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The existence and uniqueness of the solution for nonlinear Fredholm and Volterra integral equations together with nonlinear fractional differential equations via w-distances

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Abstract

In this work, we establish new fixed point theorems for w-generalized weak contraction mappings with respect to w-distances in complete metric spaces by using the concept of an altering distance function. As an application, we use the obtained results to aggregate the existence and uniqueness of the solution for nonlinear Fredholm integral equations and Volterra integral equations together with nonlinear fractional differential equations of Caputo type.

MSC: 47H10; 54H25

Keywords: w-distance; altering distance function; nonlinear Fredholm integral equation; nonlinear Volterra integral equation; nonlinear fractional differential equation

1 Introduction

In 1984, the notion of an altering distance function was introduced and studied by Khan et al. [1], applying it to define weak contractions. They also proved the existence and uniqueness of a fixed point for mappings satisfying such a contraction condition. Afterward, some fixed point results for generalized weak contraction mappings were proved by Choudhury et al. [2] by using some control function along with the notion of an altering distance function. Moreover, the notion of weak contraction mappings was extended in many different directions (see [3, 4] and references therein).

On the other hand, the notion of a w-distance on a metric space was introduced and investigated by Kada et al. [5]. Using this concept, they also improved the following famous results:

- Caristi's fixed point theorem;
- Ekland's variational principle;
- Takahashi's existence theorem.

Afterward, Du [6] proved the existence of a fixed point for some nonlinear mappings by using a specific w-distance, called a w^0-distance. From this trend, several mathematicians extended fixed point results for weak contraction mappings and generalized weak contraction mappings with respect to w-distances on metric spaces (see [7, 8] and references therein).

In this paper, we introduce the concept of a special w-distance, the so-called ceiling distance, and use this concept for proving fixed point theorems for generalized contraction mappings with respect to w-distances in complete metric spaces via the concept of an altering distance function. As an application, the obtained results are used in the warrant of the existence and uniqueness of the solution for nonlinear Fredholm integral equations and Volterra integral equations together with nonlinear fractional differential equations of Caputo type.

2 Preliminaries

In this section, we recall some important notation, definitions, and primary results together with references.

Definition 2.1 ([5]) Let (X,d) be a metric space. A function $q : X \times X \to [0,\infty)$ is called a w-distance on X if it satisfies the following three conditions for all $x,y,z \in X$:

(W1) $q(x,y) \leq q(x,z) + q(z,y)$;

(W2) $q(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;

(W3) for each $\epsilon > 0$, there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \epsilon$.

It is well known that each metric on a nonempty set X is a w-distance on X. Here, we give some other examples of w-distances.

Example 2.2 Let (X,d) be a metric space. A function $q : X \times X \to [0,\infty)$ defined by $q(x,y) = c$ for every $x,y \in X$, where c is a positive real number, is a w-distance on X. However, q is not a metric since $q(x,x) = c \neq 0$ for any $x \in X$.

Example 2.3 Let $(X,\|\cdot\|)$ be a normed space. Then the function $q : X \times X \to [0,\infty)$ defined by

$$q(x,y) = \|y\|$$

for all $x,y \in X$ is a w-distance.

Definition 2.4 ([6]) Let (X,d) be a metric space. A function $q : X \times X \to [0,\infty)$ is called a w^0-distance if it is a w-distance on X with $q(x,x) = 0$ for all $x \in X$.

Note that each metric is a w^0-distance. Next, we give another example of a w^0-distance that is not a metric.

Example 2.5 ([6]) Let $X = \mathbb{R}$ with the metric $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|$ for all $x,y \in X$, and let $a,b > 1$. Define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{a(y-x), b(x-y)\}$$

for all $x, y \in X$. Then q is nonsymmetric, and hence q is not a metric. It is easy to see that q is a w^0-distance.

The following lemma will be used in the next section.

Lemma 2.6 ([5]) *Let (X,d) be a metric space, q be a w-distance on X, $\{x_n\}$ and $\{y_n\}$ be two sequences in X, and $x, y, z \in X$.*

(i) *If $\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x_n, y) = 0$, then $x = y$. In particular, if $q(z,x) = q(z,y) = 0$, then $x = y$.*

(ii) *If $q(x_n, y_n) \le \alpha_n$ and $q(x_n, y) \le \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,\infty)$ converging to 0, then $\{y_n\}$ converges to y.*

(iii) *If for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $m > n > N_\epsilon$ implies $q(x_n, x_m) < \epsilon$, then $\{x_n\}$ is a Cauchy sequence.*

Next, we give the definition of an altering distance function.

