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Boundary value problems for a coupled system of second-order nonlinear difference equations

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Abstract
We discuss the existence of nontrivial solutions to the boundary value problems for a coupled system of second-order nonlinear difference equations by using the critical point theory. The nontrivial solutions where neither of the components is identically zero are achieved under some sufficient conditions.

Keywords: boundary value problem; coupled system; nonlinear difference equation; critical point theory

1 Introduction

Put $\mathbb{N}^+$, $\mathbb{Z}$, $\mathbb{R}$ to be the sets of positive integers, integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a,b) = \{a, a+1, \ldots, b\}$ when $a \leq b$.

In this paper, we study the following boundary value problems for a coupled system of second-order nonlinear difference equations:

$$\begin{cases} -\Delta^2\phi(k-1) = \omega_1\phi(k) - b_{1k}\phi(k) + a_1\phi^3(k) + a_3\psi^2(k)\phi(k), \\ -\Delta^2\psi(k-1) = \omega_2\psi(k) - b_{2k}\psi(k) + a_2\psi^3(k) + a_3\phi^2(k)\psi(k), \\ \phi(0) = \phi(N+1) = \psi(0) = \psi(N+1) = 0, \end{cases} \tag{1.1}$$

for all $k \in \mathbb{Z}(1,N)$, where $N \in \mathbb{N}^+$, ω_1, $\omega_2 \in \mathbb{R}$, $a_i \in \mathbb{R} \setminus \{0\}, i = 1,2,3$, and $\{b_{jk}\}$ is a real number sequence, $j = 1,2$. Δ is the forward difference operator defined by $\Delta\phi(k) = \phi(k+1) - \phi(k)$, $\Delta^2\phi(k) = \Delta(\Delta\phi(k))$.

Boundary value problems for a coupled system of nonlinear differential equations have been the subject of many investigations [1–7]. Among the main methods are the Schauder fixed point theorem, the Banach fixed point theorem, the synchronization manifold approach and the coincidence degree theory. However, the results on boundary value problems for a coupled system of second-order nonlinear difference equations are relatively rare. The critical point theory is a strong tool to study the periodic solutions [8–10], the homoclinic solutions [11, 12] and the boundary value problems [13, 14] for difference equations. Recently, Bonanno et al. [15] developed a new approach to discuss the boundary value problems for a second-order difference equation by using critical point theory. Motivated by [15], we try to use the variants of the mountain pass theorem, the local minimum theorem and its variant to study system (1.1).

System (1.1) arises from the discretization of the two-component system of time-dependent nonlinear Gross-Pitaevskii system (see [16] for more detail) as

$$\begin{cases} i\frac{du_k}{dt} = -\Delta^2 u_{k-1} + b_{1k}u_k - a_1|u_k|^2 u_k - a_3|v_k|^2 u_k, \\ i\frac{dv_k}{dt} = -\Delta^2 v_{k-1} + b_{2k}v_k - a_2|v_k|^2 v_k - a_3|u_k|^2 v_k, \end{cases} \tag{1.2}$$

where $k \in \mathbb{N}^+$. For the general background on a coupled discrete Schrödinger system, we refer to [17].

Huang and Zhou [18, 19] considered system (1.2), with a few differences in notation, and they studied the solutions to system (1.2) of the form

$$u_k = e^{-i\omega_1 t}\phi(k), \qquad v_k = e^{-i\omega_2 t}\psi(k), \tag{1.3}$$

where $k \in \mathbb{N}^+$.

By substituting (1.3) into (1.2), a routine calculation gives

$$\begin{cases} -\Delta^2\phi(k-1) = \omega_1\phi(k) - b_{1k}\phi(k) + a_1\phi^3(k) + a_3\psi^2(k)\phi(k), \\ -\Delta^2\psi(k-1) = \omega_2\psi(k) - b_{2k}\psi(k) + a_2\psi^3(k) + a_3\phi^2(k)\psi(k), \end{cases} \tag{1.4}$$

where $k \in \mathbb{N}^+$. Using the Nehari manifold approach, Huang and Zhou have established sufficient conditions on the existence of nontrivial homoclinic solutions that have both of the components not identically zero to system (1.4), in [18] and [19]. However, we find that no similar results were obtained in the literature for system (1.1). We noticed that most works on the existence of solutions to the boundary value problems are for difference equations, and less is known for coupled difference systems. If we let $\phi(k) \equiv 0$ or $\psi(k) \equiv 0$ in system (1.1), we get a boundary value problem for a single difference equation. Therefore, it is important that we get solutions of (1.1) where both of the components are not zero. Throughout this paper, we will show that the critical point theory is very useful to demonstrate the existence of solutions of system (1.1).

An outline of this paper is as follows. In Section 2, we establish the variational framework and introduce two variants of the mountain pass theorem, the local minimum theorem and its variant. Some notations will also be given. Then, in Section 3, the coerciveness and compactness of the variational functional are given by different assumptions of the coefficients. A two critical points theorem and a three critical points theorem are stated in Section 4. The existence of nontrivial solutions with both of the components being not zero to system (1.1) are then established in Section 5. Finally, as an application, an example is presented in Section 6.

2 Preliminaries

In this section, we define some notations and give the variational functional of system (1.1). Some related fundamental theorems are presented at the end of this section.

