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Asymptotic properties of a stochastic nonautonomous competitive system with impulsive perturbations

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Abstract
In this paper, a generalized nonautonomous stochastic competitive system with impulsive perturbations is studied. By the theories of impulsive differential equations and stochastic differential equations, we have established some asymptotic properties of the system, such as the extinction, nonpersistence and persistence in the mean, weak persistence and stochastic permanence and so on. In order to show the correctness and feasibility of the theoretical results, several numerical examples are presented. Finally, the effects of different white noise perturbations and different impulsive perturbations are discussed and illustrated.

Keywords: competitive system; impulsive perturbations; stochastic perturbations; extinction; stochastic permanence

1 Introduction

It is well known that there are four kinds of relationships between the species in the population ecological systems, that is, competition, predation, mutualism and parasitism. Among these relationships, competition can always ensure the survival of species and make effective use of resources, maintain the permanence of a ecological system and keep the healthy development of the population. Thus, a competitive system has received great interest by many mathematical and ecological researchers in the last decades (see [1–10]). As far as the competition is concerned, there are usually two kinds of competitive relationship, *i.e.* one is the interspecific competition and the other is the intraspecific competition.

The basic two-species competitive system is governed by the following coupled differential equations:

$$\begin{cases} \frac{dN_1(t)}{dt} = N_1(t)[r_1 - a_{11}N_1(t) - a_{12}N_2(t)], \\ \frac{dN_2(t)}{dt} = N_2(t)[r_2 - a_{21}N_1(t) - a_{22}N_2(t)], \end{cases} \tag{1}$$

where a_{ii} is the intraspecific competition coefficient, while a_{ij} ($i \neq j$, $i,j = 1,2$) is the interspecific competition coefficient.

Based on the classic competitive system, Gopalsamy proposed a series of generalized competitive systems in the monograph [11], and one of the generalized competitive systems is as follows (see p. 168, [11]):

$$\begin{cases} \frac{dN_1(t)}{dt} = N_1(t)[r_1 - a_{11}N_1(t-\tau) - \frac{a_{12}N_2(t-\tau)}{1+N_2(t-\tau)}], \\ \frac{dN_2(t)}{dt} = N_2(t)[r_2 - a_{22}N_2(t-\tau) - \frac{a_{21}N_1(t-\tau)}{1+N_1(t-\tau)}], \end{cases} \tag{2}$$

which means two species are allowed to cohabit in a common community, and each species inhibits the average growth rate of the other.

Recently, Wang and Liu (see [12]) studied the following nonautonomous competitive system:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - \frac{b_1(t)x_2(t)}{1+x_2(t)}], \\ \frac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_2(t) - \frac{b_2(t)x_1(t)}{1+x_1(t)}], \end{cases} \tag{3}$$

in which the existence and global asymptotic stability of positive almost periodic solutions is obtained. More references related to these generalized competitive systems can be also seen in [11, 13, 14].

However, most of the above mentioned references focused on the deterministic models, while the growth of the species is often affected by the interferences of the environmental noises in the real world. Thus, it is more reasonable to study ecological models. The dynamical behavior of the ecological system, and whether it will make a change to the existing results, has received wide attention in the recent several years (see references [4, 15–20] *etc.*).

Enlightened by the above mentioned references, we suppose that the random fluctuations of the environment will mainly affect the intrinsic growth rate $r_i(t)$ of the species, and they are estimated by the following form:

$$r_i(t) \to r_i(t) + \sigma_i(t)\,dB_i(t),$$

where $B_i(t)$ is Brownian motion, $\sigma_i(t)$ is a continuous and bounded function on $t \geq 0$ and $\sigma_i^2(t)$ represents the intensity of the white noise, $i = 1, 2$.

