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Distributions of zeros of solutions to first order delay dynamic equations

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Abstract
This paper is concerned with the distributions of zeros of solutions to first order delay dynamic equations on time scales. The results are obtained using iterative sequences.

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Keywords: distribution of zeros; delay equations; time scales

1 Introduction

The oscillation and distributions of zeros of solutions of first order delay differential and difference equations are studied widely in the literature; see [1–18] and the references therein. However, there are only a few papers considering the distribution of zeros of solutions of first order delay and advanced dynamic equations on time scales (see [8, 19]). In [12], Zhou considered the first order delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \quad \text{for } t \in [t_0, \infty), \tag{1.1}$$

where

$$\int_{t-\tau}^{t} p(s)\,ds \geq \rho > \frac{1}{e}, \quad \text{and} \quad \rho < 1, \tag{1.2}$$

and established lower and upper bounds for the quotient $x(t-\tau)/x(t)$. In particular the author proved that $x(t-\tau)/x(t) \geq f_n(\rho)$ and $x(t-\tau)/x(t) < g_m(\rho)$, where the sequences $f_n(\rho)$ and $g_n(\rho)$ are defined by

$$\begin{cases} f_1(\rho) = e^{\rho}, & f_{n+1}(\rho) = e^{\rho f_n(\rho)}, & n = 1, 2, \dots, \\ g_1(\rho) = \frac{2(1-\rho)}{\rho^2}, & g_{m+1}(\rho) = \frac{2(1-\rho)g_m^2(\rho)}{g_m^2(\rho)\rho^2+2}, & m = 1, 2, \dots, \end{cases}$$

and using these sequences the author studied the distribution of zeros of solutions of (1.1). In [13], Zhang and Zhou considered the first order delay differential equation

$$x'(t) + p(t)x\bigl(\tau(t)\bigr) = 0, \quad \text{for } t \in [t_0, \infty), \tag{1.3}$$

and studied the distribution of zeros of solutions using the two sequences $f_n(\rho)$ and $g_m(\rho)$ where

$$\begin{cases} f_0(\rho) = 1, \qquad f_{n+1}(\rho) = e^{\rho f_n(\rho)}, \qquad n = 0, 1, 2, \ldots, \\ g_1(\rho) = \frac{2(1-\rho)}{\rho^2}, \qquad g_{m+1}(\rho) = \frac{2(1-\rho)g_m^2(\rho)}{g_m^2(\rho)\rho^2+2}, \quad m = 1, 2, \ldots, \end{cases}$$

and

$$\int_{\tau(t)}^{t} p(s)\,ds \geq \rho > 0, \quad \text{and} \quad 0 < \rho < 1. \tag{1.4}$$

Zhang and Lian in [19] initiated the study of the distribution of zeros of dynamic equations on time scales and in particular, they considered the first order delay dynamic equation

$$x^{\Delta}(t) + p(t)x\bigl(\tau(t)\bigr) = 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.5}$$

on a time scale $\mathbb{T}$, where $p \in \mathbf{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^+)$ is a non-negative rd-continuous function, $\tau \in \mathbf{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$ is strictly increasing, $\tau(t) < t$ for $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$. In [19] the authors established lower and the upper bounds for the quotient $x(\tau(t))/x(t)$ using the sequences f_n and g_m where

$$f_0(\rho) = 1, \qquad f_n(\rho) = e^{(1-\rho)f_{n-1}(\rho)}, \quad n = 1, 2, \ldots, \tag{1.6}$$

and

$$\begin{cases} g_1(\rho) = \frac{2\rho}{(1-\rho)^2-2M(1-\rho)}, \\ g_m(\rho) = \frac{2\rho}{(1-\rho)^2-2M(1-\rho)+\frac{2}{g_{m-1}^2(\rho)}}, \quad m = 2, 3, \ldots, \end{cases} \tag{1.7}$$

and where $M < (1-\rho)/2$ and $0 \leq \rho < 1$ satisfies the condition

$$\sup_{\lambda\in E}\left\{\lambda \exp\left\{\int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-\lambda p(s)\bigr)\Delta s\right\}\right\} \leq \rho,$$

where $E = \{\lambda : \lambda > 0, 1 - \lambda p(t)\mu(t) > 0\}$; $\zeta_{\mu(s)}$ and $\mu(s)$ will be defined later.

Motivated by these papers, we study the distribution of zeros of oscillatory solutions of the delay dynamic equation (1.5) on a time scale $\mathbb{T}$ by considering new sequences f_n and g_m. In the next section, we present some basic ideas on time scales. In Section 3, we establish lower and upper bounds for $x(\tau(t))/x(t)$ and in Section 4, we study the distribution of zeros of solutions of (1.5).

2 Some preliminaries and lemmas

In this section, we present some preliminaries; see [20, 21]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and backward jump operators are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

with $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. The graininess function μ on a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. A function f is said to be Δ-differentiable if its Δ-derivative exists. A useful formula is $f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, and here $g^{\sigma} = g \circ \sigma$) of two Δ-differentiable functions f and g:

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \qquad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the chain rule,

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'\bigl(g(t) + h\mu(t)g^{\Delta}(t)\bigr)\, dh \right\} g^{\Delta}(t), \tag{2.1}$$

holds. A special case of (2.1) is

$$\bigl[x^{\lambda}(t)\bigr]^{\Delta} = \lambda \int_0^1 \bigl[hx^{\sigma} + (1-h)x\bigr]^{\lambda-1} x^{\Delta}(t)\, dh.$$

For $s, t \in \mathbb{T}$, a function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta} = f(t)$ holds for all $t \in \mathbb{T}$. In this case we define the integral of f by $\int_s^t f(\tau)\Delta\tau = F(t) - F(s)$. For $a, b \in \mathbb{T}$, and a Δ-differentiable function f, the Cauchy integral of f^{Δ} is defined by $\int_a^b f^{\Delta}(\tau)\Delta\tau = f(b) - f(a)$. The integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = \bigl[f(t)g(t)\bigr]_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t)\Delta t = \lim_{b \to \infty} \int_a^b f(t)\Delta t.$$

A function $p: \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for $t \in \mathbb{T}$. A function $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive (we write $p \in \mathcal{R}^{+}$) if it is rd-continuous function and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$. Hilger in [20] showed that for $p(t)$ rd-continuous and regressive, the solution of the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \quad y(t_0) = 1,$$

is given by the generalized exponential function $e_p(t,t_0)$, which is defined by

$$e_p(t,t_0) = \exp\left\{\int_{t_0}^{t} \zeta_{\mu(s)}\bigl(p(s)\bigr)\Delta s\right\},$$

where $t_0, t \in \mathbb{T}$, and the cylinder transformation $\zeta_h(z)$ is defined by

$$\zeta_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0, \end{cases}$$

where $z \in \mathbb{R}$ and $h \in \mathbb{R}^+$.