Let S be the N-dimensional Banach space as follows:

$$S := \{\phi : \mathbb{Z}(0, N+1) \to \mathbb{R} \mid \phi(0) = \phi(N+1) = 0\},$$

which is equipped with the norm

$$\|\phi\|_2 := \left(\sum_{k=1}^{N}\phi(k)^2\right)^{\frac{1}{2}}, \quad \forall\phi \in S.$$

Define $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|(\cdot,\cdot)\|_\infty$ as

$$\|\phi\| = \left(\sum_{k=1}^{N+1}\bigl(\Delta\phi(k-1)\bigr)^2\right)^{\frac{1}{2}}, \quad \forall\phi \in S,$$

$$\|\phi\|_\infty = \max_{k\in\mathbb{Z}(1,N)}\bigl|\phi(k)\bigr|, \quad \forall\phi \in S$$

and

$$\bigl\|(\phi,\psi)\bigr\|_\infty = \max\bigl\{\|\phi\|_\infty, \|\psi\|_\infty\bigr\}, \quad \forall(\phi,\psi) \in S \times S,$$

respectively.

Define a linear map $L : S \to \mathbb{R}^N$ by

$$L\phi = \bigl(\phi(1), \phi(2), \ldots, \phi(N)\bigr)^T,$$

where $\cdot^T$ denotes the transpose of $\cdot$.

Clearly,

$$\|\phi\|^2 = (L\phi)^T C(L\phi), \quad \forall\phi \in S,$$

where

$$C := \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}_{N\times N}.$$

The eigenvalues of matrix C (see, for instance, [15], Section 2, and [20], Section 2) are

$$\lambda_k := 4\sin^2\frac{k\pi}{2(N+1)}, \quad k \in \mathbb{Z}(1,N),$$

one has

$$\sqrt{\lambda_1}\|\phi\|_2 \le \|\phi\| \le \sqrt{\lambda_N}\|\phi\|_2, \quad \forall\phi \in S$$

and

$$\frac{\|\phi\|}{\sqrt{N\lambda_N}} \le \|\phi\|_\infty, \quad \forall\phi \in S.$$

From [15], Proposition 2.1, we know that

$$\|\phi\|_\infty \le K_2\|\phi\|, \quad \forall \phi \in S,$$

where

$$K_2 = \begin{cases} \frac{\sqrt{N+1}}{2}, & \text{if } N \text{ is odd}, \\ (\frac{2}{N} + \frac{2}{N+2})^{-\frac{1}{2}}, & \text{if } N \text{ is even}. \end{cases}$$

Let

$$f_{1k}(x,y) := \omega_1 x - b_{1k}x + a_1x^3 + a_3y^2x,$$
$$f_{2k}(x,y) := \omega_2 y - b_{2k}y + a_2y^3 + a_3x^2y,$$
$$F_k(x,y) := \frac{1}{2}(\omega_1 - b_{1k})x^2 + \frac{1}{2}(\omega_2 - b_{2k})y^2 + \frac{1}{4}a_1x^4 + \frac{1}{4}a_2y^4 + \frac{1}{2}a_3x^2y^2,$$

for all $x, y \in \mathbb{R}$ and $k \in \mathbb{Z}(1,N)$, obviously,

$$\frac{\partial F_k(x,y)}{\partial x} = f_{1k}(x,y), \qquad \frac{\partial F_k(x,y)}{\partial y} = f_{2k}(x,y).$$

The functional $I(\phi,\psi)$ is defined by

$$I(\phi,\psi) = A(\phi,\psi) - B(\phi,\psi),$$

where

$$A(\phi,\psi) := \frac{1}{2}\sum_{k=1}^{N+1}\big(\Delta\phi(k-1)\big)^2 + \frac{1}{2}\sum_{k=1}^{N+1}\big(\Delta\psi(k-1)\big)^2,$$
$$B(\phi,\psi) := \sum_{k=1}^{N} F_k\big(\phi(k),\psi(k)\big),$$

for all $(\phi,\psi) \in S \times S$.

A standard argument gives that the critical points of $I(\phi,\psi)$ are the solutions of system (1.1).

The following definition and lemma are taken from [21, 22].

Definition 2.1 The Gâteaux differentiable function I satisfies the Palais-Smale condition (PS) if every sequence $\{x_j\}$ such that $I(x_j)$ is bounded and $I'(x_j) \to 0$ for $j \to \infty$ contains a convergent subsequence.

Lemma 2.1 *Let E be a finite dimensional Banach space. Suppose that $I : E \to \mathbb{R}$ is lower semicontinuous and coercive. Then I admits a global minimum.*

We state two variants of the mountain pass theorem (see [23], Theorem 2.2) due to [15], Corollary 3.2, and [24], Theorem 1.1, Chapter II.

Lemma 2.2 *Let E be a real finite dimensional Banach space. Suppose that $I : E \to \mathbb{R}$ is continuously Gâteaux differentiable, unbounded from below and satisfies* (PS). *Further suppose that I possesses a local minimum x_1. Then I possesses a distinct second critical point.*

Lemma 2.3 *Let E be a real finite dimensional Banach space. Suppose that $I : E \to \mathbb{R}$ is continuously Gâteaux differentiable and coercive. Further suppose that I possesses two distinct local minima x_1 and x_2. Then I possesses a third critical point x_3 which is distinct from x_1 and x_2.*

The local minimum theorem and its variant are presented below.

Lemma 2.4 (Local minimum theorem) *Put $r > 0$ such that*

$$\frac{\sup_{A^{-1}([0,r])} B(\phi, \psi)}{r} < 1.$$

Then the functional $I(\phi, \psi) = A(\phi, \psi) - B(\phi, \psi)$ has at least a local minimum $(\phi^*, \psi^*) \in S \times S$ such that $A(\phi^*, \psi^*) < r$, $I(\phi^*, \psi^*) \le I(\phi, \psi)$ for all $(\phi, \psi) \in A^{-1}([0, r])$ and $I'(\phi^*, \psi^*) = 0$.