On the other hand, many natural or man-made factors, such as crop-dusting, planting, hunting, harvesting, drought, flooding and so on, will lead to sudden changes to the system. From the viewpoint of mathematical modeling, these sudden changes could be described by impulsive effects or perturbations to the models (see [21, 22]). Thus, if we introduce both impulsive perturbations and stochastic perturbations of white noises on the previous system (3), we can obtain the following system:

$$\begin{cases} \left.\begin{array}{l} dx_1(t) = x_1(t)[r_1(t) - a_1(t)x_1(t) - \frac{c_1(t)x_2(t)}{1+x_2(t)}]\,dt + \sigma_1(t)x_1(t)\,dB_1(t) \\ dx_2(t) = x_2(t)[r_2(t) - a_2(t)x_2(t) - \frac{c_2(t)x_1(t)}{1+x_1(t)}]\,dt + \sigma_2(t)x_2(t)\,dB_2(t) \end{array}\right\}, \quad t \neq t_k, k \in N, \\ \left.\begin{array}{l} x_1(t_k^+) = (1+h_{1k})x_1(t_k) \\ x_2(t_k^+) = (1+h_{2k})x_2(t_k) \end{array}\right\}, \quad t = t_k, k \in N, \end{cases} \tag{4}$$

where $x_i(t)$ is the population density of the ith population, $r_i(t)$ and $a_i(t)$ are the intrinsic growth rate and the intraspecific competing rate, respectively, and $c_i(t)$ represents the interspecific competing rate. $r_i(t)$, $a_i(t)$, $c_i(t)$, $t \in R^+ = [0,\infty)$ are positive, continuous and bounded. $0 < t_1 < t_2 < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$. $h_{ik} > -1$, $i = 1,2$, $k \in N$, when $h_{ik} > 0$, the impulsive effects represent planting, while $h_{ik} < 0$ denote harvesting.

Throughout the present paper, we denote

$$f^l = \inf_{t\in R^+} f(t), \qquad f^u = \sup_{t\in R^+} f(t)$$

for any positive, bounded function $f(t)$ defined on $R^+ = [0,+\infty)$.

The rest of this paper is organized as follows. In Section 2 we demonstrate and prove the main results of the paper, such as the existence of a unique positive solution of the system, sufficient conditions for the extinction, nonpersistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the system. In Section 3, several numerical examples are presented to support the theoretical results. Moreover, effects on the impulsive and stochastic perturbations are also analyzed and discussed at the end of the paper.

2 Preliminaries

In this section, based on the methods proposed by Yan and Zhao (see [23]), the corresponding stochastic differential equations without impulses are studied, and we will discuss the existence of a positive solution of above system (4) firstly. Further, by the definitions proposed by Liu and Wang (see [18]), we will derive some asymptotic behavior of this system, such as the extinction, nonpersistence and persistence in the mean, weak persistence and stochastic permanence and so on.

Theorem 2.1 *For any initial conditions* $(x_{10},x_{20})^T \in R^2_+ = \{(x,y)^T \in R^2 | x > 0, y > 0\}$, *system* (4) *has a unique positive solution* $x(t) = (x_1(t),x_2(t))^T$ *on* $[0,+\infty)$, *and the solution will remain in* R^2_+ *almost surely.*

Proof Consider the following stochastic differential equations (SDEs) without impulses:

$$\begin{cases} dy_1(t) = y_1(t)[r_1(t) - a_1(t)y_1(t)\prod_{0<t_k<t}(1+h_{1k})y_1(t) - \frac{c_1(t)\prod_{0<t_k<t}(1+h_{2k})y_2(t)}{1+\prod_{0<t_k<t}(1+h_{2k})y_2(t)}]\,dt \\ \qquad + \sigma_1(t)y_1(t)\,dB_1(t), \\ dy_2(t) = y_2(t)[r_2(t) - a_2(t)y_2(t)\prod_{0<t_k<t}(1+h_{2k})y_2(t) - \frac{c_2(t)\prod_{0<t_k<t}(1+h_{1k})y_1(t)}{1+\prod_{0<t_k<t}(1+h_{1k})y_1(t)}]\,dt \\ \qquad + \sigma_2(t)y_2(t)\,dB_2(t) \end{cases} \tag{5}$$

with the initial value $(y_{10},y_{20})^T = (x_{10},x_{20})^T$.

It is easy to prove that there is a unique global positive solution $y(t) = (y_1(t),y_2(t))^T$ of system (5) by the theory of non-impulsive stochastic differential equations (see [18]).

Denote $x_i(t) = \prod_{0<t_k<t}(1+h_{1k})y_i(t)$ $(i = 1,2)$, then we claim that $x(t) = (x_1(t),x_2(t))^T$ is the solution of system (4) with the initial data $(x_{10},x_{20})^T$.