The next lemma can be found in [11].

Lemma 2.1 *Assume $t_0, t \in \mathbb{T}$.*

(i) *For a non-negative φ with $-\varphi \in \mathcal{R}^+$, we have the following inequality*:

$$1 - \int_{t_0}^{t} \varphi(s)\Delta s \le e_{-\varphi}(t,t_0) \le \exp\left\{-\int_{t_0}^{t} \varphi(s)\Delta s\right\}. \tag{2.2}$$

(ii) *If φ is rd-continuous and non-negative, then*

$$1 + \int_{t_0}^{t} \varphi(s)\Delta s \le e_{\varphi}(t,t_0) \le \exp\left\{\int_{t_0}^{t} \varphi(s)\Delta s\right\}. \tag{2.3}$$

Lemma 2.2 *Assume that $\mathbb{T}$ is a time scale with $t_0 \in \mathbb{T}$. If $f(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$, then*

$$\bigl[\ln f(t)\bigr]^{\Delta} \le \frac{f^{\Delta}(t)}{f(t)}, \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}.$$

Proof Fix t. We consider two cases: (i) $f^{\Delta}(t) \le 0$ and (ii) $f^{\Delta}(t) \ge 0$.

In the first case, we see that

$$h\mu(t)f^{\Delta}(t) + f(t) \le f(t).$$

Now recall $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$ so $h\mu(t)f^{\Delta}(t) + f(t) = hf(\sigma(t)) + (1-h)f(t) > 0$ and as a result

$$\frac{1}{h\mu(t)f^{\Delta}(t) + f(t)} \ge \frac{1}{f(t)}.$$

Apply the chain rule (2.1), and we get (note $f^{\Delta}(t) \le 0$)

$$\bigl[\ln f(t)\bigr]^{\Delta} = \left\{\int_0^1 \frac{1}{h\mu(t)f^{\Delta}(t) + f(t)}\,dh\right\} f^{\Delta}(t) \le \frac{f^{\Delta}(t)}{f(t)}.$$

In the second case, we see that

$$h\mu(t)f^{\Delta}(t) + f(t) \ge f(t).$$

Applying the chain rule (2.1), we get

$$\left[\ln f(t)\right]^{\Delta}=\left\{\int_0^1\frac{1}{h\mu(t)f^{\Delta}(t)+f(t)}\,dh\right\}f^{\Delta}(t)\le\frac{f^{\Delta}(t)}{f(t)}.$$

Thus, we deduce in both cases that

$$\left[\ln f(t)\right]^{\Delta}\le\frac{f^{\Delta}(t)}{f(t)}.$$

The proof is complete. □

Lemma 2.3 *Assume that $\mathbb{T}$ is a time scale with $t_0\in\mathbb{T}$. If $f(t)>0$ and $f^{\Delta}(t)\ge 0$ for $t\in[t_0,\infty)_{\mathbb{T}}$, then for $\alpha>0$*

$$\left[e^{-\alpha f(t)}\right]^{\Delta}\ge -\alpha e^{-\alpha f(t)}f^{\Delta}(t).$$

Proof Since $f(t)>0$ and $f^{\Delta}(t)\ge 0$ for $t\in[t_0,\infty)_{\mathbb{T}}$, we have for $h\in(0,1)$

$$f(t)\le h\mu(t)f^{\Delta}(t)+f(t). \tag{2.4}$$

Applying the chain rule (2.1) and using (2.4), we see that

$$\left[e^{-\alpha f(t)}\right]^{\Delta}=\left\{-\alpha\int_0^1 e^{-\alpha(f(t)+h\mu(t)f^{\Delta}(t))}\,dh\right\}f^{\Delta}(t)\ge -\alpha e^{-\alpha f(t)}f^{\Delta}(t).$$

The proof is complete. □

3 Lower and upper bounds for $x(\tau(t))/x(t)$

In this section, we establish lower and upper bounds for $x(\tau(t))/x(t)$ where $x(t)$ is a solution of equation (1.5). We use the notation $\tau^0(t)=t$ and inductively define the iterates of $\tau^{-i}(t)$ by

$$\tau^{-i}(t)=\left(\tau^{-1}\circ\tau^{-(i-1)}\right)(t),\quad \text{for } i=1,2,\ldots,$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$. From the definition it is clear that

$$\tau(t)<t<\tau^{-1}(t)<\cdots<\tau^{-(n-1)}(t)<\tau^{-n}(t)<\cdots.$$

To find the lower bound for $x(\tau(t))/x(t)$ we define for $0<\rho<1$ a sequence $f_n(\rho)$ by

$$\begin{cases} f_0(\rho)=1, \qquad f_1(\rho)=1/\rho,\\ f_{n+2}(\rho)=\frac{f_n(\rho)}{f_n(\rho)+1-e^{(1-\rho)f_n(\rho)}}, \qquad n=0,1,2,3,\ldots. \end{cases} \tag{3.1}$$

We note some properties of $f_n(\rho)$ for the reader's interest (see [9] or use an elementary argument using $\frac{x}{x+1-e^{(1-\rho)x}}$). For $0<1-\rho\le 1/e$, we have

$$1\le f_n(\rho)\le f_{n+2}(\rho)\le e,\quad n=0,1,2,\ldots,$$

so there exists a function $f(\rho)$ such

$$\lim_{n\to\infty} f_n(\rho) = f(\rho), \quad 1 \le f(\rho) \le e,$$

where $f(\rho)$ satisfies

$$f(\rho) = e^{(1-\rho)f(\rho)}. \tag{3.2}$$

If $(1-\rho) > 1/e$, then either $f_n(\rho)$ is nondecreasing and $\lim_{n\to\infty} f_n(\rho) = +\infty$ or $f_n(\rho)$ is negative or $f_n(\rho)$ is ∞ after a finite numbers of terms.