Proof First, we deal with the case in which $\sup_{A^{-1}([0,r])} B(\phi, \psi) = 0$. We find $0 \le A(\phi, \psi) = A(\phi, \psi) - \sup_{A^{-1}([0,r])} B(u, v) \le A(\phi, \psi) - B(\phi, \psi) = I(\phi, \psi)$ for all $(\phi, \psi) \in A^{-1}([0, r])$. It is easy to show that $(\phi^*, \psi^*) = (0, 0)$ satisfies our conclusion. Now, suppose $\sup_{A^{-1}([0,r])} B(\phi, \psi) > 0$. Remind that $A(\phi, \psi)$ is continuous and coercive, therefore the set $A^{-1}([0, r])$ is closed and bounded. Hence, there exists $(\phi^*, \psi^*) \in A^{-1}([0, r])$ such that

$$I\bigl(\phi^*, \psi^*\bigr) = \min_{A^{-1}([0,r])} I(\phi, \psi).$$

Lemma 2.4 will be proved if we can show that $(\phi^*, \psi^*) \in A^{-1}([0, r[)$. For the sake of contradiction, suppose $A(\phi^*, \psi^*) = r$. Using $\frac{\sup_{A^{-1}([0,r])} B(\phi,\psi)}{r} < 1$, we obtain $\frac{B(\phi^*,\psi^*)}{r} < 1$, that is, $\frac{B(\phi^*,\psi^*)}{A(\phi^*,\psi^*)} < 1$, so $I(0, 0) = 0 < I(\phi^*, \psi^*)$ and this leads to a contradiction. Hence, $(\phi^*, \psi^*) \in A^{-1}([0, r[)$. This completes the proof of Lemma 2.4. □

Lemma 2.5 *Suppose that there exist $r > 0$ and $(w, w) \in S \times S$, with $0 < A(w, w) < r$, such that*

$$\frac{\sup_{A^{-1}([0,r])} B(\phi, \psi)}{r} < 1 < \frac{B(w, w)}{A(w, w)}.$$

Then the functional $I(\phi, \psi) = A(\phi, \psi) - B(\phi, \psi)$ has at least a local minimum $(\phi^*, \psi^*) \in S \times S$ such that $(\phi^*, \psi^*) \neq (0, 0)$, $A(\phi^*, \psi^*) < r$, $I(\phi^*, \psi^*) \le I(\phi, \psi)$ for all $(\phi, \psi) \in A^{-1}([0, r])$ and $I'(\phi^*, \psi^*) = 0$.

Proof As stated in Lemma 2.4, we obtain that $I(\phi^*, \psi^*) = \min_{A^{-1}([0,r])} I(\phi, \psi)$ holds. If $A(\phi^*, \psi^*) = 0$, then $(\phi^*, \psi^*) = (0, 0)$. It follows from $1 < \frac{B(w,w)}{A(w,w)}$ that $I(w, w) < 0 = I(\phi^*, \psi^*)$, which leads to a contradiction. If $A(\phi^*, \psi^*) = r$, from $\frac{\sup_{A^{-1}([0,r])} B(\phi,\psi)}{r} < 1$, we have $\frac{B(\phi^*,\psi^*)}{r} < 1$, that is, $\frac{B(\phi^*,\psi^*)}{A(\phi^*,\psi^*)} < 1$, so $I(0, 0) = 0 < I(\phi^*, \psi^*)$. Again, this leads to a contradiction. Hence, $(\phi^*, \psi^*) \in A^{-1}(]0, r[)$. Lemma 2.5 is proved. □

3 Compactness and coerciveness of the variational functional

We point out the following two lemmas which will be used in the next section.

Lemma 3.1 *Assume that the following condition holds.*

(J_1) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ *or* $a_3^2 < a_1 a_2$ *when* $a_3 < 0$.

Then $I(\phi, \psi)$ *satisfies the* (PS) *condition and it is unbounded from below.*

Proof Taking into consideration that (J_1) holds, we have

$$\lim_{x^2+y^2 \to +\infty} \frac{F_k(x,y)}{x^2+y^2} = +\infty$$

for all $k \in \mathbb{Z}(1,N)$. There exist $\alpha > \frac{1}{2}\lambda_N$ and $\beta > 0$ such that $F_k(x,y) \geq \alpha(x^2+y^2) - \beta$ for all $k \in \mathbb{Z}(1,N)$. Let $\{(\phi_n, \psi_n)\}$ be a (PS) sequence such that $|I(\phi_n, \psi_n)| \leq M$, where $M > 0$.