In fact, since $x_1(t)$ is continuous on $(0, t_1)$ and each interval $(t_k, t_{k+1}) \subset [0, +\infty)$ and for $t \neq t_k$, $k \in N$,

$$\begin{aligned}
dx_1(t) &= \prod_{0<t_k<t} (1 + h_{1k})\, dy_1(t) \\
&= \prod_{0<t_k<t} (1 + h_{1k}) y_1(t) \\
&\quad \times \left[r_1(t) - a_1(t) \prod_{0<t_k<t} (1 + h_{1k}) y_1(t) - \frac{c_1(t) \prod_{0<t_k<t} (1 + h_{2k}) y_2(t)}{1 + \prod_{0<t_k<t} (1 + h_{2k}) y_2(t)} \right] dt \\
&\quad + \sigma_1(t) \prod_{0<t_k<t} (1 + h_{1k}) y_1(t)\, dB_1(t) \\
&= x_1(t) \left[r_1(t) - a_1(t) x_1(t) - \frac{c_1(t) x_2(t)}{1 + x_2(t)} \right] dt + \sigma_1(t) x_1(t)\, dB_1(t).
\end{aligned}$$

Similarly, we can check that

$$dx_2(t) = x_2(t) \left[r_2(t) - a_2(t) x_1(t) - \frac{c_2(t) x_1(t)}{1 + x_1(t)} \right] dt + \sigma_2(t) x_2(t)\, dB_2(t). \tag{6}$$

And for every $t_k \in R^+$, $k \in N$,

$$\begin{aligned}
x_i(t_k^+) &= \lim_{t \to t_k^+} x_i(t) = \prod_{0<t_j \leq t_k} (1 + h_{ij}) y_i(t_k) \\
&= (1 + h_{ik}) \prod_{0<t_j<t_k} (1 + h_{ij}) y_i(t_k) = (1 + h_{ik}) x_i(t_k).
\end{aligned} \tag{7}$$

$$\begin{aligned}
x_i(t_k^-) &= \lim_{t \to t_k^-} x_i(t) = \lim_{t \to t_k^-} \prod_{0<t_j<t_k} (1 + h_{ij}) y_i(t_k^-) \\
&= \prod_{0<t_j<t_k} (1 + h_{ij}) y_i(t_k) = x_i(t_k).
\end{aligned} \tag{8}$$

These mean that $x(t) = (x_1(t), x_2(t))^T$ is the unique global positive solution of system (4), so we complete the proof of this theorem. □

In Theorem 2.1, we can see that solutions of system (4) will remain in the first quadrant, but how do they vary in this quadrant? In the following part, we will discuss the sufficient conditions for several cases, such as extinction and weak persistence, nonpersistence and persistence in the mean and so on.

Theorem 2.2 *Denote by* $x(t) = (x_1(t), x_2(t))^T$ *a solution of system* (4), *then*

$$\limsup_{t \to +\infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t \to +\infty} \frac{1}{t} \left[\sum_{0<t_k<t} \ln(1 + h_{ik}) + \int_0^t b_i(s)\, ds \right] := b_i^*, \quad a.s.,$$

where $b_i(t) = r_i(t) - 0.5\sigma_i^2(t)$, $i = 1, 2$.

Proof For the non-impulsive system (5), by Itô's formula, we obtain

$$\begin{aligned} d\ln y_i(t) &= \frac{dy_i(t)}{y_i(t)} - \frac{(dy_i(t))^2}{2y_i^2(t)} \\ &= \left[r_i(t) - 0.5\sigma_i^2(t) - a_i(t) \prod_{0<t_k<t} (1+h_{ik}) y_i(t) - \frac{c_i(t)\prod_{0<t_k<t}(1+h_{jk})y_j(t)}{1+\prod_{0<t_k<t}(1+h_{jk})y_j(t)} \right] dt \\ &\quad + \sigma_i(t)\,dB_i(t), \end{aligned} \tag{9}$$

where $j = 1, 2$, $j \neq i$, and this leads to

$$d\ln y_i(t) \leq \left[b_i(t) - a_i(t)x_i(t) \right] dt + \sigma_i(t)\,dB_i(t). \tag{10}$$

Integrating both sides of inequality (10) on the interval $[0, t]$ yields

$$\ln y_i(t) - \ln y_i(0) \leq \int_0^t b_i(s)\,ds - \int_0^t a_i(s)x_i(s)\,ds + M_i(t), \tag{11}$$

where $M_i(t) = \int_0^t \sigma_i(s)\,dB_i(s)$.