Theorem 3.1 *Assume that $\mathbb{T}$ is a time scale and t', t_0, $t_1 \in \mathbb{T}$, $t_0 \ge t'$, $t_1 \ge \tau^{-3}(t_0)$, $x(t)$ is a solution of* (1.5) *on $[t',\infty)_{\mathbb{T}}$, $x(t)$ is positive on $[t_0,t_1]_{\mathbb{T}}$ and there exists $\rho \in (0,1)$ with $\infty > f_n(\rho) > 0$ for $n \in \{2,3,\ldots\}$ and*

$$\sup_{\lambda\in E}\left\{\lambda \exp\left\{\int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-\lambda p(s)\bigr)\Delta s\right\}\right\} \le \rho \quad \textit{for } t \in \bigl[\tau^{-3}(t_0), t_1\bigr]_{\mathbb{T}}; \tag{3.3}$$

here $E = \{\lambda : \lambda > 0, 1-\lambda p(t)\mu(t) > 0 \text{ for } t \in [\tau^{-2}(t_0),t_1]_{\mathbb{T}}\}$. Then for $n \ge 0$ when $\tau^{-(2+n)}(t_0) \le t_1$ we have

$$\frac{x(\tau(t))}{x(t)} \ge f_n(\rho), \quad \textit{for } t \in \bigl[\tau^{-(2+n)}(t_0), t_1\bigr]_{\mathbb{T}},$$

where $f_n(\rho)$ is defined in (3.1).

Proof From (1.5), we see that

$$x^{\Delta}(t) = -p(t)x\bigl(\tau(t)\bigr) \le 0, \quad \text{for } t \in \bigl[\tau^{-1}(t_0), t_1\bigr]_{\mathbb{T}}, \tag{3.4}$$

so since $x(t)$ is nonincreasing on $[\tau^{-1}(t_0),t_1]_{\mathbb{T}}$ we have

$$\frac{x(\tau(t))}{x(t)} \ge 1, \quad \text{for } t \in \bigl[\tau^{-2}(t_0), t_1\bigr]_{\mathbb{T}}. \tag{3.5}$$

Note (3.5) and the fact that x is positive on $[t_0,t_1]_{\mathbb{T}}$, so for $t \in [\tau^{-2}(t_0),t_1]_{\mathbb{T}}$ we have (note $x(\sigma(t)) > 0$ since $\sigma(t) \ge t \ge \tau^{-2}(t_0) > t_0$)

$$\begin{aligned} 0 &= -\mu(t)\bigl[x^{\Delta}(t) + p(t)x\bigl(\tau(t)\bigr)\bigr] \\ &= x(t) - x\bigl(\sigma(t)\bigr) - \mu(t)p(t)x\bigl(\tau(t)\bigr) \\ &< x(t) - \mu(t)p(t)x(t) \\ &= \bigl[1-\mu(t)p(t)\bigr]x(t). \end{aligned}$$

Hence $1-\mu(t)p(t) > 0$ for $t \in [\tau^{-2}(t_0),t_1]_{\mathbb{T}}$ so $-p \in \mathcal{R}^+$ on the interval $[\tau^{-2}(t_0),t_1]_{\mathbb{T}}$. Using Lemma 2.1 (with the time scale $[\tau^{-2}(t_0),t_1]_{\mathbb{T}}$) and (3.3), we have for $t \in [\tau^{-3}(t_0),t_1]_{\mathbb{T}}$ (note

$\tau(t) \in [\tau^{-2}(t_0), t_1]_{\mathbb{T}})$

$$\begin{aligned}\int_{\tau(t)}^{t} p(s)\Delta s &\geq 1 - \exp\left\{\int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-p(s)\bigr)\Delta s\right\} \\ &\geq 1 - \sup_{\lambda\in E}\left\{\lambda \exp\left\{\int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-\lambda p(s)\bigr)\Delta s\right\}\right\} \\ &\geq 1 - \rho. \end{aligned} \tag{3.6}$$

Integrating (1.5) from $\tau(t)$ to t, we get

$$x\bigl(\tau(t)\bigr) = x(t) + \int_{\tau(t)}^{t} p(s)x\bigl(\tau(s)\bigr)\Delta s, \tag{3.7}$$

and hence, for $t \in [\tau^{-3}(t_0), t_1]_{\mathbb{T}}$, we get

$$x\bigl(\tau(t)\bigr) = x(t) + \int_{\tau(t)}^{t} p(s)x\bigl(\tau(s)\bigr)\Delta s \geq x(t) + x\bigl(\tau(t)\bigr)\int_{\tau(t)}^{t} p(s)\Delta s \geq x(t) + x\bigl(\tau(t)\bigr)(1-\rho),$$

so

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{\rho} = f_1(\rho) > 0, \quad \text{for } t \in \bigl[\tau^{-3}(t_0), t_1\bigr].$$

When $\tau^{-4}(t_0) \leq t_1$, note, for $t \in [\tau^{-4}(t_0), t_1]_{\mathbb{T}}$ and $\tau(t) \leq s \leq t$, that

$$\int_{\tau(s)}^{\tau(t)} \frac{x^{\Delta}(\xi)}{x(\xi)}\Delta\xi + \int_{\tau(s)}^{\tau(t)} p(\xi)\frac{x(\tau(\xi))}{x(\xi)}\Delta\xi = 0, \tag{3.8}$$

so from Lemma 2.2 we have

$$\int_{\tau(s)}^{\tau(t)} \bigl[\ln x(\xi)\bigr]^{\Delta}\Delta\xi + \int_{\tau(s)}^{\tau(t)} p(\xi)\frac{x(\tau(\xi))}{x(\xi)}\Delta\xi \leq 0, \tag{3.9}$$

which implies that

$$\frac{x(\tau(s))}{x(\tau(t))} \geq \exp\left\{\int_{\tau(s)}^{\tau(t)} p(\xi)\frac{x(\tau(\xi))}{x(\xi)}\Delta\xi\right\},$$

and so using (3.5), we have (note $\xi \in [\tau^{-2}(t_0), t_1]_{\mathbb{T}}$ since $\tau(t) \leq s \leq t$ and $t \in [\tau^{-4}(t_0), t_1]_{\mathbb{T}}$)