First, we need to verify that $\{(\phi_n, \psi_n)\}$ is bounded. We have

$$\begin{aligned}
-M &\leq I(\phi_n, \psi_n) \\
&= \frac{1}{2}\|\phi_n\|^2 + \frac{1}{2}\|\psi_n\|^2 - \sum_{k=1}^{N} F_k\bigl(\phi_n(k), \psi_n(k)\bigr) \\
&\leq \frac{1}{2}\lambda_N \|\phi_n\|_2^2 + \frac{1}{2}\lambda_N \|\psi_n\|_2^2 - \alpha \sum_{k=1}^{N} \bigl(\phi_n^2(k) + \psi_n^2(k)\bigr) + N\beta \\
&= \frac{1}{2}(\lambda_N - 2\alpha)\bigl(\|\phi_n\|_2^2 + \|\psi_n\|_2^2\bigr) + N\beta \\
&\leq \frac{1}{2}(\lambda_N - 2\alpha)\frac{1}{K_2^2\lambda_N}\bigl(\|\phi_n\|_\infty^2 + \|\psi_n\|_\infty^2\bigr) + N\beta \\
&\leq \frac{\lambda_N - 2\alpha}{2K_2^2\lambda_N}\bigl\|(\phi_n, \psi_n)\bigr\|_\infty^2 + N\beta.
\end{aligned}$$

We obtain $\frac{2\alpha-\lambda_N}{2K_2^2\lambda_N}\|(\phi_n,\psi_n)\|_\infty^2 \leq M + N\beta$. Notice that $2\alpha - \lambda_N > 0$, hence, $\{(\phi_n, \psi_n)\}$ is bounded and admits a convergent subsequence.

The next thing to do in the proof is to verify that $I(\phi, \psi)$ is unbounded from below. From the above discussion, one has $I(\phi,\psi) \leq \frac{\lambda_N - 2\alpha}{2K_2^2\lambda_N}\|(\phi,\psi)\|_\infty^2 + N\beta$. Since $\lambda_N - 2\alpha < 0$, one has

$$\lim_{\|(\phi,\psi)\|_\infty \to +\infty} I(\phi, \psi) = -\infty.$$

This completes the proof of Lemma 3.1. □

Lemma 3.2 *Assume that the following condition holds.*

(J_2) $a_1 < 0$, $a_2 < 0$, $a_3 < 0$ *or* $a_3^2 < a_1 a_2$ *when* $a_3 > 0$.

Then $I(\phi, \psi)$ *is coercive.*

Proof Notice that (J_2) holds, one has

$$\lim_{x^2+y^2 \to +\infty} \frac{F_k(x,y)}{x^2+y^2} = -\infty$$

for all $k \in \mathbb{Z}(1,N)$. There exist $\alpha < 0$ and $\beta > 0$ such that $F_k(x,y) \leq \alpha(x^2 + y^2) + \beta$ for all $k \in \mathbb{Z}(1,N)$. Then

$$\begin{aligned} I(\phi,\psi) &= \frac{1}{2}\|\phi\|^2 + \frac{1}{2}\|\psi\|^2 - \sum_{k=1}^{N} F_k\big(\phi(k),\psi(k)\big) \\ &\geq \frac{1}{2}\lambda_1\|\phi\|_2^2 + \frac{1}{2}\lambda_1\|\psi\|_2^2 - \alpha\sum_{k=1}^{N}\big(\phi^2(k)+\psi^2(k)\big) - N\beta \\ &= \frac{1}{2}(\lambda_1 - 2\alpha)\big(\|\phi\|_2^2 + \|\psi\|_2^2\big) - N\beta \\ &\geq \frac{1}{2}(\lambda_1 - 2\alpha)\frac{1}{K_2^2\lambda_N}\big(\|\phi\|_\infty^2 + \|\psi\|_\infty^2\big) - N\beta \\ &\geq \frac{\lambda_1 - 2\alpha}{2K_2^2\lambda_N}\big\|(\phi,\psi)\big\|_\infty^2 - N\beta. \end{aligned}$$

Since $\lambda_1 - 2\alpha > 0$, we have

$$\lim_{\|(\phi,\psi)\|_\infty \to +\infty} I(\phi,\psi) = +\infty.$$

Therefore, $I(\phi,\psi)$ is coercive. □

4 Multiple critical points theorems

Two consequences of the local minimum theorem were discussed in [15] (see [15], Section 4). In this section, motivated by [15], we state two consequences of Lemma 2.4 as follows. The first one is a two critical points theorem and the second one is a three critical points theorem.

Theorem 4.1 *Assume that* (J_1) *holds. Moreover, there exists* $r > 0$ *such that*

$$\frac{\sup_{A^{-1}([0,r])} B(\phi,\psi)}{r} < 1.$$

Then $I(\phi,\psi)$ *has at least two distinct critical points.*

Proof Taking into consideration that (J_1) holds, from Lemma 3.1 we obtain that $I(\phi,\psi)$ satisfies the (PS) condition and it is unbounded from below. Notice that $\frac{\sup_{A^{-1}([0,r])} B(\phi,\psi)}{r} < 1$, it follows from Lemma 2.4 that $I(\phi,\psi)$ admits a local minimum. $I(\phi,\psi)$ satisfies the conditions in Lemma 2.2, then $I(\phi,\psi)$ admits a distinct second critical point. Hence, the proof is completed. □

Theorem 4.2 *Assume that* (J_2) *holds. Furthermore, there exist* $r > 0$ *and* $(w,w) \in S \times S$, *with* $r < A(w,w)$, *such that*

$$\frac{\sup_{A^{-1}([0,r])} B(\phi,\psi)}{r} < 1 < \frac{B(w,w)}{A(w,w)}.$$

Then the functional $I(\phi,\psi) = A(\phi,\psi) - B(\phi,\psi)$ *has at least three distinct critical points.*

Proof Taking into account that (J_2) holds, we know that $I(\phi,\psi)$ is coercive from Lemma 3.2. Since

$$\frac{\sup_{A^{-1}([0,r])} B(\phi,\psi)}{r} < 1,$$

from Lemma 2.4, $I(\phi,\psi)$ admits a local minimum (ϕ_1,ψ_1) such that $A(\phi_1,\psi_1) < r$.

