl(\Lambda_s - \mu_s S^*\bigr)\rho_e(a) + \int_0^\infty \Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds. \end{aligned} \tag{5.4}$$

Similarly, the fourth equation of (5.1) and the boundary condition $r^*(0) = kI^*$ lead to

$$r^*(b) = \frac{k(\Lambda_s - \mu_s S^*)}{\beta S^*}\rho_r(b) + \int_0^\infty \Lambda_r(s)\frac{\rho_r(b)}{\rho_r(s)}\,ds. \tag{5.5}$$

Inserting (5.4) and (5.5) into the third equation of (5.1), we have

$$f\bigl(S^*\bigr) = B_1\bigl(S^*\bigr)^2 + B_2 S^* + B_3 = 0,$$

where

$$\begin{aligned} &B_1 = \beta\mu_s\theta_e > 0, \\ &B_2 = -\bigl[\mu_s(\mu_i + k - k\theta_r) + \beta(\Lambda_s\theta_e + \Lambda_i + M_1 + M_2)\bigr] < 0, \\ &B_3 = \Lambda_s(\mu_i + k - k\theta_r) > 0. \end{aligned}$$

Here, $\theta_r \in [0,1]$ denotes the probability of entering the removed compartment alive and $M_i > 0$ ($i = 1,2$) are defined as $M_1 = \int_0^\infty \sigma(a)\int_0^\infty \Lambda_e(s)\rho_e(a)/\rho_e(s)\,ds\,da$ and $M_2 = \int_0^\infty \gamma(a)\int_0^\infty \Lambda_r(s)\rho_r(a)/\rho_r(s)\,ds\,da$. Clearly, $f(0) > 0$ and $f(\Lambda_s/\mu_s) < 0$. Thus, there exists a unique solution S^* for $f(S^*) = 0$ in the interval $[0, \Lambda_s/\mu_s]$. It follows from equations (5.3)-(5.5) that I^*, and the functions $e^*(\cdot)$ and $r^*(\cdot)$ uniquely depend on S^*. Therefore, the unique steady state T^* is determined. Obviously, $S^* \in (0, \Lambda_s/\mu_s)$ implies that I^*, $e^*(\cdot)$ and $r^*(\cdot)$ are both positive. Thus, T^* is strictly positive. Clearly, $\{T^*\}$ is an invariant bounded set.

From the above discussions, we have the following theorem.

Theorem 5.1 *System* (2.1) *always has a unique endemic steady state* $T^* = (S^*, e^*(\cdot), I^*, r^*(\cdot))$, *and* $T^* \in \mathcal{A}$.

5.2 Global stability

This subsection is devoted to showing the global stability of T^*, which implies that the attractor $\mathcal{A}$ only contains the unique endemic steady state.

The following lemma will be useful in the proof of our main result.

Lemma 5.1 *Let* $g(x) = x - 1 - \ln x$ *for each* $x \geq 0$. *Each solution of system* (2.1) *satisfies*

$$\begin{aligned} &\frac{\partial}{\partial a} g\left(\frac{e(t,a)}{e^*(a)}\right) = -\frac{\partial}{\partial t} g\left(\frac{e(t,a)}{e^*(a)}\right), \\ &\frac{\partial}{\partial b} g\left(\frac{r(t,b)}{r^*(b)}\right) = -\frac{\partial}{\partial t} g\left(\frac{r(t,b)}{r^*(b)}\right). \end{aligned}$$

Proof It follows from the fact that $g'(x) = 1 - 1/x$ and the second equation of (5.1) that we have

$$\begin{aligned}
\frac{\partial}{\partial a} g\left(\frac{e(t,a)}{e^*(a)}\right) &= \left(1 - \frac{e^*(a)}{e(t,a)}\right) \frac{\partial}{\partial a}\left(\frac{e(t,a)}{e^*(a)}\right) \\
&= \left(1 - \frac{e^*(a)}{e(t,a)}\right)\left[\frac{1}{e^*(a)}\frac{\partial}{\partial a} e(t,a) + e(t,a)\frac{\partial}{\partial a}\frac{1}{e^*(a)}\right] \\
&= \left(\frac{e(t,a)}{e^*(a)} - 1\right)\left[\frac{e_a(t,a)}{e(t,a)} + \varepsilon(a) - \frac{\Lambda_e(a)}{e^*(a)}\right] \\
&= -\frac{\partial}{\partial t} g\left(\frac{e(t,a)}{e^*(a)}\right),
\end{aligned}$$

where $e_a(t,a)$ denotes $\partial e(t,a)/\partial a$. Similarly, we have

$$\begin{aligned}
\frac{\partial}{\partial b} g\left(\frac{r(t,b)}{r^*(b)}\right) &= \left(\frac{r(t,b)}{r^*(b)} - 1\right)\left[\frac{r_b(t,b)}{r(t,b)} + \eta(b) - \frac{\Lambda_r(b)}{r^*(b)}\right] \\
&= -\frac{\partial}{\partial t} g\left(\frac{r(t,b)}{r^*(b)}\right).
\end{aligned}$$

□

Based on the above preparations, we show our main result.