$$\frac{x(\tau(s))}{x(\tau(t))} \geq \exp\left\{f_0(\rho)\int_{\tau(s)}^{\tau(t)} p(\xi)\Delta\xi\right\}; \tag{3.10}$$

we write $f_0(\rho)$ (which is of course 1 here) to indicate the general procedure. Now applying Lemma 2.3 and using (3.6), (3.7) and (3.10), we get (here $t \in [\tau^{-4}(t_0), t_1]_{\mathbb{T}}$)

$$\begin{aligned} x\bigl(\tau(t)\bigr) &= x(t) + \int_{\tau(t)}^{t} p(s)x\bigl(\tau(s)\bigr)\Delta s \\ &\geq x(t) + x\bigl(\tau(t)\bigr)\int_{\tau(t)}^{t} p(s)\exp\left\{f_0(\rho)\int_{\tau(s)}^{\tau(t)} p(\xi)\Delta\xi\right\}\Delta s \end{aligned}$$

$$\begin{aligned}
&= x(t) + x\big(\tau(t)\big) \int_{\tau(t)}^{t} p(s) \exp\biggl\{ f_0(\rho) \biggl(\int_{\tau(s)}^{s} p(\xi)\Delta\xi - \int_{\tau(t)}^{s} p(\xi)\Delta\xi \biggr) \biggr\} \Delta s \\
&\geq x(t) + x\big(\tau(t)\big) e^{(1-\rho)f_0(\rho)} \int_{\tau(t)}^{t} p(s) \exp\biggl\{ -f_0(\rho) \int_{\tau(t)}^{s} p(\xi)\Delta\xi \biggr\} \Delta s \\
&\geq x(t) + x\big(\tau(t)\big) e^{(1-\rho)f_0(\rho)} \int_{\tau(t)}^{t} \frac{-[\exp\{-f_0(\rho) \int_{\tau(t)}^{s} p(\xi)\Delta\xi\}]^{\Delta}}{f_0(\rho)} \Delta s \\
&= x(t) + x\big(\tau(t)\big) e^{(1-\rho)f_0(\rho)} \biggl[\frac{1 - \exp\{-f_0(\rho) \int_{\tau(t)}^{t} p(\xi)\Delta\xi\}}{f_0(\rho)} \biggr] \Delta s \\
&\geq x(t) + x\big(\tau(t)\big) \biggl(\frac{e^{(1-\rho)f_0(\rho)} - 1}{f_0(\rho)} \biggr).
\end{aligned}$$

Thus, for $t \in [\tau^{-4}(t_0), t_1]_{\mathbb{T}}$, we get

$$\frac{x(\tau(t))}{x(t)} \geq \frac{f_0(\rho)}{f_0(\rho) + 1 - e^{(1-\rho)f_0(\rho)}} = f_2(\rho) > 0.$$

Repeating the above procedure, when $\tau^{-(2+n)}(t_0) \leq t_1$ we get for $t \in [\tau^{-(2+n)}(t_0), t_1]_{\mathbb{T}}$

$$\frac{x(\tau(t))}{x(t)} \geq \frac{f_{n-2}(\rho)}{f_{n-2}(\rho) + 1 - e^{(1-\rho)f_{n-2}(\rho)}} = f_n(\rho) > 0.$$

The proof is complete. □

Remark 3.1 From the proof of Theorem 3.1 notice in the statement of Theorem 3.1 we could replace $\infty > f_n(\rho) > 0$ for $n \in \{2, 3, \ldots\}$ with $\infty > f_n(\rho) > 0$ for $n \in \{2, 3, \ldots, N-2\}$ if $\tau^{-(2+N)}(t_0) < t_1 < \tau^{-(3+N)}(t_0)$ or $\infty > f_n(\rho) > 0$ for $n \in \{2, 3, \ldots, N-3\}$ if $\tau^{-(2+N)}(t_0) = t_1 < \tau^{-(3+N)}(t_0)$.

To establish the upper bound for $x(\tau(t))/x(t)$, we define a sequence $g_m(\rho)$ by

$$\begin{cases} g_1(\rho) := \frac{2\rho}{(1-\rho)^2 - 2M(1-\rho)}, \\ g_{m+1}(\rho) := \frac{2(\rho - \frac{1}{g_m(\rho)})}{[(1-\rho)^2 - 2M(1-\rho)]}, \end{cases} \tag{3.11}$$

where $0 \leq \rho < 1$, $m = 1, 2, 3, \ldots$, and $0 \leq M < (1-\rho)/2$.

We note some properties of $g_m(\rho)$ for the reader's interest. Note $g_{m+1}(\rho) < g_m(\rho)$, for $m = 1, 2, 3, \ldots$, and trivially

$$g_1(\rho) > \frac{\rho}{(1-\rho)^2 - 2M(1-\rho)}.$$

More generally when $0 < 1 - \rho \leq 1/e$ using an induction argument (*i.e.* assuming $g_m(\rho) > \frac{\rho}{(1-\rho)^2 - 2M(1-\rho)}$) then

$$\begin{aligned}
g_{m+1}(\rho) &= \frac{2(\rho g_m(\rho) - 1)}{g_m(\rho)[(1-\rho)^2 - 2M(1-\rho)]} \\
&> \frac{2\rho}{(1-\rho)^2 - 2M(1-\rho)} - \frac{2}{\rho} > \frac{\rho}{(1-\rho)^2 - 2M(1-\rho)};
\end{aligned}$$

thus $g_k(\rho) > \frac{\rho}{(1-\rho)^2-2M(1-\rho)}$ where $k = 1, 3, \ldots$. Then there exists a function $g(\rho)$ with

$$\lim_{m\to\infty} g_m(\rho) = g(\rho) = \frac{2}{\rho - \sqrt{2(2M-1) - 4(M-1)\rho - \rho^2}}.$$

for $0 < 1-\rho \leq 1/e$ (note $2(2M-1) - 4(M-1)\rho - \rho^2 > 0$ if $0 < 1-\rho \leq 1/e$).