Let

$$A^r(\phi,\psi) = \begin{cases} r, & \text{if } A(\phi,\psi) \le r, \\ A(\phi,\psi), & \text{if } A(\phi,\psi) > r. \end{cases} \tag{4.1}$$

Apparently, $I^r(\phi,\psi) = A^r(\phi,\psi) - B(\phi,\psi)$ is continuous. Since $A^r(\phi,\psi) - B(\phi,\psi) \ge A(\phi,\psi) - B(\phi,\psi)$ for all $(\phi,\psi) \in S \times S$, then $I^r(\phi,\psi)$ is coercive.

It follows from Lemma 2.1 that $I^r(\phi,\psi)$ has a global minimum (ϕ_2,ψ_2), that is to say,

$$A^r(\phi_2,\psi_2) - B(\phi_2,\psi_2) \le A^r(\phi,\psi) - B(\phi,\psi), \quad \forall(\phi,\psi) \in S \times S.$$

We assert that $A(\phi_2,\psi_2) > r$. If otherwise, then $A(\phi_2,\psi_2) \le r$.

It follows from $\frac{\sup_{A^{-1}([0,r])} B(\phi,\psi)}{r} < 1 < \frac{B(w,w)}{A(w,w)}$ that

$$\frac{B(w,w) - \sup_{A^{-1}([0,r])} B(\phi,\psi)}{A(w,w) - r} > \frac{B(w,w) - r\frac{B(w,w)}{A(w,w)}}{A(w,w) - r} = \frac{B(w,w)}{A(w,w)} > 1,$$

then $r - \sup_{A^{-1}([0,r])} B(\phi,\psi) > A(w,w) - B(w,w)$. Since $A(w,w) > r$ and $A(\phi_2,\psi_2) \le r$, we obtain that $A^r(\phi_2,\psi_2) - B(\phi_2,\psi_2) > A^r(w,w) - B(w,w)$ and this is contrary to (4.1). Our assertion is proved.

From inequality (4.1), one has $A(\phi_2,\psi_2) - B(\phi_2,\psi_2) \le A(\phi,\psi) - B(\phi,\psi)$ for all $(\phi,\psi) \in A^{-1}(]r,+\infty[)$. Since $A^{-1}(]r,+\infty[)$ is an open set, we obtain that (ϕ_2,ψ_2) is a local minimum of $I(\phi,\psi)$.

To conclude, $I(\phi,\psi)$ has a local minimum (ϕ_1,ψ_1) such that $A(\phi_1,\psi_1) < r$ and a local minimum (ϕ_2,ψ_2) such that $A(\phi_2,\psi_2) > r$. According to Lemma 2.3, the statement in Theorem 4.2 is proved. □

5 Existence of nontrivial solutions

In this section, we establish our main results. The existence of four nontrivial solutions where both of the components are not zero to system (1.1) is ensured by some sufficient conditions.

To prove the main results, we need the following four lemmas.

Lemma 5.1 *Assume that the following condition holds.*

(J_3) $a_1 > 0$ *and there exists a constant* $c > 0$ *such that*

$$\omega_1 - \min_{k\in\mathbb{Z}(1,N)} b_{1k} + a_1 c^2 \le 0.$$

Then the boundary value problem

$$\begin{cases} -\Delta^2\phi(k-1) = f_{1k}(\phi(k),0), & k \in \mathbb{Z}(1,N), \\ \phi(0) = \phi(N+1) = 0 \end{cases} \tag{5.1}$$

has no nontrivial solution ϕ^* *such that* $\|\phi^*\|_\infty < c$.

Proof If the conclusion is not true, then suppose that the boundary value problem (5.1) has a nontrivial solution ϕ^* such that $\|\phi^*\|_\infty < c$, hence, $I(\phi,0)$ has a nontrivial critical point ϕ^*. Since

$$I'(\phi,0) = C(L\phi) - \nabla B(\phi,0),$$

where ∇ denotes the gradient. One has

$$\langle \nabla B(\phi^*,0), L\phi^* \rangle = \langle C(L\phi^*), L\phi^* \rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^N$. By matrix theory, C is positive definite, we have

$$\langle C(L\phi^*), L\phi^* \rangle > 0,$$

that is,

$$\langle \nabla B(\phi^*,0), L\phi^* \rangle > 0.$$

Using (J_3), we find

$$\begin{aligned} \langle \nabla B(\phi^*,0), L\phi^* \rangle &= \sum_{k=1}^{N} \phi^*(k) f_{1k}(\phi^*(k),0) = \sum_{k=1}^{N} (\omega_1 - b_{1k} + a_1\phi^*(k)^2)\phi^*(k)^2 \\ &\le \sum_{k=1}^{N} \Bigl(\omega_1 - \min_{k\in\mathbb{Z}(1,N)} b_{1k} + a_1\phi^*(k)^2\Bigr)\phi^*(k)^2 \\ &< \sum_{k=1}^{N} \Bigl(\omega_1 - \min_{k\in\mathbb{Z}(1,N)} b_{1k} + a_1c^2\Bigr)\phi^*(k)^2 \\ &\le 0, \end{aligned}$$

where $\phi^*(k) \in \,]-c,c[$ for all $k \in \mathbb{Z}(1,N)$.