Thus,

$$\begin{aligned} &\sum_{0<t_k<t} \ln(1+h_{ik}) + \ln y_i(t) - \ln y_i(0) \\ &\quad \leq \sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds - \int_0^t a_i(s)x_i(s)\,ds + M_i(t), \end{aligned} \tag{12}$$

which yields

$$\ln x_i(t) \leq \ln y_{i0} + \sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds + M_i(t) - \int_0^t a_i(s)x_i(s)\,ds. \tag{13}$$

Note that $M_i(t)$ is a local martingale whose quadratic variation is

$$\langle M_i(t), M_i(t) \rangle = \int_0^t \sigma_i^2(s)\,ds \leq \left(\sigma_i^u\right)^2 t.$$

By the strong law of large numbers for local martingale (see [24]), we have

$$\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0, \quad \text{a.s.}$$

If we multiply $\frac{1}{t}$ on each side of inequality (13) and take superior limit on both sides of it, we can obtain

$$\limsup_{t\to+\infty} \frac{\ln x_i(t)}{t} \leq \limsup_{t\to+\infty} \frac{1}{t}\left[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds \right] := b_i^*, \quad \text{a.s.}$$

□

Corollary 2.1 *If* $b_i^* = \limsup_{t\to+\infty} \frac{1}{t}[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds] < 0$, *then the ith species of system* (4) *is extinct.*

Theorem 2.3 *Suppose that $x(t) = (x_1(t), x_2(t))^T$ is a solution of system* (4), *then*

$$\limsup_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} \le \frac{b_i^*}{a_i^l}, \quad a.s.$$

Proof By the definition of the limit, for $\forall\varepsilon_i > 0$, there exists $T_1 > 0$ such that

$$\begin{aligned}&\frac{\ln y_i(0)}{t} = \frac{\ln y_{i0}}{t} \le \varepsilon_i/3, \qquad \frac{M_i(t)}{t} \le \varepsilon_i/3,\\ &\frac{1}{t}\left[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds\right] \le b_i^* + \varepsilon_i/3\end{aligned}$$

for $t > T_1$.

Combining inequality (13) and the above inequality, we have

$$\ln x_i(t) \le \big(b_i^* + \varepsilon_i\big)t - \int_0^t a_i(s)x_i(s)\,ds \le \lambda_i t - a_i^l \int_0^t x_i(s)\,ds \tag{14}$$

for $\forall t > T_1$ a.s., where $\lambda_i = b_i^* + \varepsilon_i$.

If we denote $h_i(t) = \int_0^t x_i(s)\,ds$, then $h_i'(t) = x_i(t)$. Then it follows from inequality (14) that

$$e^{a_i^l h_i(t)} \frac{dh_i(t)}{dt} \le e^{\lambda_i t}. \tag{15}$$

Integrating inequality (15) from T_1 to t, we have

$$e^{a_i^l h_i(t)} \le \frac{a_i^l}{\lambda_i} e^{\lambda_i t} + e^{a_i^l h_i(T_1)} - \frac{a_i^l}{\lambda_i} e^{\lambda_i T_1}. \tag{16}$$

Thus,

$$\int_0^t x_i(s)\,ds \le \frac{1}{a_i^l} \ln\left[\frac{a_i^l}{\lambda_i} e^{\lambda_i t} + e^{a_i^l h_i(T_1)} - \frac{a_i^l}{\lambda_i} e^{\lambda_i T_1}\right]. \tag{17}$$

If we multiply $\frac{1}{t}$ on each side of inequality (17) and take superior limit on both sides of it, we can obtain

$$\limsup_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} \le \limsup_{t\to+\infty} \frac{1}{a_i^l t} \ln\left[\frac{a_i^l}{\lambda_i} e^{\lambda_i t} + e^{a_i^l h_i(T_1)} - \frac{a_i^l}{\lambda_i} e^{\lambda_i T_1}\right]. \tag{18}$$

By L'Hospital's rule we have

$$\limsup_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} \le \limsup_{t\to+\infty} \frac{\lambda_i}{a_i^l} = \frac{b_i^*}{a_i^l}. \tag{19}$$

This means that we have completed the proof. □

If $b_i^* = 0$, it is easy to obtain $\lim_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} = 0$, and we can obtain the following Corollary 2.2.