Definition 2.7 ([1]) A function $\psi : [0,\infty) \to [0,\infty)$ is said to be an altering distance function if it satisfies the following conditions:

(a) ψ is continuous and nondecreasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

Example 2.8 Define $\psi_1, \psi_2, \psi_3, \psi_4 : [0,\infty) \to [0,\infty)$ by $\psi_1(t) = t^2$, $\psi_2(t) = \sqrt{t^2 + 2t}$, $\psi_3(t) = (t^2 + t)e^t$, and $\psi_4(t) = \ln(t+1)$ for all $t \ge 0$. We see that ψ_1, ψ_2, ψ_3, and ψ_4 are altering distance functions because ψ_1, ψ_2, ψ_3, and ψ_4 are continuous and nondecreasing. Moreover, $\psi_i(t) = 0$ if and only if $t = 0$ for all $i = 1,2,3,4$. (The graphs of the functions ψ_1, ψ_2, ψ_3, and ψ_4 are shown in Figure 1.)

3 Main results

In this section, we introduce the new concepts of a distance on a metric space and a generalized weak contraction mapping along with w-distance in metric spaces. Furthermore, we investigate the sufficient condition for the existence and uniqueness of a fixed point for a self-mapping on a metric space satisfying the generalized weak contractive condition.

First, we introduce the new definition of a ceiling distance on a metric space.

Definition 3.1 A w-distance q on a metric space (X,d) is said to be a ceiling distance of d if and only if

$$q(x,y) \ge d(x,y) \tag{3.1}$$

for all $x, y \in X$.

Now we give some examples of a ceiling distance.

Example 3.2 Each metric on a nonempty set X is a ceiling distance of itself.

Example 3.3 Let $X = \mathbb{R}$ with the metric $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x - y|$ for all $x, y \in X$, and let $a, b \ge 1$. Define the w-distance $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \max\{a(y-x), b(x-y)\}$$

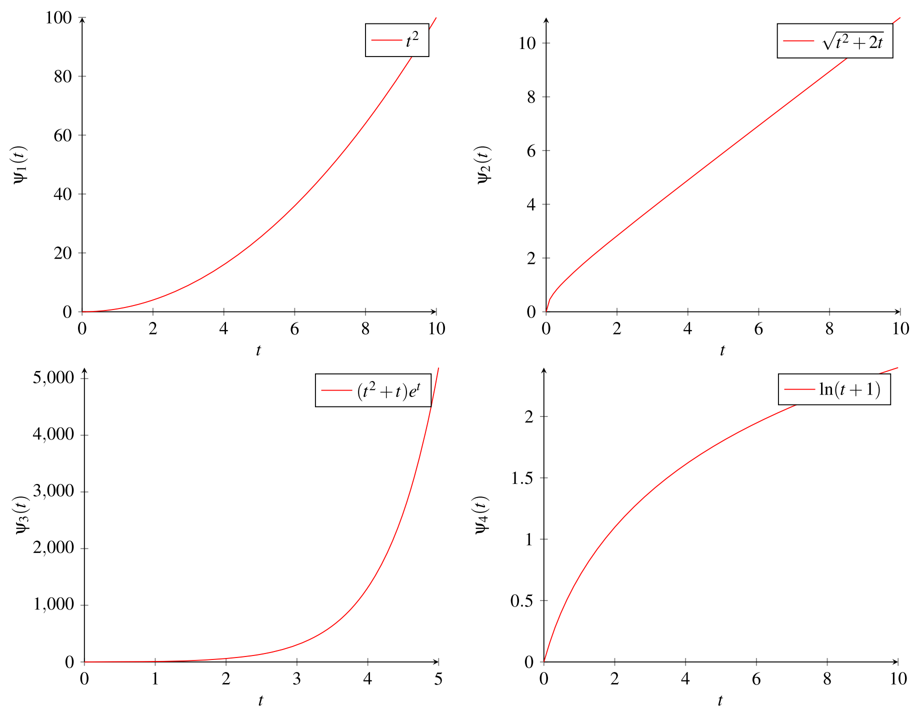

Figure 1 Graphs of $\psi_1, \psi_2, \psi_3, \psi_4$ in Example 2.8.

for all $x, y \in X$. For all $x, y \in X$, we get

$$\begin{aligned} d(x,y) &= |x-y| \\ &= \begin{cases} x-y, x \geq y, \\ y-x, x \leq y, \end{cases} \\ &\leq \max\bigl\{a(y-x), b(x-y)\bigr\} \\ &= q(x,y). \end{aligned}$$

Thus q is a ceiling distance of d.