This leads to a contradiction. This completes the proof. □

Lemma 5.2 *Assume that the following condition holds.*

(J_4) $a_2 > 0$ *and there exists a constant* $c > 0$ *such that*

$$\omega_2 - \min_{k\in\mathbb{Z}(1,N)} b_{2k} + a_2c^2 \le 0.$$

Then the boundary value problem

$$\begin{cases} -\Delta^2\psi(k-1) = f_{2k}(0,\psi(k)), \quad k \in \mathbb{Z}(1,N), \\ \psi(0) = \psi(N+1) = 0 \end{cases} \tag{5.2}$$

has no nontrivial solution ψ^* *such that* $\|\psi^*\|_\infty < c$.

Proof The proof of this lemma is analogous to that in Lemma 5.1 and so is omitted. □

Lemma 5.3 *Assume that the following condition holds.*

(J_5) $a_1 < 0$, $\min_{k\in\mathbb{Z}(1,N)} b_{1k} - \omega_1 \le 0$ *and there exists a constant* $c > 0$ *such that*

$$\sqrt{\frac{N(\min_{k\in\mathbb{Z}(1,N)} b_{1k} - \omega_1)}{a_1}} \le \frac{c}{\sqrt{2N\lambda_N}K_2}.$$

Then the boundary value problem (5.1) *has no nontrivial solution* ϕ^* *such that*

$$\|\phi^*\|_\infty > \frac{c}{\sqrt{2N\lambda_N}K_2}.$$

Proof It follows from Lemma 5.1 that

$$\langle \nabla B(\phi^*,0), L\phi^*\rangle = \langle C(L\phi^*), L\phi^*\rangle > 0.$$

Taking (J_5) into consideration, we have

$$\begin{aligned} \langle \nabla B(\phi^*,0), L\phi^*\rangle &= \sum_{k=1}^N \phi^*(k) f_{1k}(\phi^*(k),0) = \sum_{k=1}^N (\omega_1 - b_{1k} + a_1\phi^*(k)^2)\phi^*(k)^2 \\ &\le \sum_{k=1}^N \Bigl(\omega_1 - \min_{k\in\mathbb{Z}(1,N)} b_{1k} + a_1\phi^*(k)^2\Bigr)\phi^*(k)^2 \\ &= \Bigl(\omega_1 - \min_{k\in\mathbb{Z}(1,N)} b_{1k}\Bigr)\sum_{k=1}^N \phi^*(k)^2 + a_1\sum_{k=1}^N \phi^*(k)^4 \\ &\le \Bigl(\omega_1 - \min_{k\in\mathbb{Z}(1,N)} b_{1k}\Bigr) N\|\phi^*\|_\infty^2 + a_1\|\phi^*\|_\infty^4 \\ &= \Bigl(\Bigl(\omega_1 - \min_{k\in\mathbb{Z}(1,N)} b_{1k}\Bigr)N + a_1\|\phi^*\|_\infty^2\Bigr)\|\phi^*\|_\infty^2 \\ &\le 0, \end{aligned}$$

where $\|\phi^*\|_\infty > \frac{c}{\sqrt{2N\lambda_N}K_2} \ge \sqrt{\frac{N(\min_{k\in\mathbb{Z}(1,N)} b_{1k}-\omega_1)}{a_1}}$.

This leads to a contradiction. We have thus proved the lemma. □

Lemma 5.4 *Assume that the following condition holds.*

(J_6) $a_2 < 0$, $\min_{k\in\mathbb{Z}(1,N)} b_{2k} - \omega_2 \le 0$ *and there exists a constant* $c > 0$ *such that*

$$\sqrt{\frac{N(\min_{k\in\mathbb{Z}(1,N)} b_{2k} - \omega_2)}{a_2}} \le \frac{c}{\sqrt{2N\lambda_N}K_2}.$$

Then the boundary value problem (5.2) *has no nontrivial solution* ψ^* *such that*

$$\|\psi^*\|_\infty > \frac{c}{\sqrt{2N\lambda_N}K_2}.$$

Proof The proof of this lemma is quite similar to Lemma 5.3 and so is omitted. □

Theorem 5.1 *Assume that there exist two constants* c, d, *with* $0 < d < c$, *such that*

$$2K_2^2\frac{\sum_{k=1}^N \max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y)}{c^2} < 1 < \frac{\sum_{k=1}^N F_k(d,d)}{2d^2}.$$

Then system (1.1) *has at least one nontrivial solution* (ϕ^*,ψ^*) *such that* $\|(\phi^*,\psi^*)\|_\infty < c$. *Furthermore, if* ($J_3$) *and* ($J_4$) *hold* (*and* c *as above*), *then system* (1.1) *has at least one nontrivial solution* (ϕ^*,ψ^*) *with* $\phi^* \not\equiv 0$ *and* $\psi^* \not\equiv 0$ *such that* $\|(\phi^*,\psi^*)\|_\infty < c$, *there exist other three nontrivial solutions* $(-\phi^*,\psi^*)$, $(\phi^*,-\psi^*)$ *and* $(-\phi^*,-\psi^*)$ *to system* (1.1).