Corollary 2.2 *If* $b_i^* = \limsup_{t\to+\infty} \frac{1}{t}[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds] = 0$, *then system* (4) *is nonpersistent in the mean.*

Theorem 2.4 *If* $b_i^* = \limsup_{t\to+\infty} \frac{1}{t}[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds] > 0$, *then at least one of the species in system* (4) *is weakly persistent.*

Proof It follows from (9) that

$$d\ln y_i(t) = \left[b_i(t) - a_i(t)x_i(t) - \frac{c_i(t)x_j(t)}{1+x_j(t)}\right]dt + \sigma_i(t)\,dB_i(t), \quad i = 1,2, j \neq i.$$

If we integrate on each side of the above equation, we have

$$\ln x_i(t) - \ln x_{i0} = \sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t b_i(s)\,ds - \int_0^t a_i(s)x_i(s)\,ds - \int_0^t \frac{c_i(s)x_j(s)}{1+x_j(s)}\,ds + M_i(t).$$

Set $S = \{\lim_{t\to+\infty} \sup x_i(t) = 0\}$, if the assertion of this theorem is not true, then $\mathcal{P}(S) > 0$, and for $\omega \in S$, $\lim_{t\to+\infty} x_i(t,\omega) = 0$.

Note that $\limsup_{t\to+\infty} \frac{M_i(t)}{t} = 0$. Further, it follows from the boundedness of $a_i(t)$ and $c_i(t)$ that

$$\begin{aligned} &\limsup_{t\to+\infty} \frac{1}{t}\int_0^t a_i(s)x_i(s)\,ds = 0, \\ &\limsup_{t\to+\infty} \frac{1}{t}\int_0^t \frac{c_i(s)x_j(s)}{1+x_j(s)}\,ds = 0, \\ &\limsup_{t\to+\infty} \frac{1}{t}\bigl[\ln x_i(t,w) - \ln x_{i0}\bigr] \leq 0. \end{aligned}$$

Thus,

$$0 \geq \limsup_{t\to+\infty} \frac{1}{t}\bigl[\ln x_i(t,w) - \ln x_{i0}\bigr] = b_i^* > 0,$$

which is a contradiction, and this completes the proof of this theorem. □

Theorem 2.5 *Denote* $b_{i*} = \liminf_{t\to+\infty} \frac{1}{t}[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t \bar{b}_i(s)\,ds]$, *then the solution of system* (4) *satisfies*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \geq \frac{b_{i*}}{a_i^u}, \quad a.s.,$$

where $\bar{b}_i(t) = r_i(t) - c_i(t) - 0.5\sigma_i^2(t)$, $i = 1,2$.

Proof It follows from (9) again that

$$d\ln y_i(t) \geq \bigl[r_i(t) - c_i(t) - 0.5\sigma_i^2(t) - a_i(t)x_i(t)\bigr]dt + \sigma_i(t)\,dB_i(t). \tag{20}$$

Integrating both sides of inequality (20) from 0 to t yields

$$\ln y_i(t) - \ln y_i(0) \geq \int_0^t \bar{b}_i(s)\,ds - \int_0^t a_i(s)x_i(s)\,ds + M_i(t). \tag{21}$$

Then

$$\ln x_i(t) \geq \ln x_{i0} + \sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t \bar{b}_i(s)\,ds + M_i(t) - a_i^u \int_0^t x_i(s)\,ds. \tag{22}$$

Note the definition of $M_i(t)$ and b_{i*}, according to the property of the limit again, for any $\epsilon_i > 0$, $i = 1,2$, there exists $T_2 > 0$ such that, for $t > T_2$,

$$\begin{aligned} &\frac{\ln x_{i0}}{t} \geq -\epsilon_i/3, \qquad \frac{M_i(t)}{t} \geq -\epsilon_i/3, \\ &\frac{1}{t}\left[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t \bar{b}_i(s)\,ds\right] \geq \bar{b}_i - \epsilon_i/3. \end{aligned}$$

Substituting above inequalities into (22) yields

$$\ln x_i(t) \geq \mu_i t - a_i^u \int_0^t x_i(s)\,ds \tag{23}$$

for all $t > T_1$ almost surely, where $\mu_i = b_{i*} - \epsilon_i$.