Example 3.4 Let $a, b \in \mathbb{R}$ with $a < b$, and $X = C[a,b]$ (the set of all continuous functions from $[a,b]$ into $\mathbb{R}$) with the metric $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|$ for all $x, y \in X$. Define the w-distance $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} \bigl|x(t)\bigr| + \sup_{t\in[a,b]} \bigl|y(t)\bigr|$$

for all $x, y \in X$. For all $x, y \in X$ and $t \in [a,b]$, we have

$$\begin{aligned} \bigl|x(t) - y(t)\bigr| &\leq \bigl|x(t)\bigr| + \bigl|y(t)\bigr| \\ &\leq \sup_{t\in[a,b]} \bigl|x(t)\bigr| + \sup_{t\in[a,b]} \bigl|y(t)\bigr|, \end{aligned}$$

which yields

$$d(x,y) = \sup_{t\in[a,b]} \left|x(t) - y(t)\right| \leq \sup_{t\in[a,b]} \left|x(t)\right| + \sup_{t\in[a,b]} \left|y(t)\right| = q(x,y)$$

for all $x, y \in X$. So q is a ceiling distance of d.

Next, we introduce the new type of generalized weak contraction mappings, the so-called w-generalized weak contraction mappings.

Definition 3.5 Let q be a w-distance on a metric space (X,d). A mapping $T : X \to X$ is said to be a w-generalized weak contraction mapping if

$$\psi\left(q(Tx, Ty)\right) \leq \psi\left(m(x,y)\right) - \phi\left(q(x,y)\right) \tag{3.2}$$

for all $x, y \in X$, where

$$m(x,y) := \max\left\{q(x,y), \frac{1}{2}\left[q(x, Ty) + q(Tx, y)\right]\right\},$$

$\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. If $q = d$, then the mapping T is said to be a generalized weak contraction mapping.

Our main result is the following:

Theorem 3.6 *Let (X,d) be a complete metric space, and $q : X \times X \to [0,\infty)$ be a w^0-distance on X and a ceiling distance of d. Suppose that $T : X \to X$ is a continuous w-generalized weak contraction mapping. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$ converges to a unique fixed point of T.*

Proof Suppose that $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions in the contractive condition (3.2). Starting from a fixed arbitrary point $x_0 \in X$, we put $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then x_{n^*} is a fixed point of T. Thus we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, that is, $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since q is a ceiling distance of d, we obtain $q(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. From the contractive condition (3.2), for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
&\psi\left(q(x_{n+1}, x_{n+2})\right) \\
&\quad = \psi\left(q(Tx_n, Tx_{n+1})\right) \\
&\quad \leq \psi\left(m(x_n, x_{n+1})\right) - \phi\left(q(x_n, x_{n+1})\right) \\
&\quad = \psi\left(\left\{q(x_n, x_{n+1}), \frac{1}{2}\left[q(x_n, Tx_{n+1}) + q(Tx_n, x_{n+1})\right]\right\}\right) - \phi\left(q(x_n, x_{n+1})\right) \\
&\quad \leq \psi\left(\left\{q(x_n, x_{n+1}), \frac{1}{2}\left[q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2})\right]\right\}\right) - \phi\left(q(x_n, x_{n+1})\right).
\end{aligned} \tag{3.3}$$

Suppose that

$$q(x_n, x_{n+1}) \leq q(x_{n+1}, x_{n+2})$$

for some $n \in \mathbb{N} \cup \{0\}$. From (3.3) we have

$$\psi\left(q(x_{n+1}, x_{n+2})\right) \leq \psi\left(q(x_{n+1}, x_{n+2})\right) - \phi\left(q(x_n, x_{n+1})\right),$$

which yields that $\phi(q(x_n, x_{n+1})) = 0$, and so $q(x_n, x_{n+1}) = 0$, a contradiction. Therefore,

$$q(x_{n+1}, x_{n+2}) < q(x_n, x_{n+1})$$

for all $n \in \mathbb{N} \cup \{0\}$, and hence $\{q(x_n, x_{n+1})\}$ is decreasing and bounded below. Therefore, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = r. \tag{3.4}$$