Proof Let $r = \frac{c^2}{2K_2^2}$. From $\|\phi\|_\infty \le K_2\|\phi\|$ and $\|\psi\|_\infty \le K_2\|\psi\|$, one has $\|(\phi,\psi)\|_\infty = \max\{\|\phi\|_\infty,\|\psi\|_\infty\} \le \max\{K_2\|\phi\|,K_2\|\psi\|\} = K_2\max\{\|\phi\|,\|\psi\|\} \le K_2\sqrt{2r} = c$ for all $(\phi,\psi) \in S\times S$ such that $A(\phi,\psi) \le r$. Hence,

$$\begin{aligned} B(\phi,\psi) &= \sum_{k=1}^N F_k\bigl(\phi(k),\psi(k)\bigr) \\ &\le \sum_{k=1}^N \max_{(x,y)\in[-c,c]\times[-c,c]} F_k(x,y) \\ &= \sum_{k=1}^N \max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y) \end{aligned}$$

for all $(\phi,\psi) \in S\times S$ such that $A(\phi,\psi) \le r$. As a result,

$$\frac{\sup_{A(\phi,\psi)\le r} B(\phi,\psi)}{r} \le 2K_2^2\frac{\sum_{k=1}^N \max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y)}{c^2}.$$

Put $(w,w) \in \mathbb{R}^{N+2}\times\mathbb{R}^{N+2}$ to be such that $w(k) = d$ for all $k \in \mathbb{Z}(1,N)$ and $w(0) = w(N+1) = 0$. Obviously, $(w,w) \in S\times S$. Furthermore, we obtain

$$\begin{aligned} A(w,w) &= \frac{1}{2}\sum_{k=1}^{N+1}\bigl(\Delta w(k-1)\bigr)^2 + \frac{1}{2}\sum_{k=1}^{N+1}\bigl(\Delta w(k-1)\bigr)^2 = 2d^2, \\ B(w,w) &= \sum_{k=1}^N F_k(d,d). \end{aligned}$$

Consequently,

$$\frac{B(w,w)}{A(w,w)} = \frac{\sum_{k=1}^N F_k(d,d)}{2d^2}.$$

It follows that

$$\frac{\sup_{A(\phi,\psi)\le r} B(\phi,\psi)}{r} < 1 < \frac{B(w,w)}{A(w,w)}.$$

According to $0 < d < c$, we have

$$\frac{d^2}{\sum_{k=1}^{N} F_k(d,d)} < \frac{1}{4K_2^2}\frac{c^2}{\sum_{k=1}^{N}\max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y)} \le \frac{1}{4K_2^2}\frac{c^2}{\sum_{k=1}^{N} F_k(d,d)},$$

so, $0 < d < \frac{1}{2K_2}c$. Hence, $0 < A(w,w) < r$. Lemma 2.5 ensures that system (1.1) has at least one nontrivial solution (ϕ^*,ψ^*) such that $\|(\phi^*,\psi^*)\|_\infty < c$.

Furthermore, if (J_3) and (J_4) hold, we claim that system (1.1) has at least one nontrivial solution (ϕ^*,ψ^*) with $\phi^* \not\equiv 0$ and $\psi^* \not\equiv 0$ such that $\|(\phi^*,\psi^*)\|_\infty < c$. For the sake of contradiction, assume that $(\phi^*,0)$ is a nontrivial solution of system (1.1), that is to say, ϕ^* is a nontrivial solution of the boundary value problem (5.1) such that $\|\phi^*\|_\infty < c$, this is contrary to the conclusion of Lemma 5.1. Similarly, we can show that $(0,\psi^*)$ is not a nontrivial solution of system (1.1). Our claim is proved.

Apparently, $(-\phi^*,\psi^*)$, $(\phi^*,-\psi^*)$ and $(-\phi^*,-\psi^*)$ also satisfy system (1.1). Hence, the statements are proved. □

Theorem 5.2 *Assume that* (J_1) *holds and there exists* $c > 0$ *such that*

$$2K_2^2\frac{\sum_{k=1}^{N}\max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y)}{c^2} < 1.$$

Then system (1.1) *has at least one nontrivial solution* (ϕ^*,ψ^*) *such that* $\|(\phi^*,\psi^*)\|_\infty < c$. *Moreover, if* (J_3) *and* (J_4) *hold* (*and* c *as above*), *then system* (1.1) *has at least one nontrivial solution* (ϕ^*,ψ^*) *with* $\phi^* \not\equiv 0$ *and* $\psi^* \not\equiv 0$ *such that* $\|(\phi^*,\psi^*)\|_\infty < c$. *Also,* $(-\phi^*,\psi^*)$, $(\phi^*,-\psi^*)$ *and* $(-\phi^*,-\psi^*)$ *are other three nontrivial solutions to system* (1.1).

Proof As we have stated in the proof of Theorem 5.1, it follows from

$$2K_2^2\frac{\sum_{k=1}^{N}\max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y)}{c^2} < 1$$

that

$$\frac{\sup_{A(\phi,\psi)\le r} B(\phi,\psi)}{r} \le 2K_2^2\frac{\sum_{k=1}^{N}\max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y)}{c^2} < 1.$$

Taking (J_1) into consideration, Theorem 4.1 ensures that system (1.1) has at least two solutions, that is, system (1.1) has at least one nontrivial solution. Discussing as in the proof of Theorem 5.1, the remaining conclusion is achieved. □

Theorem 5.3 *Assume that* (J_2) *holds and there exist two constants* c, d, *with* $0 < c < \sqrt{2}d$, *such that*

$$2K_2^2\frac{\sum_{k=1}^{N}\max_{(x,y)\in[0,c]\times[0,c]} F_k(x,y)}{c^2} < 1 < \frac{\sum_{k=1}^{N} F_k(d,d)}{2d^2}.$$

Then system (1.1) *has at least two nontrivial solutions* (ϕ^*, ψ^*) *and* (ϕ^{**}, ψ^{**}) *such that* $A(\phi^{**}, \psi^{**}) > r$, *where* $r = \frac{c^2}{2K_2^2}$. *Furthermore, if* (J_5) *and* (J_6) *hold* (*and c as above*), *then system* (1.1) *has at least one nontrivial solution* (ϕ^{**}, ψ^{**}) *with* $\phi^{**} \neq 0$ *and* $\psi^{**} \neq 0$ *such that* $\|(\phi^{**}, \psi^{**})\|_\infty > \frac{c}{\sqrt{2N\lambda_N}K_2}$. *Then there exist other three nontrivial solutions* $(-\phi^{**}, \psi^{**})$, $(\phi^{**}, -\psi^{**})$, *and* $(-\phi^{**}, -\psi^{**})$ *to system* (1.1).