Theorem 5.2 *The unique endemic steady state T^* is globally asymptotically stable, and $\mathcal{A} = \{T^*\}$.*

Proof Define the Lyapunov functional as follows:

$$V = W_s + W_e + W_i + W_r,$$

where

$$\begin{aligned}
W_s &= \theta_e S^* g\left(\frac{S}{S^*}\right), \qquad W_e = \int_0^\infty \omega_e(a) e^*(a) g\left(\frac{e(t,a)}{e^*(a)}\right) da, \\
W_i &= I^* g\left(\frac{I}{I^*}\right), \qquad W_r = \int_0^\infty \omega_r(b) r^*(b) g\left(\frac{r(t,b)}{r^*(b)}\right) db,
\end{aligned}$$

where

$$\begin{aligned}
\omega_e(a) &= \int_a^\infty \sigma(s) \exp\left(-\int_a^s \varepsilon(\tau)\, d\tau\right) ds, \\
\omega_r(b) &= \int_b^\infty \gamma(s) \exp\left(-\int_b^s \eta(\tau)\, d\tau\right) ds.
\end{aligned}$$

Obviously, $\omega_e(a), \omega_r(b) > 0$ for $a, b \geq 0$ and $\omega_j(0) = \theta_j$ for $j = e, r$. The derivatives of $\omega_e(a)$ and $\omega_r(b)$ satisfy

$$\frac{d\omega_e(a)}{da} = \omega_e(a)\varepsilon(a) - \sigma(a), \qquad \frac{d\omega_r(b)}{db} = \omega_r(b)\eta(b) - \gamma(b). \tag{5.6}$$

Since $\Lambda_s = \beta S^* I^* + \mu_s S^*$, the derivative of W_s along with the solutions of (2.1) is

$$\begin{aligned}\frac{dW_s}{dt} &= \theta_e S^*\left(1-\frac{S^*}{S}\right)\frac{1}{S^*}\left[\mu_s S^*\left(1-\frac{S}{S^*}\right)+\beta S^* I^*\left(1-\frac{SI}{S^*I^*}\right)\right] \\ &= \theta_e\mu_s S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\theta_e\beta S^* I^*\left(1-\frac{SI}{S^*I^*}-\frac{S^*}{S}+\frac{I}{I^*}\right).\end{aligned} \tag{5.7}$$

The derivative of W_e along with the solutions of system (2.1) is calculated as

$$\begin{aligned}\frac{dW_e}{dt} &= \int_0^\infty \omega_e(a)e^*(a)\frac{\partial}{\partial t}g\left(\frac{e(t,a)}{e^*(a)}\right)da \\ &= -\int_0^\infty \omega_e(a)e^*(a)\left(1-\frac{e^*(a)}{e(t,a)}\right)\frac{1}{e^*(a)}\left(\frac{\partial}{\partial a}e(t,a)+\varepsilon(a)e(t,a)\right)da \\ &= -\int_0^\infty \omega_e(a)e^*(a)\left(\frac{e(t,a)}{e^*(a)}-1\right)\left(\frac{e_a(t,a)}{e(t,a)}+\varepsilon(a)-\frac{\Lambda_e(a)}{e(t,a)}\right)da.\end{aligned}$$

It follows from (A_4) of Assumption 2.1 that we have $\lim_{a\to\infty} e(t,a) = 0$, $\lim_{b\to\infty} r(t,b) = 0$ for $t \geq 0$. Combining the second equation of system (5.1), Lemma 5.1 and (5.6), using integration by parts, we have