Note that $h_i(t) = \int_0^t x_i(s)\,ds$ and $\frac{dh_i(t)}{dt} = x_i(t)$, then from (23) we have

$$e^{a_i^u h_i(t)} \frac{dh_i(t)}{dt} \geq e^{\mu_i t}. \tag{24}$$

If we integrate the above inequality on the interval $[T_2, t]$, then we have

$$e^{a_i^u h_i(t)} \geq \frac{a_i^u}{\mu_i} e^{\mu_i t} + e^{a_i^u h_i(T_2)} - \frac{a_i^u}{\mu_i} e^{\mu_i T_2}. \tag{25}$$

Thus,

$$\int_0^t x_i(s)\,ds \geq \frac{1}{a_i^u} \ln\left[\frac{a_i^u}{\mu_i} e^{\mu_i t} + e^{a_i^u h_i(T_2)} - \frac{a_i^u}{\mu_i} e^{\mu_i T_2}\right]. \tag{26}$$

Taking inferior limit on (26) yields

$$\liminf_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} \geq \liminf_{t\to+\infty} \frac{1}{a_i^u t} \ln\left[\frac{a_i^u}{\mu_i} e^{\mu_i t} + e^{a_i^u h_i(T_2)} - \frac{a_i^u}{\mu_i} e^{\mu_i T_2}\right]. \tag{27}$$

By L'Hospital's rule again, we have

$$\liminf_{t\to+\infty} \frac{\int_0^t x_i(s)\,ds}{t} \geq \liminf_{t\to+\infty} \frac{\mu_i}{a_i^u} = \frac{b_{i*}}{a_i^u}. \tag{28}$$

Thus, we complete the proof of the above theorem. □

Further, if $b_{i*} > 0$, we have the following Corollary 2.3.

Corollary 2.3 *If* $b_{i*} = \liminf_{t\to+\infty} \frac{1}{t}[\sum_{0<t_k<t} \ln(1+h_{ik}) + \int_0^t \bar{b}_i(s)\,ds] > 0$, *then system* (4) *is persistent in the mean a.s.*

Theorem 2.6 *If system* (4) *satisfies the following two conditions*:

(H1) *there exist positive constants* m_i *and* M_i *such that* $m_i < \prod_{0<t_k<t}(1+h_{ik}) < M_i$;

(H2) $(\sigma_i^u)^2 < \bar{b}_i^l$;

then system (4) *is stochastically permanent, where* $\bar{b}_i(t) = r_i(t) - c_i(t) - 0.5\sigma_i^2(t)$, $i = 1, 2$.

Proof Applying Itô's integration by parts formula, we can derive that

$$\begin{aligned} d\bigl(e^t y_i^2(t)\bigr) &= e^t y_i^2(t)\,dt + 2e^t y_i(t)dy_i(t) + e^t\bigl(dy_i(t)\bigr)^2 \\ &= e^t y_i^2(t)\Biggl[1 + 2\Biggl(r_i(t) - a_i(t)\prod_{0<t_k<t}(1+h_{ik})y_i(t) - \frac{c_i(t)\prod_{0<t_k<t}(1+h_{jk})y_j(t)}{1+\prod_{0<t_k<t}(1+h_{jk})y_j(t)}\Biggr) \\ &\quad + \sigma_i^2(t)\Biggr]dt \\ &\quad + 2e^t y_i^2(t)\sigma_i(t)\,dB_i(t) \\ &\le e^t y_i^2(t)\Biggl[1 + 2r_i(t) + \sigma_i^2(t) - 2a_i(t)\prod_{0<t_k<t}(1+h_{ik})y_i(t)\Biggr]dt \\ &\quad + 2e^t y_i^2(t)\sigma_i(t)\,dB_i(t). \end{aligned}$$