Theorem 3.2 *Assume that $\mathbb{T}$ is a time scale and $t', t_0 \in \mathbb{T}$, $t_0 \geq t'$, $x(t)$ is a solution of* (1.5) *on $[t', \infty)_{\mathbb{T}}$, there exists a positive integer $N \geq 4$ such that $x(t)$ is positive on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$ and there exists $\rho \in (0,1)$ with $g_m(\rho) > 0$ for $m \in \{2, 3, \ldots, N-3\}$ and*

$$\sup_{\lambda\in E}\left\{\lambda \exp\left\{\int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-\lambda p(s)\bigr)\Delta s\right\}\right\} \leq \rho \quad \text{for } t \in \bigl[\tau^{-3}(t_0), \tau^{-N}(t_0)\bigr]_{\mathbb{T}}, \tag{3.12}$$

where $E = \{\lambda : \lambda > 0, 1 - \lambda p(t)\mu(t) > 0 \text{ for } t \in [\tau^{-2}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}\}$ and

$$M = \sup_{s\in[\tau^{-3}(t_0),\tau^{-N}(t_0)]_{\mathbb{T}}} p(s)\mu(s) < \frac{1-\rho}{2}.$$

Then for $m \in \{1, \ldots, N-3\}$ we have

$$\frac{x(\tau(t))}{x(t)} < g_m(\rho), \quad \text{for } t \in \bigl[\tau^{-3}(t_0), \tau^{-(N-m)}(t_0)\bigr]_{\mathbb{T}},$$

where $g_m(\rho)$ is defined in (3.11).

Proof From (1.5), we see that

$$x^{\Delta}(t) \leq 0, \quad \text{for } t \in \bigl[\tau^{-1}(t_0), \tau^{-N}(t_0)\bigr]_{\mathbb{T}}, \tag{3.13}$$

and as in Theorem 3.1 notice $1 - \mu(t)p(t) > 0$ for $t \in [\tau^{-2}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}$ so $-p \in \mathcal{R}^+$ on the interval $[\tau^{-2}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}$. From Lemma 2.1 (with the time scale $[\tau^{-2}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}$) and (3.12), we have for $t \in [\tau^{-3}(t_0), \tau^{-(N-1)}(t_0)]_{\mathbb{T}}$ (note $\tau^{-1}(t) \leq \tau^{-N}(t_0)$)

$$\int_{\tau(t)}^{t} p(s)\Delta s \geq 1-\rho \quad \text{and} \quad \int_{t}^{\tau^{-1}(t)} p(s)\Delta s \geq 1-\rho. \tag{3.14}$$

Let $t \in [\tau^{-3}(t_0), \tau^{-(N-1)}(t_0)]_{\mathbb{T}}$ and consider

$$G(r) := \int_{t}^{r} p(s)\Delta s - 1 + \rho, \quad \text{for } r \in \bigl[t, \tau^{-1}(t)\bigr]_{\mathbb{T}}.$$

Note $G : [t, \tau^{-1}(t)] \to \mathbb{R}$ is nondecreasing, $G(t) = -1 + \rho < 0$, and

$$G\bigl(\tau^{-1}(t)\bigr) = \int_{t}^{\tau^{-1}(t)} p(s)\Delta s - 1 + \rho \geq 1 - \rho - 1 + \rho = 0.$$

If $G(\tau^{-1}(t)) = 0$, then

$$\int_{t}^{\tau^{-1}(t)} p(s)\Delta s = G\bigl(\tau^{-1}(t)\bigr) + 1 - \rho = 1 - \rho,$$

whereas if $G(\tau^{-1}(t)) > 0$ then $G(t) < 0 < G(\tau^{-1}(t))$.

In either case (from the intermediate value theorem [20]) there exists $t^* \in [t, \tau^{-1}(t)]_{\mathbb{T}}$ with $\sigma(t^*) \in [t, \tau^{-1}(t)]_{\mathbb{T}}$ such that $G(t^*)G(\sigma(t^*)) \leq 0$ and so

$$\int_t^{t^*} p(s)\Delta s \leq 1-\rho \quad \text{and} \quad \int_t^{\sigma(t^*)} p(s)\Delta s \geq 1-\rho. \tag{3.15}$$

Integrating both sides of (1.5) from t to $\sigma(t^*)$, for $t \in [\tau^{-3}(t_0), \tau^{-(N-1)}(t_0)]_{\mathbb{T}}$, we have

$$x(t) = x\bigl(\sigma\bigl(t^*\bigr)\bigr) + \int_t^{\sigma(t^*)} p(s)x\bigl(\tau(s)\bigr)\Delta s. \tag{3.16}$$

Fix $t \in [\tau^{-3}(t_0), \tau^{-(N-1)}(t_0)]_{\mathbb{T}}$. Let $s \in \mathbb{T}$ be such that $t \leq s \leq \sigma(t^*) \leq \tau^{-1}(t)$ (here t^* is as described above, and note $\tau(t) \leq \tau(s) \leq t$) and integrating (1.5) from $\tau(s)$ to t yields

$$x\bigl(\tau(s)\bigr) = x(t) + \int_{\tau(s)}^t p(u)x\bigl(\tau(u)\bigr)\Delta u,$$

and this together with x being nonincreasing on $[\tau^{-1}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}$ and (3.14) will give

$$\begin{aligned} x\bigl(\tau(s)\bigr) &\geq x(t) + x\bigl(\tau(t)\bigr)\int_{\tau(s)}^t p(u)\Delta u \\ &= x(t) + x\bigl(\tau(t)\bigr)\biggl\{\int_{\tau(s)}^s p(u)\Delta u - \int_t^s p(u)\Delta u\biggr\} \\ &\geq x(t) + x\bigl(\tau(t)\bigr)\biggl\{1-\rho - \int_t^s p(u)\Delta u\biggr\}, \end{aligned} \tag{3.17}$$

so from (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned} x(t) &= x\bigl(\sigma\bigl(t^*\bigr)\bigr) + \int_t^{\sigma(t^*)} p(s)x\bigl(\tau(s)\bigr)\Delta s \\ &\geq x\bigl(\sigma\bigl(t^*\bigr)\bigr) + \int_t^{\sigma(t^*)} p(s)\biggl\{x(t) + x\bigl(\tau(t)\bigr)\biggl\{1-\rho - \int_t^s p(u)\Delta u\biggr\}\biggr\}\Delta s \\ &\geq x\bigl(\sigma\bigl(t^*\bigr)\bigr) + (1-\rho)x(t) + (1-\rho)^2 x\bigl(\tau(t)\bigr) \\ &\quad - x\bigl(\tau(t)\bigr)\biggl\{\int_t^{t^*} p(s)\biggl\{\int_t^s p(u)\Delta u\biggr\}\Delta s \\ &\quad + \int_{t^*}^{\sigma(t^*)} p(s)\biggl\{\int_t^s p(u)\Delta u\biggr\}\Delta s\biggr\}. \end{aligned} \tag{3.18}$$