$$\begin{aligned}\frac{dW_e}{dt} &= -\int_0^\infty \omega_e(a)e^*(a)\frac{\partial}{\partial a}g\left(\frac{e(t,a)}{e^*(a)}\right)da + \int_0^\infty \omega_e(a)e^*(a)\left(\frac{\Lambda_e(a)}{e^*(a)}-\frac{\Lambda_e(a)}{e(t,a)}\right)da \\ &= -\omega_e(a)e^*(a)g\left(\frac{e(t,a)}{e^*(a)}\right)\Big|_{a=0}^{a=\infty} - \int_0^\infty g\left(\frac{e(t,a)}{e^*(a)}\right)\frac{d}{da}\left(\omega_e(a)e^*(a)\right)da \\ &\quad + \int_0^\infty \omega_e(a)\Lambda_e(a)\left(2-\frac{e(t,a)}{e^*(a)}-\frac{e^*(a)}{e(t,a)}\right)da \\ &= \omega_e(0)e^*(0)g\left(\frac{e(t,0)}{e^*(0)}\right) - \int_0^\infty \sigma(a)e^*(a)g\left(\frac{e(t,a)}{e^*(a)}\right)da \\ &\quad + \int_0^\infty w_e(a)\Lambda_e(a)g\left(\frac{e(t,a)}{e^*(a)}\right)da \\ &\quad + \int_0^\infty \omega_e(a)\Lambda_e(a)\left(2-\frac{e(t,a)}{e^*(a)}-\frac{e^*(a)}{e(t,a)}\right)da \\ &= \omega_e(0)e^*(0)g\left(\frac{e(t,0)}{e^*(0)}\right) - \int_0^\infty \sigma(a)e^*(a)g\left(\frac{e(t,a)}{e^*(a)}\right)da \\ &\quad - \int_0^\infty w_e(a)\Lambda_e(a)g\left(\frac{e^*(a)}{e(t,a)}\right)da.\end{aligned}$$

Note that $\omega_e(0) = \theta_e$, $e^*(0) = \beta S^* I^*$ and $e(t,0) = \beta SI$, thus

$$\begin{aligned}\frac{dW_e}{dt} &= \theta_e\beta S^* I^* g\left(\frac{SI}{S^*I^*}\right) - \int_0^\infty \sigma(a)e^*(a)g\left(\frac{S(t,a)}{S^*(a)}\right)da \\ &\quad - \int_0^\infty w_e(a)\Lambda_e(a)g\left(\frac{e^*(a)}{e(t,a)}\right)da.\end{aligned} \tag{5.8}$$

Similarly, combining $\omega_r(0) = \theta_r$, $r^*(0) = kI^*$ and $r(t,0) = kI$ yields the derivative of W_r along with the solutions of system (2.1) as follows:

$$\begin{aligned}\frac{dW_r}{dt} &= \theta_r kI^* g\left(\frac{I}{I^*}\right) - \int_0^\infty \gamma(b) r^*(b) g\left(\frac{r(t,b)}{r^*(b)}\right) db \\ &\quad - \int_0^\infty w_r(b)\Lambda_r(b) g\left(\frac{r^*(b)}{r(t,b)}\right) db.\end{aligned} \tag{5.9}$$

Applying $\int_0^\infty \sigma(a)e^*(a)\,da + \int_0^\infty \gamma(b)r^*(b)\,db = (\mu_i + k)I^*$, we see that the derivative of W_i along with the solutions of system (2.1) has the following form:

$$\begin{aligned}\frac{dW_i}{dt} &= I^*\left(1 - \frac{I^*}{I}\right)\frac{1}{I^*}\left(\Lambda_i + \int_0^\infty \sigma(a)e(t,a)\,da + \int_0^\infty \gamma(b)r(t,b)\,db - (\mu_i + \delta_i + k)I\right) \\ &= \left(1 - \frac{I}{I^*}\right)\left[\Lambda_i\left(1 - \frac{I^*}{I}\right) + \int_0^\infty \sigma(a)e^*(a)\left(\frac{e(t,a)}{e^*(a)} - \frac{I}{I^*}\right) da \right. \\ &\quad \left. + \int_0^\infty \gamma(b)r^*(b)\left(\frac{r(t,b)}{r^*(b)} - \frac{I}{I^*}\right) db\right] \\ &= \Lambda_i\left(2 - \frac{I^*}{I} - \frac{I}{I^*}\right) + \int_0^\infty \sigma(a)e^*(a)\left(1 - \frac{I}{I^*} - \frac{I^*e(t,a)}{Ie^*(a)} + \frac{e(t,a)}{e^*(a)}\right) da \\ &\quad + \int_0^\infty \gamma(b)r^*(b)\left(1 - \frac{I}{I^*} - \frac{I^*r(t,b)}{Ir^*(b)} + \frac{r(t,b)}{r^*(b)}\right) db.\end{aligned} \tag{5.10}$$