Now, from (3.3) we have that, for all $n \in \mathbb{N} \cup \{0\}$,

$$\psi\left(q(x_{n+1}, x_{n+2})\right) \leq \psi\left(q(x_n, x_{n+1})\right) - \phi\left(q(x_n, x_{n+1})\right).$$

Taking the limit as $n \to \infty$ in this inequality and using the continuity of ϕ and ψ, we have

$$\psi(r) \leq \psi(r) - \phi(r),$$

which is a contradiction unless $r = 0$. Hence

$$\lim_{n\to\infty} q(x_n, x_{n+1}) = 0. \tag{3.5}$$

Similarly, we can prove that

$$\lim_{n\to\infty} q(x_{n+1}, x_n) = 0. \tag{3.6}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose by contradiction with Lemma 2.6 (iii) that there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ with $n_k > m_k \geq k$ such that

$$q(x_{m_k}, x_{n_k}) \geq \epsilon \quad \text{for all } k \in \mathbb{N}. \tag{3.7}$$

Choosing n_k to be the smallest integer exceeding m_k for which (3.7) holds, we obtain that

$$q(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{3.8}$$

From (3.7), (3.8), and (W1) we obtain

$$\epsilon \leq q(x_{m_k}, x_{n_k}) \leq q(x_{m_k}, x_{n_k-1}) + q(x_{n_k-1}, x_{n_k}) < \epsilon + q(x_{n_k-1}, x_{n_k}).$$

Taking the limit as $k \to \infty$ in this inequality and using (3.5), we have

$$\lim_{n\to\infty} q(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.9}$$

By using (W1) we have

$$q(x_{m_k}, x_{n_k}) \le q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k})$$

and

$$q(x_{m_k+1}, x_{n_k+1}) \le q(x_{m_k+1}, x_{m_k}) + q(x_{m_k}, x_{n_k}) + q(x_{n_k}, x_{n_k+1}).$$

Taking the limit as $k \to \infty$ in the last two inequalities and using (3.5), (3.6), and (3.9), we have

$$\lim_{n\to\infty} q(x_{m_k+1}, x_{n_k+1}) = \epsilon. \tag{3.10}$$

Again, by using (W1) we obtain

$$\begin{aligned} q(x_{m_k}, x_{n_k}) &\le q(x_{m_k}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k}) \\ &\le q(x_{m_k}, x_{n_k}) + q(x_{n_k}, x_{n_k+1}) + q(x_{n_k+1}, x_{n_k}) \end{aligned}$$

and

$$\begin{aligned} q(x_{m_k}, x_{n_k}) &\le q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{n_k}) \\ &\le q(x_{m_k}, x_{m_k+1}) + q(x_{m_k+1}, x_{m_k}) + q(x_{m_k}, x_{n_k}). \end{aligned}$$

Taking the limit as $k \to \infty$ in the last two inequalities and using (3.5), (3.6), and (3.9), we have

$$\lim_{n\to\infty} q(x_{m_k}, x_{n_k+1}) = \epsilon, \qquad \lim_{n\to\infty} q(x_{m_k+1}, x_{n_k}) = \epsilon. \tag{3.11}$$

Substituting of $x = x_{m_k}$ and $y = x_{n_k}$ into (3.2), we have

$$\begin{aligned} &\psi\bigl(q(x_{m_k+1}, x_{n_k+1})\bigr) \\ &\quad \le \psi\left(\max\left\{q(x_{m_k}, x_{n_k}), \frac{1}{2}\bigl[q(x_{m_k}, x_{n_k+1}) + q(x_{m_k+1}, x_{n_k})\bigr]\right\}\right) - \phi\bigl(q(x_{m_k}, x_{n_k})\bigr). \end{aligned}$$

Letting $k \to \infty$ and using (3.9), (3.10), (3.11), and the continuity of ϕ and ψ, we have

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction with the property of ϕ. Hence by Lemma 2.6(iii) we can conclude that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. From the continuity of T we get $x_{n+1} = Tx_n \to Tp$ as $n \to \infty$, that is, $p = Tp$. Thus, T has a fixed point.