Let $F(s) = \int_t^s p(u)\Delta u$, and note

$$\begin{aligned} \bigl[F^2(s)\bigr]^{\Delta} &= 2\int_0^1 \bigl[hF^{\sigma}(s) + (1-h)F(s)\bigr]F^{\Delta}(s)\,dh \\ &= 2\int_0^1 \bigl[hF^{\sigma}(s) + (1-h)F(s)\bigr]p(s)\,dh \\ &\geq 2F(s)p(s). \end{aligned}$$

Hence,

$$\int_t^{t^*} p(s)\left\{\int_t^s p(u)\Delta u\right\}\Delta s = \int_t^{t^*} p(s)F(s)\Delta s \le \frac{1}{2}F^2\big(t^*\big) \\ = \frac{1}{2}\left(\int_t^{t^*} p(u)\Delta u\right)^2 \le \frac{(1-\rho)^2}{2}, \tag{3.19}$$

and so we obtain

$$\begin{aligned}&\int_t^{t^*} p(s)\Delta s\int_t^s p(u)\Delta u + \int_{t^*}^{\sigma(t^*)} p(s)\Delta s\int_t^s p(u)\Delta u \\ &\quad\le \frac{(1-\rho)^2}{2} + \mu\big(t^*\big)p\big(t^*\big)\int_t^{t^*} p(u)\Delta u \\ &\quad\le \frac{(1-\rho)^2}{2} + (1-\rho)M.\end{aligned} \tag{3.20}$$

Note $\sigma(t^*) \in [t, \tau^{-1}(t)]_{\mathbb{T}}$, $t \in [\tau^{-3}(t_0), \tau^{-(N-1)}(t_0)]_{\mathbb{T}}$, and x is positive on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$ (so $x(\sigma(t^*)) > 0$). Thus from (3.18) and (3.20), we obtain

$$\begin{aligned}x(t) &\ge x\big(\sigma\big(t^*\big)\big) + (1-\rho)x(t) \\ &\quad + (1-\rho)^2 x\big(\tau(t)\big) - \left[\frac{(1-\rho)^2}{2} + (1-\rho)M\right]x\big(\tau(t)\big) \\ &= x\big(\sigma\big(t^*\big)\big) + (1-\rho)x(t) \\ &\quad + \left[\frac{(1-\rho)^2}{2} - (1-\rho)M\right]x\big(\tau(t)\big),\end{aligned} \tag{3.21}$$

and so we have

$$\frac{x(\tau(t))}{x(t)} < \frac{2\rho}{(1-\rho)^2 - 2M(1-\rho)} = g_1(\rho), \quad \text{for } t \in \big[\tau^{-3}(t_0), \tau^{-(N-1)}(t_0)\big]_{\mathbb{T}}. \tag{3.22}$$

Fix $t \in [\tau^{-3}(t_0), \tau^{-(N-2)}(t_0)]_{\mathbb{T}}$ and with t^* as described above we have $t \le \sigma(t^*) \le \tau^{-1}(t) \le \tau^{-(N-1)}(t_0)$, so from (3.22) we have

$$x\big(\sigma\big(t^*\big)\big) > \frac{1}{g_1(\rho)}x\big(\tau\big(\sigma\big(t^*\big)\big)\big),$$

and since x is nonincreasing on $[\tau^{-1}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}$ and $\tau(\sigma(t^*)) \le t \le \tau^{-(N-1)}(t_0)$ we have

$$x\big(\sigma\big(t^*\big)\big) > \frac{1}{g_1(\rho)}x\big(\tau\big(\sigma\big(t^*\big)\big)\big) \ge \frac{1}{g_1(\rho)}x(t). \tag{3.23}$$

Substituting (3.23) into (3.21), we obtain for $t \in [\tau^{-3}(t_0), \tau^{-(N-2)}(t_0)]_{\mathbb{T}}$ that

$$x(t) > \frac{1}{g_1(\rho)}x(t) + (1-\rho)x(t) + \left[\frac{(1-\rho)^2}{2} - (1-\rho)M\right]x\big(\tau(t)\big),$$

and so we have

$$\frac{x(\tau(t))}{x(t)} < \frac{2(\rho - \frac{1}{g_1(\rho)})}{(1-\rho)^2 - 2M(1-\rho)} := g_2(\rho).$$

Repeating the above procedure, we obtain for $t \in [\tau^{-3}(t_0), \tau^{-(N-m)}(t_0)]_{\mathbb{T}}$

$$\frac{x(\tau(t))}{x(t)} < \frac{2(\rho - \frac{1}{g_{m-1}(\rho)})}{(1-\rho)^2 - 2M(1-\rho)} := g_m(\rho).$$

The proof is complete. □

4 Distributions of zeros of solutions

In this section, we study the distribution of zeros of solutions of (1.5) using the lower and upper bounds for $x(\tau(t))/x(t)$ in Section 3.

Theorem 4.1 *Assume that $\mathbb{T}$ is a time scale and $t', t_0 \in \mathbb{T}$, $t_0 \geq t'$, $x(t)$ is a solution of* (1.5) *on $[t', \infty)_{\mathbb{T}}$, and there exist $\rho \in (0,1)$ and $n_0, m_0 \in \{1, 2, \ldots\}$ with $f_{n_0}(\rho) \geq g_{m_0}(\rho)$, and with*

$$N = 2 + \min_{n \geq 1, m \geq 1} \{n + m : f_n(\rho) \geq g_m(\rho)\} = 2 + n^\star + m^\star$$

assume $\infty > f_k(\rho) > 0$, $g_k(\rho) > 0$ for $n \in \{2, 3, \ldots, N-3\}$ and

$$\sup_{\lambda \in E} \left\{ \lambda \exp\left\{ \int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-\lambda p(s)\bigr) \Delta s \right\} \right\} \leq \rho \quad \text{for } t \in \bigl[\tau^{-3}(t_0), \tau^{-N}(t_0)\bigr]_{\mathbb{T}},$$

where $E = \{\lambda : \lambda > 0, 1 - \lambda p(t)\mu(t) > 0$ for $t \in [\tau^{-2}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}\}$ and

$$M = \sup_{s \in [\tau^{-3}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}} p(s)\mu(s) < \frac{1-\rho}{2}.$$