Summarizing (5.7)-(5.10), we have

$$\begin{aligned}\frac{dV}{dt} &= \theta_e\mu_s S^*\left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + \Lambda_i\left(2 - \frac{I^*}{I} - \frac{I}{I^*}\right) - \int_0^\infty w_e(a)\Lambda_e(a) g\left(\frac{e^*(a)}{e(t,a)}\right) da \\ &\quad + \theta_r kI^* g\left(\frac{I}{I^*}\right) - \int_0^\infty w_r(b)\Lambda_r(b) g\left(\frac{r^*(b)}{r(t,b)}\right) db + H_{si} + H_e + H_r,\end{aligned} \tag{5.11}$$

where

$$\begin{aligned}H_{si} &:= \theta_e\beta S^*I^*\left(1 - \frac{SI}{S^*I^*} - \frac{S^*}{S} + \frac{I}{I^*} + \frac{SI}{S^*I^*} - 1 - \ln\frac{SI}{S^*I^*}\right), \\ H_e &:= \int_0^\infty \sigma(a)e^*(a)\left(1 - \frac{I}{I^*} - \frac{I^*e(t,a)}{Ie^*(a)} + \frac{e(t,a)}{e^*(a)} + 1 - \frac{e(t,a)}{e^*(a)} + \ln\frac{e(t,a)}{e^*(a)}\right) da, \\ H_r &:= \int_0^\infty \gamma(b)r^*(b)\left(1 - \frac{I}{I^*} - \frac{I^*r(t,b)}{Ir^*(b)} + \frac{r(t,b)}{r^*(b)} + 1 - \frac{r(t,b)}{r^*(b)} + \ln\frac{r(t,b)}{r^*(b)}\right) db.\end{aligned}$$

Note that

$$\begin{aligned}\ln\frac{SI}{S^*I^*} &= \ln\frac{I}{I^*} - \ln\frac{S^*}{S}, \\ \ln\frac{e(t,a)}{e^*(a)} &= \ln\frac{I^*e(t,a)}{Ie^*(a)} + \ln\frac{I}{I^*}, \\ \ln\frac{r(t,b)}{r^*(b)} &= \ln\frac{I^*r(t,b)}{Ir^*(b)} + \ln\frac{I}{I^*},\end{aligned}$$

and

$$\int_0^\infty \sigma(a)e^*(a)\,da = \theta_e\beta S^*I^* + \int_0^\infty\int_0^a \sigma(a)\Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da,$$
$$\int_0^\infty \gamma(b)r^*(b)\,db = \theta_r kI^* + \int_0^\infty\int_0^b \gamma(b)\Lambda_r(s)\frac{\rho_r(b)}{\rho_r(s)}\,ds\,db.$$

Thus, we have

$$\begin{aligned}
H_{si} &= \theta_e\beta S^*I^*\left[\left(1-\frac{S^*}{S}+\ln\frac{S^*}{S}\right)+\left(\frac{I}{I^*}-1-\ln\frac{I}{I^*}\right)\right]\\
&= -\theta_e\beta S^*I^*g\left(\frac{S^*}{S}\right)+\theta_e\beta S^*I^*g\left(\frac{I}{I^*}\right), \qquad (5.12)\\
H_e &= \int_0^\infty \sigma(a)e^*(a)\left[\left(1-\frac{I}{I^*}+\ln\frac{I}{I^*}\right)+\left(1-\frac{I^*e(t,a)}{Ie^*(a)}+\ln\frac{I^*e(t,a)}{Ie^*(a)}\right)\right]da\\
&= -\int_0^\infty \sigma(a)e^*(a)g\left(\frac{I}{I^*}\right)da-\int_0^\infty \sigma(a)e^*(a)g\left(\frac{I^*e(t,a)}{Ie^*(a)}\right)da\\
&= -\theta_e\beta S^*I^*g\left(\frac{I}{I^*}\right)-g\left(\frac{I}{I^*}\right)\int_0^\infty\int_0^a \sigma(a)\Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da\\
&\quad -\int_0^\infty \sigma(a)e^*(a)g\left(\frac{I^*e(t,a)}{Ie^*(a)}\right)da, \qquad (5.13)\\
H_r &= \int_0^\infty \gamma(b)r^*(b)\left[\left(1-\frac{I}{I^*}+\ln\frac{I}{I^*}\right)+\left(1-\frac{I^*r(t,b)}{Ir^*(b)}+\ln\frac{I^*r(t,b)}{Ir^*(b)}\right)\right]db\\
&= -\int_0^\infty \gamma(b)r^*(b)g\left(\frac{I}{I^*}\right)db-\int_0^\infty \gamma(b)r^*(b)g\left(\frac{I^*r(t,b)}{Ir^*(b)}\right)db\\
&= -\theta_r kI^*g\left(\frac{I}{I^*}\right)-g\left(\frac{I}{I^*}\right)\int_0^\infty\int_0^b \gamma(b)\Lambda_r(s)\frac{\rho_r(b)}{\rho_r(s)}\,ds\,db\\
&\quad -\int_0^\infty \gamma(b)r^*(b)g\left(\frac{I^*r(t,b)}{Ir^*(b)}\right)db. \qquad (5.14)
\end{aligned}$$