Finally, we will show that the fixed point is unique. Suppose that p and p^* are two distinct fixed points of T. Putting $x = p$ and $y = p^*$ in (3.2), we obtain

$$\psi\left(q\left(Tp, Tp^*\right)\right) \leq \psi\left(\max\left\{q\left(p,p^*\right), \frac{1}{2}\left[q\left(p, Tp^*\right) + q\left(Tp,p^*\right)\right]\right\}\right) - \phi\left(q\left(p,p^*\right)\right),$$

that is, $\psi(q(p,p^*)) \leq \psi(q(p,p^*)) - \phi(q(p,p^*))$, which is a contradiction by the property of ϕ. Therefore, $p = p^*$, and hence the fixed point is unique. This completes the proof. □

In the next theorem, we replace the continuity hypothesis of T in Theorem 3.6 by another condition.

Theorem 3.7 *Let (X,d) be a complete metric space, and $q : X \times X \to [0,\infty)$ be a continuous w^0-distance on X and a ceiling distance of d. Suppose that $T : X \to X$ is a w-generalized weak contraction mapping. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$ converges to a unique fixed point of T.*

Proof Suppose that $\psi, \phi : [0,\infty) \to [0,\infty)$ are two functions in the contractive condition (3.2). Let x_0 be an arbitrary point in X, and $\{x_n\}$ be a Picard sequence in X defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N} \cup \{0\}$, then x_{n^*} is a fixed point of T. So we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Following the proof of Theorem 3.6, we know that $\{x_n\}$ is a Cauchy sequence in X. The completeness of (X,d) ensures that there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Assume that $p \neq Tp$. Putting $x = x_n$ and $y = p$ in (3.2), we have

$$\begin{aligned}\psi\left(q(x_{n+1}, Tp)\right) &= \psi\left(q(Tx_n, Tp)\right)\\ &\leq \psi\left(\max\left\{q(x_n,p), \frac{1}{2}\left[q(x_n, Tp) + q(x_{n+1},p)\right]\right\}\right) - \phi\left(q(x_n,p)\right)\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking the limit as $n \to \infty$ and using the continuity of ϕ, ψ, and q, we have

$$\psi\left(q(p, Tp)\right) \leq \psi\left(\frac{1}{2}q(p, Tp)\right),$$

which is a contradiction. Thus $p = Tp$, that is, p is a fixed point of T. Following the proof of Theorem 3.6, we know that p is a unique fixed point of T. This completes the proof. □

Taking $q = d$ in Theorem 3.7, we obtain the following result.

Corollary 3.8 *Let (X,d) be a complete metric space. Suppose that $T : X \to X$ is a generalized weak contraction mapping. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of T.*

We can extend the condition of w^0-distances in Theorems 3.6 and 3.7 to w-distances if we replace the contractive condition (3.2) by some stronger condition. Here we give the results.

Theorem 3.9 *Let (X,d) be a complete metric space, and $q : X \times X \to [0,\infty)$ be a w-distance on X and a ceiling distance of d. Suppose that $T : X \to X$ is a continuous mapping such that, for all $x, y \in X$,*

$$\psi\big(q(Tx,Ty)\big) \leq \psi\big(q(x,y)\big) - \phi\big(q(x,y)\big), \tag{3.12}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$ converges to a unique fixed point of T.

Theorem 3.10 *Let (X,d) be a complete metric space,and $q : X \times X \to [0,\infty)$ be a continuous w-distance on X and a ceiling distance of d. Suppose that $T : X \to X$ is a mapping such that, for all $x, y \in X$,*

$$\psi\big(q(Tx,Ty)\big) \leq \psi\big(q(x,y)\big) - \phi\big(q(x,y)\big), \tag{3.13}$$

where $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ defined by $x_n = T^n x_0$ for all $n \in \mathbb{N}$ converges to a unique fixed point of T.

4 Existence of a solution for nonlinear integral equations and fractional differential equations

The aim of this section is to present an application of our theoretical results in the previous section for guaranteeing the existence and uniqueness of a solution for various problems regarded by the following equations:

- nonlinear Fredholm integral equations;
- nonlinear Volterra integral equations;
- fractional differential equations of Caputo type.

4.1 Nonlinear integral equations

In this subsection, we prove the existence and uniqueness of a solution for nonlinear Fredholm integral equations and nonlinear Volterra integral equations by using Theorem 3.9.