Then every solution of (1.5) *cannot be totally positive or totally negative on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$.*

Proof Note

$$f_{n^\star}(\rho) \geq g_{m^\star}(\rho). \tag{4.1}$$

Without loss of generality assume x is positive on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$. From Theorem 3.1 we have

$$\frac{x(\tau(t))}{x(t)} \geq f_{n^\star}(\rho), \quad \text{for } t \in \bigl[\tau^{-(2+n^\star)}(t_0), \tau^{-N}(t_0)\bigr]_{\mathbb{T}}$$

and from Theorem 3.2 we have (note $m^\star = N - (2 + n^\star) \leq N - 3$)

$$\frac{x(\tau(t))}{x(t)} < g_{m^\star}(\rho), \quad \text{for } t \in \bigl[\tau^{-3}(t_0), \tau^{-(N-m^\star)}(t_0)\bigr]_{\mathbb{T}}.$$

Note since $N = 2 + n^\star + m^\star$ we have (take $t = \tau^{-(N-m^\star)}(t_0)$)

$$f_{n^\star}(\rho) \leq \frac{x(\tau^{-(1+n^\star)}(t_0))}{x(\tau^{-(2+n^\star)}(t_0))} < g_{m^\star}(\rho),$$

which contradicts (4.1). The proof is complete. □

Theorem 4.2 *Assume that $\mathbb{T}$ is a time scale and $t', t_0 \in \mathbb{T}$, $t_0 \geq t'$, $x(t)$ is a solution of* (1.5) *on $[t',\infty)_{\mathbb{T}}$, and there exist $\rho \in (0,1)$ and a positive integer $N \geq 4$ and $m_0 \in \{1,2,\dots,N-3\}$ with*

$$\int_{\tau(t_{m_0})}^{t_{m_0}} p(s)\Delta s > 1 - \frac{1}{g_{m_0}(\rho)} \quad \textit{where } t_{m_0} = \tau^{-(N-m_0)}(t_0),$$

and with

$$m^\star = \min_{m\in\{1,\dots,N-3\}} \left\{ m : \int_{\tau(t_m)}^{t_m} p(s)\Delta s > 1 - \frac{1}{g_m(\rho)} \right\} \quad \textit{where } t_m = \tau^{-(N-m)}(t_0)$$

assume $\infty > f_k(\rho) > 0$, $g_k(\rho) > 0$ for $n \in \{2,3,\dots,N-3\}$ and

$$\sup_{\lambda\in E} \left\{ \lambda \exp\left\{ \int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-\lambda p(s)\bigr)\Delta s \right\} \right\} \leq \rho \quad \textit{for } t \in \left[\tau^{-3}(t_0), \tau^{-N}(t_0)\right]_{\mathbb{T}},$$

where $E = \{\lambda : \lambda > 0, 1 - \lambda p(t)\mu(t) > 0$ for $t \in [\tau^{-2}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}\}$ and

$$M = \sup_{s\in[\tau^{-3}(t_0),\tau^{-N}(t_0)]_{\mathbb{T}}} p(s)\mu(s) < \frac{1-\rho}{2}.$$

Then every solution of (1.5) *cannot be totally positive or totally negative on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$.*

Proof Note

$$\int_{\tau(t_{m^\star})}^{t_{m^\star}} p(s)\Delta s > 1 - \frac{1}{g_{m^\star}(\rho)} \quad \text{where } t_{m^\star} = \tau^{-(N-m^\star)}(t_0). \tag{4.2}$$

Without loss of generality assume x is positive on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$. From Theorem 3.2, we have

$$\frac{x(\tau(t))}{x(t)} < g_{m^\star}(\rho), \quad \text{for } t \in \left[\tau^{-3}(t_0), \tau^{-(N-m^\star)}(t_0)\right]_{\mathbb{T}},$$

so in particular

$$\frac{x(\tau(t_{m^\star}))}{x(t_{m^\star})} < g_{m^\star}(\rho). \tag{4.3}$$

Integrating (1.5) from $\tau(t_{m^\star})$ to $t_{m^\star}$, we obtain

$$x\bigl(\tau(t_{m^\star})\bigr) - x(t_{m^\star}) = \int_{\tau(t_{m^\star})}^{t_{m^\star}} p(s)x\bigl(\tau(s)\bigr)\Delta s \geq x\bigl(\tau(t_{m^\star})\bigr) \int_{\tau(t_{m^\star})}^{t_{m^\star}} p(s)\Delta s,$$

and this together with (4.3) gives

$$\int_{\tau(t_{m^\star})}^{t_{m^\star}} p(s)\Delta s \leq 1 - \frac{x(t_{m^\star})}{x(\tau(t_{m^\star}))} \leq 1 - \frac{1}{g_{m^\star}(\rho)},$$

which contradicts (4.2). The proof is complete. □

Theorem 4.3 *Assume that $\mathbb{T}$ is a time scale and $t', t_0 \in \mathbb{T}$, $t_0 \ge t'$, $x(t)$ is a solution of* (1.5) *on $[t', \infty)_{\mathbb{T}}$, and there exist $\rho \in (0,1)$, a constant L and $n_0, m_0 \in \{1, 2, \dots\}$ with*

$$\frac{1 + \ln f_{n_0-1}(\rho)}{f_{n_0-1}(\rho)} - \frac{1}{g_{m_0}(\rho)} < L$$

and with

$$N = 2 + \min_{n\ge1, m\ge1} \left\{ n + m : L > \left(\frac{1 + \ln f_{n-1}(\rho)}{f_{n-1}(\rho)} - \frac{1}{g_m(\rho)} \right) \right\} = 2 + n^\star + m^\star,$$

assume $\infty > f_k(\rho) > 0$, $g_k(\rho) > 0$ for $n \in \{2, 3, \dots, N-3\}$ and

$$\sup_{\lambda \in E} \left\{ \lambda \exp\left\{ \int_{\tau(t)}^{t} \zeta_{\mu(s)}\bigl(-\lambda p(s)\bigr) \Delta s \right\} \right\} \le \rho \quad \text{for } t \in \left[\tau^{-3}(t_0), \tau^{-N}(t_0)\right]_{\mathbb{T}},$$

where $E = \{\lambda : \lambda > 0, 1 - \lambda p(t)\mu(t) > 0 \text{ for } t \in [\tau^{-2}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}\}$ and

$$M = \sup_{s \in [\tau^{-3}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}} p(s)\mu(s) < \frac{1-\rho}{2}.$$