Proof Let $r = \frac{c^2}{2K_2^2}$. Furthermore, put $(w, w) \in \mathbb{R} \times \mathbb{R}$ to be such that $w(k) = d$ for all $k \in \mathbb{Z}(1, N)$ and $w(0) = w(N+1) = 0$. Apparently, $(w, w) \in S \times S$. From the above discussion of Theorem 5.1, we have

$$\frac{\sup_{A(\phi,\psi)\leq r} B(\phi, \psi)}{r} < 1 < \frac{B(w, w)}{A(w, w)}.$$

It follows from $0 < c < \sqrt{2}d$ and $\sqrt{2}K_2 \geq 1$ that $0 < c < 2K_2 d$. Hence, $0 < r < A(w, w)$. Taking (J_2) into consideration, Theorem 4.2 ensures that system (1.1) has at least two nontrivial solutions (ϕ^*, ψ^*) and (ϕ^{**}, ψ^{**}) such that $A(\phi^{**}, \psi^{**}) > r$. From $\|\phi\| \leq \sqrt{N\lambda_N}\|\phi\|_\infty$ and $\|\psi\| \leq \sqrt{N\lambda_N}\|\psi\|_\infty$, one has $\|(\phi, \psi)\|_\infty = \sqrt{\|(\phi, \psi)\|_\infty^2} = \sqrt{\max\{\|\phi\|_\infty^2, \|\psi\|_\infty^2\}} \geq \sqrt{\frac{1}{2}(\|\phi\|_\infty^2 + \|\psi\|_\infty^2)} \geq \sqrt{\frac{1}{2N\lambda_N}(\|\phi\|^2 + \|\psi\|^2)} > \sqrt{\frac{r}{N\lambda_N}} = \frac{c}{\sqrt{2N\lambda_N}K_2}$ for all $A(\phi, \psi) > r$. Hence, $\|(\phi^{**}, \psi^{**})\|_\infty > \frac{c}{\sqrt{2N\lambda_N}K_2}$.

If (J_5) and (J_6) hold, according to Lemma 5.3 and Lemma 5.4, we assert that system (1.1) has at least one nontrivial solution (ϕ^{**}, ψ^{**}) with $\phi^{**} \neq 0$ and $\psi^{**} \neq 0$ such that $\|(\phi^{**}, \psi^{**})\|_\infty > \frac{c}{\sqrt{2N\lambda_N}K_2}$. Arguing by contradiction, suppose that $(\phi^{**}, 0)$ is a nontrivial solution of system (1.1), that is to say, ϕ^{**} is a nontrivial solution of the boundary value problem (5.1) such that $\|\phi^{**}\|_\infty > \frac{c}{\sqrt{2N\lambda_N}K_2}$, this is contrary to the conclusion of Lemma 5.3. Similarly, we can show that $(0, \psi^{**})$ is not a nontrivial solution of system (1.1). Our assertion is proved.

It is obvious that $(-\phi^{**}, \psi^{**})$, $(\phi^{**}, -\psi^{**})$ and $(-\phi^{**}, -\psi^{**})$ also satisfy system (1.1). This completes the proof. □

6 Application

Example 6.1 Fix $N = 1$, $\omega_1 = -9$, $\omega_2 = -11$, $b_{11} = 1$, $b_{21} = 3$, $a_1 = 2$, $a_2 = 5$ and $a_3 = 4$. Theorem 5.2 ensures that the system

$$\begin{cases} -\Delta^2\phi(k-1) = \ -9\phi(k) - \phi(k) \ + 2\phi^3(k) + 4\psi^2(k)\phi(k), \\ -\Delta^2\psi(k-1) = -11\psi(k) - 3\psi(k) + 5\psi^3(k) + 4\phi^2(k)\psi(k), \\ \phi(0) = \phi(2) = \psi(0) = \psi(2) = 0, \end{cases} \tag{6.1}$$

where $k = 1$, admits at least four nontrivial solutions where both of the components are not zero. We observe that $K_2 = \frac{\sqrt{2}}{2}$ and $F_1(x, y) = -5x^2 - 7y^2 + \frac{1}{2}x^4 + \frac{5}{4}y^4 + x^2y^2$. Choosing $c = \frac{33}{20}$, we have $2K_2^2 \frac{\max_{(x,y)\in[0,c]\times[0,c]} F_1(x,y)}{c^2} = 2K_2^2 \frac{F_1(0,0)}{c^2} = 0 < 1$. Furthermore, $\omega_1 - b_{11} + a_1c^2 < 0$, $\omega_2 - b_{21} + a_2c^2 < 0$ show that conditions (J_3) and (J_4) hold. The conclusion follows from Theorem 5.2. Indeed, a simple calculation establishes $((0, 0), (\frac{\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}), (0, 0))$, $((0, 0), (-\frac{\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}), (0, 0))$, $((0, 0), (\frac{\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}), (0, 0))$ and $((0, 0), (-\frac{\sqrt{6}}{3}, -\frac{2\sqrt{6}}{3}), (0, 0))$ as the four nontrivial solutions of system (6.1).

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