Theorem 4.1 *Consider the nonlinear Fredholm integral equation*

$$x(t) = \varphi(t) + \int_a^b K\big(t,s,x(s)\big)\,ds, \tag{4.1}$$

where $a, b \in \mathbb{R}$ with $a < b$, and $\varphi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Suppose that the following conditions hold:

(i) *the mapping $T : C[a,b] \to C[a,b]$ defined by*

$$(Tx)(t) = \varphi(t) + \int_a^b K\big(t,s,x(s)\big)\,ds \quad \textit{for all } x \in C[a,b] \textit{ and } t \in [a,b]$$

is a continuous mapping;

(ii) *there are two functions $\psi, \phi : [0,\infty) \to [0,\infty)$ with ψ is an altering distance function and ϕ is a continuous function such that $\psi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$, and for all $x, y \in C[a,b]$, we have*

$$\begin{aligned}&\big|K\big(t,s,x(s)\big)\big| + \big|K\big(t,s,y(s)\big)\big| \\ &\quad\le \frac{[\psi(|x(s)| + |y(s)|)] - [\phi(\sup_{l\in[a,b]} |x(l)| + \sup_{l\in[a,b]} |y(l)|)] - 2|\varphi(t)|}{b-a}\end{aligned}$$

for all $t, s \in [a,b]$.

Then the nonlinear integral equation (4.1) *has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$ defined by*

$$(x_n)(t) = \varphi(t) + \int_a^b K\big(t,s,x_{n-1}(s)\big)\,ds$$

for all $n \in \mathbb{N}$ converges to a unique solution of the nonlinear integral equation (4.1).

Proof Let $X = C[a,b]$. Clearly, X with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} \big|x(t) - y(t)\big|$$

for all $x, y \in X$ is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} \big|x(t)\big| + \sup_{t\in[a,b]} \big|y(t)\big|$$

for all $x, y \in X$. Clearly, q is a w-distance on X and a ceiling distance of d. Here, we will show that T satisfies the contractive condition (3.12). Assume that $x, y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned}&\big|(Tx)(t)\big| + \big|(Ty)(t)\big| \\ &\quad= \left|\varphi(t) + \int_a^b K\big(t,s,x(s)\big)\,ds\right| + \left|\varphi(t) + \int_a^b K\big(t,s,y(s)\big)\,ds\right| \\ &\quad\le \big|\varphi(t)\big| + \left|\int_a^b K\big(t,s,x(s)\big)\,ds\right| \\ &\qquad + \big|\varphi(t)\big| + \left|\int_a^b K\big(t,s,y(s)\big)\,ds\right| \\ &\quad\le 2\big|\varphi(t)\big| + \int_a^b \big|K\big(t,s,x(s)\big)\big|\,ds + \int_a^b \big|K\big(t,s,y(s)\big)\big|\,ds \\ &\quad= 2\big|\varphi(t)\big| + \int_a^b \big(\big|K\big(t,s,x(s)\big)\big| + \big|K\big(t,s,y(s)\big)\big|\big)\,ds \\ &\quad\le 2\big|\varphi(t)\big| + \int_a^b \left(\frac{[\psi(|x(s)| + |y(s)|)] - \phi(q(x,y)) - 2|\varphi(t)|}{b-a}\right) ds \\ &\quad\le 2\big|\varphi(t)\big| + \frac{1}{b-a}\int_a^b \big(\psi\big(q(x,y)\big) - \phi\big(q(x,y)\big) - 2\big|\varphi(t)\big|\big)\,ds \\ &\quad= \psi\big(q(x,y)\big) - \phi\big(q(x,y)\big).\end{aligned}$$

This implies that

$$\sup_{t\in[a,b]} \left|(Tx)(t)\right| + \sup_{t\in[a,b]} \left|(Ty)(t)\right| \leq \psi\left(q(x,y)\right) - \phi\left(q(x,y)\right),$$

and so

$$q(Tx, Ty) \leq \psi\left(q(x,y)\right) - \phi\left(q(x,y)\right)$$

for all $x, y \in X$. Hence we have

$$\psi\left(q(Tx, Ty)\right) \leq q(Tx, Ty) \leq \psi\left(q(x,y)\right) - \phi\left(q(x,y)\right)$$

for all $x, y \in X$. It follows that T satisfies condition (3.12). Therefore, all conditions of Theorem 3.9 are satisfied, and thus T has a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (4.1). This completes the proof. □

By using the identical method in the proof of Theorem 4.1, we get the following result.