Suppose $f_{n^\star-1}(\rho) \ge 1$, $f_{n^*}(\rho) > f_{n^*-1}(\rho)$ and for $t^* \in [\tau(t_1), t_1]_{\mathbb{T}}$ (here $t_1 = \tau^{-(N-m^\star)}(t_0)$) that

$$\int_{\tau(t_1)}^{t^*} p(s)\Delta s + \int_{\sigma(t^*)}^{t_1} p(s)\Delta s \ge L. \tag{4.4}$$

Then every solution of (1.5) *cannot be totally positive or totally negative on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$.*

Proof Note

$$L > \left(\frac{1 + \ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)} - \frac{1}{g_{m^*}(\rho)} \right). \tag{4.5}$$

Without loss of generality assume x is positive on $[t_0, \tau^{-N}(t_0)]_{\mathbb{T}}$. From Theorem 3.1, we have

$$\frac{x(\tau(t))}{x(t)} \ge f_{n^*}(\rho), \quad t \in \left[\tau^{-(2+n^\star)}(t_0), \tau^{-N}(t_0)\right]_{\mathbb{T}}, \tag{4.6}$$

$$\frac{x(\tau(t))}{x(t)} \ge f_{n^*-1}(\rho), \quad t \in \left[\tau^{-(1+n^\star)}(t_0), \tau^{-N}(t_0)\right]_{\mathbb{T}}, \tag{4.7}$$

and from Theorem 3.2, we have

$$\frac{x(\tau(t))}{x(t)} < g_{m^*}(\rho), \quad t \in \left[\tau^{-3}(t_0), \tau^{-(N-m^*)}(t_0)\right]_{\mathbb{T}},$$

so in particular (with $t_1 = \tau^{-(N-m^\star)}(t_0) = \tau^{-(2+n^\star)}(t_0)$) we have

$$\frac{x(\tau(t_1))}{x(t_1)} < g_{m^*}(\rho). \tag{4.8}$$

From (4.6) and $f_{n^*}(\rho) > f_{n^*-1}(\rho)$ we have

$$\frac{x(\tau(t_1))}{x(t_1)} > f_{n^*-1}(\rho).$$

Now since x is nonincreasing on $[\tau^{-1}(t_0), \tau^{-N}(t_0)]_{\mathbb{T}}$ and $f_{n^\star-1}(\rho) \geq 1$ (and trivially note $\frac{x(\tau(t_1))}{x(\tau(t_1))} = 1$) there exists a $t^* \in [\tau(t_1), t_1]_{\mathbb{T}}$ with

$$\frac{x(\tau(t_1))}{x(t^*)} \leq f_{n^*-1}(\rho) \quad \text{and} \quad \frac{x(\tau(t_1))}{x(\sigma(t^*))} \geq f_{n^*-1}(\rho). \tag{4.9}$$

Integrating (1.5) from $\sigma(t^*)$ to t_1, we obtain

$$x(\sigma(t^*)) - x(t_1) = \int_{\sigma(t^*)}^{t_1} p(s)x(\tau(s))\Delta s \geq x(\tau(t_1)) \int_{\sigma(t^*)}^{t_1} p(s)\Delta s,$$

which implies

$$\int_{\sigma(t^*)}^{t_1} p(s)\Delta s \leq \left(\frac{x(\sigma(t^*))}{x(\tau(t_1))} - \frac{x(t_1)}{x(\tau(t_1))}\right). \tag{4.10}$$

From (4.8), (4.9) and (4.10), we obtain

$$\int_{\sigma(t^*)}^{t_1} p(s)\Delta s \leq \left(\frac{1}{f_{n^*-1}(\rho)} - \frac{1}{g_{m^*}(\rho)}\right). \tag{4.11}$$

Divide (1.5) by x and integrate from $\tau(t_1)$ to t^*, and we get

$$\int_{\tau(t_1)}^{t^*} \frac{x^{\Delta}(s)}{x(s)}\Delta s = -\int_{\tau(t_1)}^{t^*} p(s)\frac{x(\tau(s))}{x(s)}\Delta s \leq -f_{n^*-1}(\rho)\int_{\tau(t_1)}^{t^*} p(s)\Delta s,$$

which implies

$$\int_{\tau(t_1)}^{t^*} p(s)\Delta s \leq -\frac{1}{f_{n^*-1}(\rho)}\int_{\tau(t_1)}^{t^*} \frac{x^{\Delta}(s)}{x(s)}\Delta s. \tag{4.12}$$

From (4.9), (4.12) and Lemma 2.2, we obtain

$$\begin{aligned}\int_{\tau(t_1)}^{t^*} p(s)\Delta s &\leq -\frac{1}{f_{n^*-1}(\rho)}\int_{\tau(t_1)}^{t^*} \left[\ln x(s)\right]^{\Delta}\Delta s = \frac{1}{f_{n^*-1}(\rho)}\ln\left(\frac{x(\tau(t_1))}{x(t^*)}\right) \\ &\leq \frac{\ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)},\end{aligned} \tag{4.13}$$

and from (4.5), (4.11) and (4.13) we have

$$\int_{\tau(t_1)}^{t^*} p(s)\Delta s + \int_{\sigma(t^*)}^{t_1} p(s)\Delta s \leq \left(\frac{1+\ln f_{n^*-1}(\rho)}{f_{n^*-1}(\rho)} - \frac{1}{g_{m^*}(\rho)}\right) < L,$$

which contradicts (4.4). The proof is complete. □

Remark 4.1 When $\mathbb{T} = \mathbb{R}$ equation (1.5) is the delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{R}.$$

Theorem 3.1 and Theorem 3.2 are related to the results in [9], Lemma 2.1 and Lemma 2.2, and Theorem 4.3 is motivated from results in [13], Theorem 3.

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The authors declare that they have no competing interests.

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