Consequently, we derive

$$\begin{aligned}
\frac{dV}{dt} &= \theta_e\mu_s S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\Lambda_i\left(2-\frac{I^*}{I}-\frac{I}{I^*}\right)\\
&\quad -\int_0^\infty w_e(a)\Lambda_e(a)g\left(\frac{e^*(a)}{e(t,a)}\right)da-\int_0^\infty \sigma(a)e^*(a)g\left(\frac{I^*e(t,a)}{Ie^*(a)}\right)da\\
&\quad -\int_0^\infty w_r(b)\Lambda_r(b)g\left(\frac{r^*(b)}{r(t,b)}\right)db-\int_0^\infty \gamma(b)r^*(b)g\left(\frac{I^*r(t,b)}{Ir^*(b)}\right)db\\
&\quad -g\left(\frac{I}{I^*}\right)\int_0^\infty\int_0^a \sigma(a)\Lambda_e(s)\frac{\rho_e(a)}{\rho_e(s)}\,ds\,da\\
&\quad -g\left(\frac{I}{I^*}\right)\int_0^\infty\int_0^b \gamma(b)\Lambda_r(s)\frac{\rho_r(b)}{\rho_r(s)}\,ds\,db,
\end{aligned}$$

which implies that $dV/dt \le 0$ holds true. Furthermore, the strict equality holds only if $(S, e(\cdot), I, r(\cdot)) = T^*$. It is easy to obtain that the largest invariant subset of $\{(S, e(\cdot), I, r(\cdot)) :$

$dV/dt = 0\}$ is $\{T^*\} \subset \Omega$, and the Lyapunov-LaSalle invariance principle implies that the unique endemic steady state T^* is globally asymptotically stable. □

6 Numerical simulations

In this section, we give some numerical simulations to show the epidemiological insights. In our application, we provide simulations of system (2.1) by using tuberculosis data from [4, 20, 26–28] to investigate the effects of immigration level on disease transmission. Here, we take

$$\mu_s = \mu_e = \mu_r = 0.02, \qquad \mu_i = 0.159, \qquad k = 0.058, \qquad \beta = 0.7 \times 10^{-4}.$$

Furthermore, we set the maximum age for the upper bound of latent and relapse age as 10 years. Then

$$\begin{aligned} \sigma(a) &= 0.003\left(1 + \sin\frac{(a-5)\pi}{10}\right), \\ \gamma(b) &= 0.01\left(1 + \sin\frac{(b-5)\pi}{10}\right), \quad \text{for } 0 \le a, b \le 10. \end{aligned}$$

Thus, the averages of $\sigma(a)$ and $\gamma(b)$ are 0.003 and 0.01, which are the same as in [28] and [27], respectively. Set $\Lambda_s = 7.5$, $\Lambda_i = 0.2$ and

$$\begin{aligned} \Lambda_e(a) &= 0.8\left(1 + \sin\frac{(a-5)\pi}{10}\right), \\ \Lambda_r(b) &= 1.5\left(1 + \sin\frac{(b-5)\pi}{10}\right), \quad \text{for } 0 \le a, b \le 10. \end{aligned}$$

Hence, $\bar{\Lambda}_e = 0.8$ and $\bar{\Lambda}_r = 1.5$. We have $\Lambda^* = 10$.

6.1 The long time behaviors of system (2.1)

First, Theorem 5.2 asserts that the unique endemic steady state T^* is globally asymptotically stable. This fact is revealed by Figure 2. From Figure 2, one can observe that the levels of all compartmental individuals tend to stable values, where $S(t)$, $e(t,a)$, $I(t)$ and $r(t,b)$ converge to the positive steady states S^*, $e^*(a)$, I^* and $r^*(b)$. Easily, we can get $S^* = 1.4 \times 10^4$ and $I^* = 1.9 \times 10^4$. Obviously, it follows from Figure 2 that the numbers of the exposed and removed compartments are the distribution functions of a and b, respectively, particularly, $e^*(0) = 1.96 \times 10^4$, $e^*(10) = 1.26 \times 10^4$ and $r^*(0) = 1{,}116$, $r^*(10) = 693$. Furthermore, we show that the distribution of $e(t,a)$ and $r(t,a)$ for all age and at $a = 5$. Figure 3 shows that this is a stationary distribution along all time.

6.2 The age distribution of the latent and removed populations

Then we observe how the age distribution change the prevalence of the latent and removed populations. It follows from Figure 4 that with the increasing of age, the latent and removed populations prevalence decrease.