Theorem 4.2 *Consider the nonlinear Volterra integral equation*

$$x(t) = \varphi(t) + \int_a^t K\left(t, s, x(s)\right) ds, \tag{4.2}$$

where $a, b \in \mathbb{R}$ *with* $a < b$, *and* $\varphi : [a,b] \to \mathbb{R}$ *and* $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ *are given continuous mappings. Suppose that the following conditions hold*:

(i) *the mapping* $T : C[a,b] \to C[a,b]$ *defined by*

$$(Tx)(t) = \varphi(t) + \int_a^t K\left(t, s, x(s)\right) ds \quad \text{for all } x \in C[a,b] \text{ and } t \in [a,b]$$

is a continuous mapping;

(ii) *there are two functions* $\psi, \phi : [0,\infty) \to [0,\infty)$ *with* ψ *is an altering distance function and* ϕ *is a continuous function such that* $\psi(t) < t$ *for all* $t > 0$, $\phi(t) = 0$ *if and only if* $t = 0$, *and for each* $x, y \in C[a,b]$, *we have*

$$\begin{aligned} &\left|K\left(t, s, x(s)\right)\right| + \left|K\left(t, s, y(s)\right)\right| \\ &\quad \leq \frac{[\psi(|x(s)| + |y(s)|)] - [\phi(\sup_{l\in[a,b]} |x(l)| + \sup_{l\in[a,b]} |y(l)|)] - 2|\varphi(t)|}{b-a} \end{aligned}$$

for all $t, s \in [a,b]$.

Then the nonlinear integral equation (4.2) *has a unique solution. Moreover, for each* $x_0 \in C[a,b]$, *the Picard iteration* $\{x_n\}$ *defined by*

$$(x_n)(t) = \varphi(t) + \int_a^t K\left(t, s, x_{n-1}(s)\right) ds$$

for all $n \in \mathbb{N}$ *converges to a unique solution of the nonlinear integral equation* (4.2).

4.2 Nonlinear fractional differential equations

The theory of nonlinear fractional differential equations nowadays is a large subject of mathematics, which found numerous applications of many branches such as physics, engineering, and other fields connected with real-world problems. Based on this fact, many authors studied various results on this theory (see [9–13]).

First, let us recall some basic definitions of fractional calculus (see [14, 15]). For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of g of order $\beta > 0$ is defined as

$$^{C}D^{\beta}\big(g(t)\big) := \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\, ds,$$

where $n := [\beta] + 1$ with $[\beta]$ denoting the integer part of a positive real number β, and Γ is the gamma function.

The aim of this subsection is to present an application of Theorem 3.9 for proving the existence and uniqueness of a solution for the following nonlinear fractional differential equation of Caputo type:

$$^{C}D^{\beta}\big(x(t)\big) = f\big(t, x(t)\big) \tag{4.3}$$

with integral boundary conditions

$$x(0) = 0, \qquad x(1) = \int_0^{\eta} x(s)\, ds,$$

where $1 < \beta \leq 2$, $0 < \eta < 1$, $x \in C[0,1]$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function (see [12]). It is well known that if f is continuous, then (4.3) is immediately inverted as the very familiar integral equation

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\big(s, x(s)\big)\, ds \\ & - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\big(s, x(s)\big)\, ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f\big(m, x(m)\big)\, dm \right) ds. \end{aligned} \tag{4.4}$$

Now, we prove the following existence theorem.

Theorem 4.3 *Consider the nonlinear fractional differential equation* (4.3). *Suppose that there are two functions* $\psi, \phi : [0,\infty) \to [0,\infty)$ *with* ψ *is an altering distance function and* ϕ *is a continuous function such that* $\psi(t) < t$ *for all* $t > 0$ *and* $\phi(t) = 0$ *if and only if* $t = 0$, *and for each* $x, y \in C[0,1]$, *we have*

$$\begin{aligned} & \big|f\big(s, x(s)\big)\big| + \big|f\big(s, y(s)\big)\big| \\ & \quad \leq \frac{\Gamma(\beta+1)}{5} \big[\psi\big(\big|x(s)\big| + \big|y(s)\big|\big)\big] - \Big[\phi\Big(\sup_{t\in[0,1]} \big|x(t)\big| + \sup_{t\in[0,1]} \big|y(t)\big|\Big)\Big] \end{aligned}$$

for all $s \in [0,1]$. *If the mapping* $T : C[0,1] \to C[0,1]$ *defined by*

$$\begin{aligned}(Tx)(t) = {}& \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\big(s, x(s)\big)\, ds \\ & - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\big(s, x(s)\big)\, ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f\big(m, x(m)\big)\, dm \right) ds\end{aligned}$$

for all $x \in C[a,b]$ *and* $t \in [a,b]$, *is a continuous mapping, then the nonlinear fractional differential equation of Caputo type* (4.3) *has a unique solution. Moreover, for each* $x_0 \in C[0,1]$, *the Picard iteration* $\{x_n\}$ *defined by*

$$\begin{aligned}(x_n)(t) = {}& \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\big(s, x_{n-1}(s)\big)\, ds \\ & - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\big(s, x_{n-1}(s)\big)\, ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f\big(m, x_{n-1}(m)\big)\, dm \right) ds\end{aligned}$$

for all $n \in \mathbb{N}$ *converges to a unique solution of the nonlinear fractional differential equation of Caputo type* (4.3).

Proof Let $X = C[0,1]$. Clearly, X with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[0,1]} \big|x(t) - y(t)\big|$$

for all $x, y \in X$ is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[0,1]} \big|x(t)\big| + \sup_{t\in[0,1]} \big|y(t)\big|$$

for all $x, y \in X$. Clearly, q is a w-distance on X and a ceiling distance of d. We will show that T satisfies the contractive condition (3.12). Assume that $x, y \in X$ and $t \in [0,1]$. Then we get

$$\begin{aligned}&\big|(Tx)(t)\big| + \big|(Ty)(t)\big| \\ &\quad = \bigg| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\big(s, x(s)\big)\, ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\big(s, x(s)\big)\, ds \\ &\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f\big(m, x(m)\big)\, dm \right) ds \bigg| \\ &\qquad + \bigg| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\big(s, y(s)\big)\, ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\big(s, y(s)\big)\, ds \\ &\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f\big(m, y(m)\big)\, dm \right) ds \bigg|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} \big(\big|f\big(s,x(s)\big)\big| + \big|f\big(s,y(s)\big)\big| \big)\, ds \\
&\quad - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \big(\big|f\big(s,x(s)\big)\big| + \big|f\big(s,y(s)\big)\big| \big)\, ds \\
&\quad + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left| \int_0^s (s-m)^{\beta-1} \big(f\big(m,x(m)\big) + f\big(m,y(m)\big) \big)\, dm \right| ds \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} \frac{\Gamma(\beta+1)}{5} \big(\big[\psi\big(\big|x(s)\big| + \big|y(s)\big|\big)\big] - \phi\big(q(x,y)\big) \big)\, ds \\
&\quad - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \frac{\Gamma(\beta+1)}{5} \big(\big[\psi\big(\big|x(s)\big| + \big|y(s)\big|\big)\big] - \phi\big(q(x,y)\big) \big)\, ds \\
&\quad + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \\
&\quad \times \int_0^\eta \left| \int_0^s (s-m)^{\beta-1} \frac{\Gamma(\beta+1)}{5} \big(\big[\psi\big(\big|x(s)\big| + \big|y(s)\big|\big)\big] - \phi\big(q(x,y)\big) \big)\, dm \right| ds \\
&\leq \frac{\Gamma(\beta+1)}{5} \big(\big[\psi\big(q(x,y)\big)\big] - \phi\big(q(x,y)\big) \big) \\
&\quad \times \sup_{t\in(0,1)} \left(\frac{1}{\Gamma(\beta)} \int_0^1 |t-s|^{\beta-1}\, ds \right. \\
&\quad \left. + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}\, ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \int_0^s |s-m|^{\beta-1}\, dm\, ds \right) \\
&\leq \psi\big(q(x,y)\big) - \phi\big(q(x,y)\big).
\end{aligned}
$$

This implies that

$$\sup_{t\in[a,b]} \big|(Tx)(t)\big| + \sup_{t\in[a,b]} \big|(Ty)(t)\big| \leq \psi\big(q(x,y)\big) - \phi\big(q(x,y)\big),$$

and so

$$q(Tx,Ty) \leq \psi\big(q(x,y)\big) - \phi\big(q(x,y)\big)$$

for all $x, y \in X$. Hence we have

$$\psi\big(q(Tx,Ty)\big) \leq q(Tx,Ty) \leq \psi\big(q(x,y)\big) - \phi\big(q(x,y)\big)$$

for all $x, y \in X$. It follows that T satisfies condition (3.12). Therefore, all conditions of Theorem 3.9 are satisfied, and thus T has a unique fixed point. This implies that there exists a unique solution of the nonlinear fractional differential equation of Caputo type (4.3). This completes the proof. □

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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