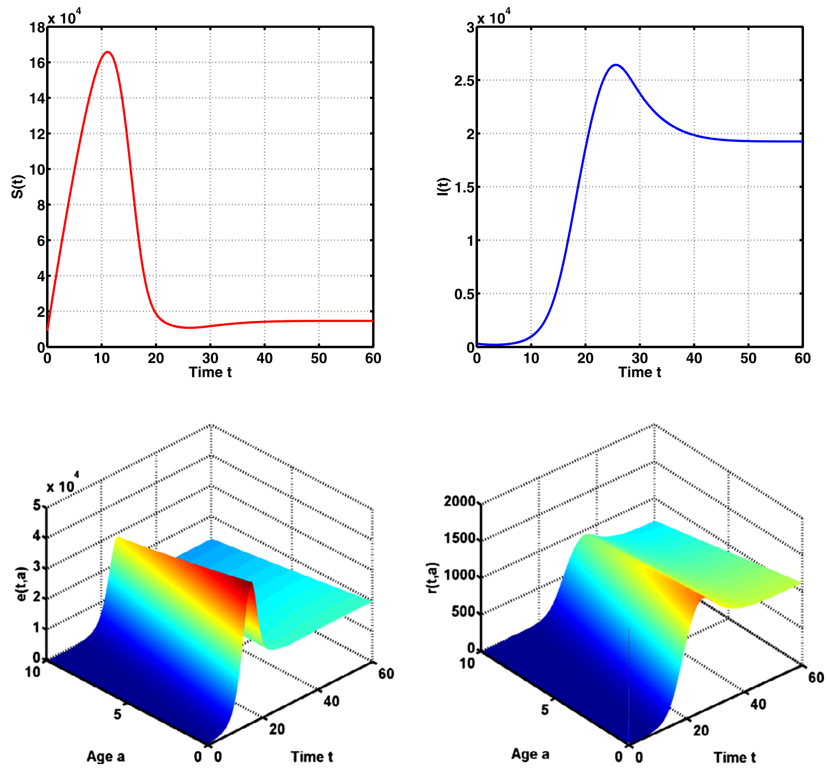


Figure 2 The long time dynamical behaviors of system (2.1).

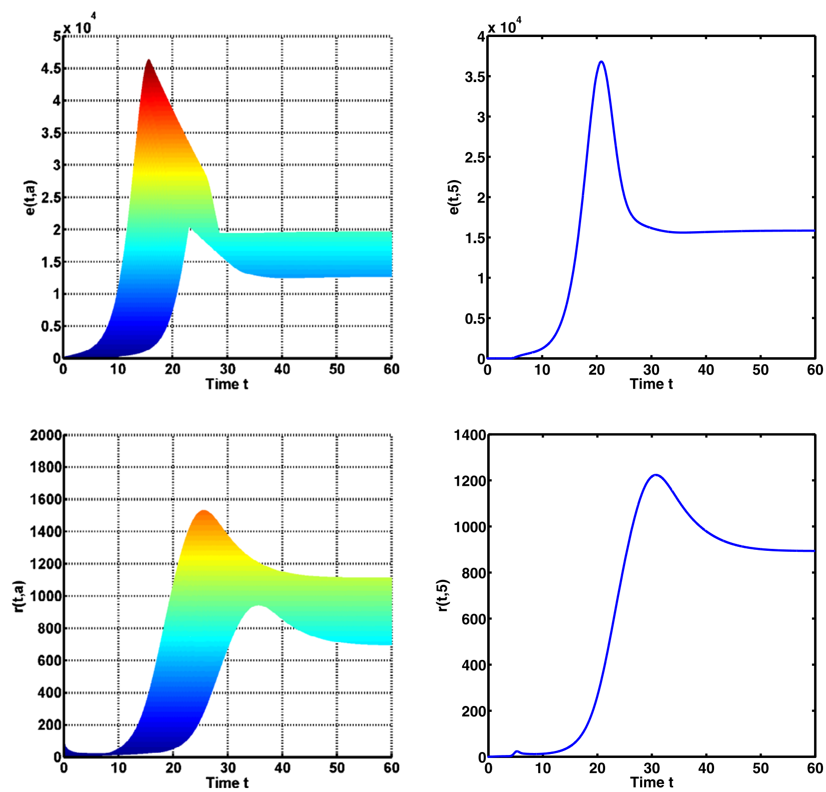


Figure 3 Distribution of the latent and removed populations prevalence for all age and at $a = 5$.

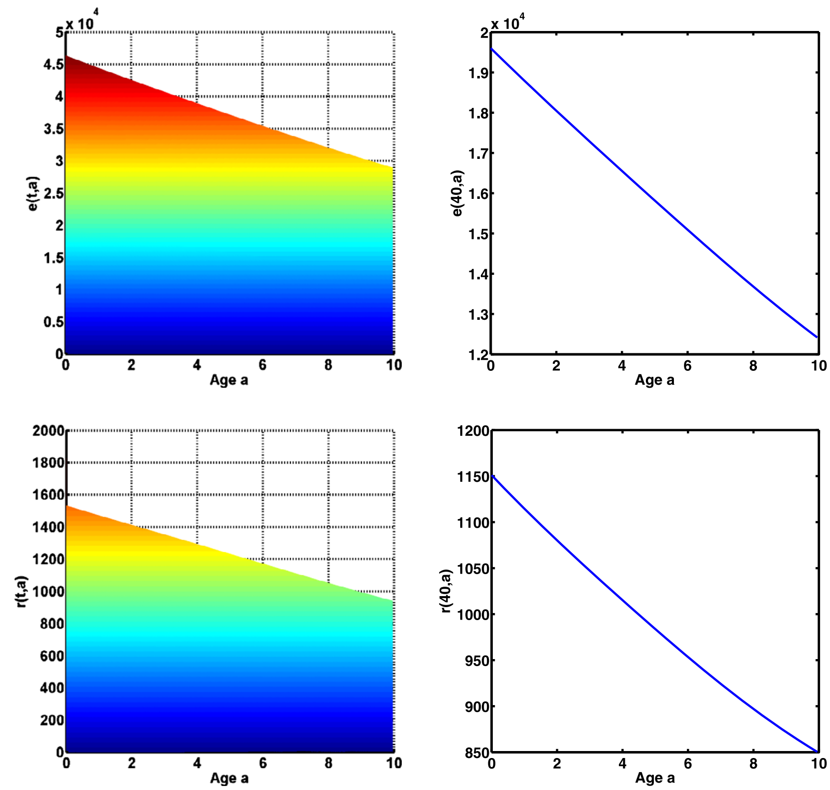


Figure 4 Distribution of the latent and removed populations prevalence for all time and at $t = 40$.

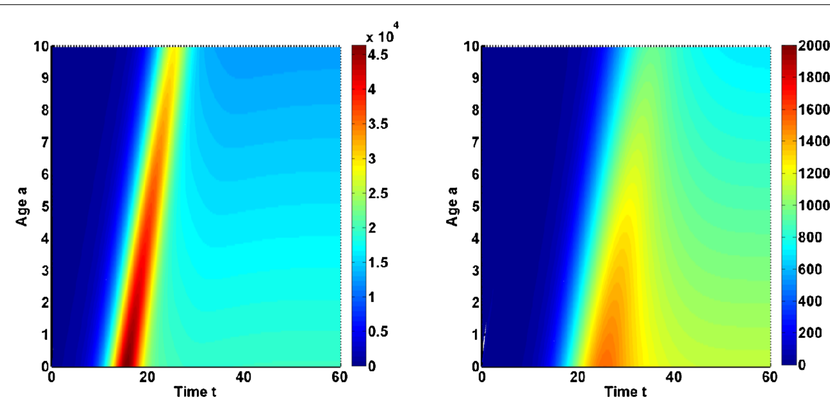


Figure 5 Distribution of the latent and removed populations prevalence for $t \in [0, 60]$ and a range of age. Results are obtained for $e(t, a)$ (left) and $r(t, b)$ (right).

6.3 The stationary distribution

Finally, we are interested to observe that whether the stationary distribution exists or not. The numerical simulation is performed over a range of age, and then the stationary distribution of the latent and removed populations is obtained in Figure 5 for $t \in [0, 60]$. From the color bar, we can observe that the distribution patterns of the latent and removed populations remain unchanged. Those distributions which do not change with time are called stationary patterns.

7 Conclusion and discussion

In this paper, we proposed and investigated a class of age-structured SEIR epidemic model with immigration. We show that, for all parameter values, the endemic steady state is unique and globally asymptotically stable by using the Lyapunov functional.

All simulation results show that the immigration of individuals leads to the insight that the TB cannot be fully eliminated from the population and will eventually reach a steady endemic level. The age of latent and removed leads to stationary patterns. This implies that TB will exist if either the immigration of population is allowed or there are TB infections in the region. Thus, in order to eradicate the TB, there are two choices: one is to prohibit the immigration of infected individuals, which is difficult to achieve; the other is to clear away the TB in any one region.

Acknowledgements
This work was supported by the National Nature Science Foundation of China under Grant Nos. 11601293 and 11331009, Shanxi Scientific Data Sharing Platform for Animal Diseases (201605D121014), and the science and technology innovation team of Shanxi province (201605D131044-06). We are very grateful to the anonymous referees and editor for their careful reading, valuable comments and helpful suggestions, which have helped to improve the presentation of this work significantly.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

Author details
[1]Complex Systems Research Center, Shanxi University, Taiyuan, Shan'xi 030006, China. [2]Key Laboratory of Eco-environments in Three Gorges Reservoir Region (Ministry of Education), School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China.

Publisher's Note

Received: 16 May 2017 Accepted: 27 July 2017 Published online: 29 August 2017

References

1. Daniel, TM, Bates, JH, Downes, KA: History of Tuberculosis: Pathogenesis, Protection, and Control. Am. Soc. Microbiol., Washington (1994)
2. World Health Organization: Global tuberculosis report 2012, Geneva, Switzerland (2012)
3. Bowong, S, Tewa, JJ: Global analysis of a dynamical model for transmission of tuberculosis with a general contact rate. Commun. Nonlinear Sci. Numer. Simul. **15**(11), 3621-3631 (2010)
4. Ziv, E, Daley, CL, Blower, SM: Early therapy for latent tuberculosis infection. Am. J. Epidemiol. **153**(4), 381-385 (2001)
5. Feng, Z, Iannelli, M, Milner, F: A two-strain tuberculosis model with age of infection. SIAM J. Appl. Math. **62**(5), 1634-1656 (2002)
6. Liu, J, Zhang, T: Global stability for a tuberculosis model. Math. Comput. Model. **54**(1), 836-845 (2011)
7. Yang, Y, Li, J, Zhou, Y: Global stability of two tuberculosis models with treatment and self-cure. Rocky Mt. J. Math. **42**(4), 1367-1386 (2012)
8. Martin, SW: Livestock Disease Eradication: Evaluation of the Cooperative State-Federal Bovine Tuberculosis Eradication Program. National Academies Washington (1994)
9. van den Driessche, P, Zou, X: Modeling relapse in infectious diseases. Math. Biosci. **207**(1), 89-103 (2007)
10. Liu, L, Wang, J, Liu, X: Global stability of an SEIR epidemic model with age-dependent latency and relapse. Nonlinear Anal., Real World Appl. **24**, 18-35 (2015)
11. Magal, P: Compact attractors for time periodic age-structured population models. Electron. J. Differ. Equ. **2001**, 65 (2001)
12. Huang, G, Liu, X, Takeuchi, Y: Lyapunov functions and global stability for age-structured HIV infection model. SIAM J. Appl. Math. **72**(1), 25-38 (2012)
13. McCluskey, CC: Global stability for an SEI epidemiological model with continuous age-structure in the exposed and infectious classes. Math. Biosci. Eng. **9**(4), 819-841 (2012)
14. Wang, J, Zhang, R, Kuniya, T: Global dynamics for a class of age-infection HIV models with nonlinear infection rate. J. Math. Anal. Appl. **432**(1), 289-313 (2015)
15. Wang, L, Liu, Z, Zhang, X: Global dynamics for an age-structured epidemic model with media impact and incomplete vaccination. Nonlinear Anal., Real World Appl. **32**, 136-158 (2016)
16. Webb, GF: Theory of Nonlinear Age-Dependent Population Dynamics. Dekker, New York (1985)
17. Iannelli, M: Mathematical Theory of Age-Structured Population Dynamics. Giardini Editori e Stampatori, Pisa (1995)

18. Smith, HL, Thieme, HR: Dynamical Systems and Population Persistence. Am. Math. Soc., Providence (2011)
19. McCluskey, CC, van den Driessche, P: Global analysis of two tuberculosis models. J. Dyn. Differ. Equ. **16**(1), 139-166 (2004)
20. Guo, H, Li, MY: Global stability of the endemic equilibrium of a tuberculosis model with immigration and treatment. Can. Appl. Math. Q. **19**, 1-18 (2011)
21. McCluskey, CC: Global stability for an SEI model of infectious disease with age structure and immigration of infecteds. Math. Biosci. Eng. **13**(2), 381-400 (2016)
22. Wang, L, Wang, X: influence of temporary migration on the transmission of infectious diseases in a migrants' home village. J. Theor. Biol. **300**, 100-109 (2012)
23. Guo, H, Wu, J: Persistent high incidence of tuberculosis among immigrants in a lowincidence country: impact of immigrants with early or late latency. Math. Biosci. Eng. **8**(2), 695-709 (2011)
24. Zhang, J, Li, J, Ma, Z: Global dynamics of an seir epidemic model with immigration of different compartments. Acta Math. Sci. **26**(3), 551-567 (2006)
25. Brauer, F, van den Driessche, P: Models for transmission of disease with immigration of infectives. Math. Biosci. **171**(2), 143-154 (2001)
26. Grzybowski, S, Enarson, DA: The fate of cases of pulmonary tuberculosis under various treatment programmes. Bull. Int. Union Against Tuberc. **53**(2), 70-75 (1978)
27. Styblo, K: Tuberculosis control and surveillance. Recent Adv. Respir. Med. **4**, 77-108 (1986)
28. Comstock, GW: Epidemiology of tuberculosis 1-3. Am. Rev. Respir. Dis. **125**(3P2), 8-15 (1982)