Integrating the above inequality on the interval $[0, t]$, we have

$$\begin{aligned} e^t y_i^2(t) - y_i^2(0) \le{}& \int_0^t e^s y_i^2(s)\bigl[1 + 2r_i(s) + \sigma_i^2(s) - 2m_i a_i(s)y_i(s)\bigr]\,dt \\ & + 2\int_0^t e^s y_i^2(s)\sigma_i(s)\,dB_i(s). \end{aligned} \tag{29}$$

Taking expectations on both sides of (29) and making some estimations lead to

$$\begin{aligned} E\bigl[e^t y_i^2(t)\bigr] &\le y_{i0}^2 + E\biggl[\int_0^t e^s y_i^2(s)\bigl[1 + 2r_i(s) + \sigma_i^2(s) - 2m_i a_i(s)y_i(s)\bigr]\,ds\biggr] \\ &\le y_{i0}^2 + E\biggl[\int_0^t e^s y_i^2(s)\bigl[1 + 2r_i^u + \bigl(\sigma_i^u\bigr)^2 - 2m_i a_i^l y_i(s)\bigr]\,ds\biggr]. \end{aligned}$$

Thus, by the maximum principle, we have

$$E\bigl[e^t y_i^2(t)\bigr] \le y_{i0}^2 + L_i E\biggl[\int_0^t e^s ds\biggr] = y_{i0}^2 + L_i\bigl(e^t - 1\bigr), \tag{30}$$

where $L_i = \frac{(1+2r_i^u+(\sigma_i^u)^2)^3}{27(m_i a_i^l)^2}$, $i = 1, 2$.

Thus,

$$\limsup_{t\to+\infty} E\bigl[y_i^2(t)\bigr] \le \limsup_{t\to+\infty} \frac{y_{i0}^2 + L_i(e^t - 1)}{t} = L_i, \tag{31}$$

which yields

$$\limsup_{t\to+\infty} E\big[x_i^2(t)\big] = \limsup_{t\to+\infty} E\bigg[\prod_{0<t_k<t}(1+h_{ik})^2 y_i^2(t)\bigg] \le L_i M_i^2. \tag{32}$$

Then, for any $\xi_i > 0$, set $\beta_i = M_i\sqrt{\frac{L_i}{\xi_i}}$, and by Chebyshev's inequality, we have

$$\limsup_{t\to+\infty} \mathcal{P}\big\{x_i(t) > \beta_i\big\} = \limsup_{t\to+\infty} \mathcal{P}\big\{x_i^2(t) > \beta_i^2\big\} \le \limsup_{t\to+\infty} \frac{E[x_i^2(t)]}{\beta_i^2} \le \frac{L_i M_i^2}{\beta_i^2} = \xi_i. \tag{33}$$

In other words,

$$\liminf_{t\to+\infty} \mathcal{P}\big\{x_i(t) \le \beta_i\big\} \ge 1 - \xi_i. \tag{34}$$

Now we will prove that for $\forall\xi_i > 0$, $\exists\eta_i > 0$, s.t. $\liminf_{t\to+\infty} \mathcal{P}\{x_i(t) \ge \eta_i\} \ge 1 - \xi_i$. In fact, it follows from condition (H2) that $b_i^l > (\sigma_i^u)^2$. If we define

$$V_1(y_i) = y_i^{-1}, \qquad V_2(y_i) = e^{kt}\big(1 + V_1(y_i)\big)^2, \quad i = 1, 2,$$

where $0 < k < 2[b_i^l - (\sigma_i^u)^2]$.

Then it follows from Itô's formula again that

$$\begin{aligned} dV_1(y_i) &= -y_i^{-2} dy_i(t) + y_i^{-3}\big(dy_i(t)\big)^2 \\ &= -V_1(y_i)\bigg[r_i(t) - a_i(t)\prod_{0<t_k<t}(1+h_{ik})y_i(t) - \frac{c_i(t)\prod_{0<t_k<t}(1+h_{jk})y_j(t)}{1+\prod_{0<t_k<t}(1+h_{jk})y_j(t)}\bigg]dt \\ &\quad - \sigma_i(t)V_1(y_i)\,dB_i(t) + \sigma_i^2(t)V_1(y_i)\,dt, \end{aligned}$$

which yields

$$dV_1(y_i) \le -V_1(y_i)\big[r_i(t) - c_i(t) - \sigma_i^2(t) - M_i a_i(t) y_i(t)\big]\,dt - \sigma_i(t)V_1(y_i)\,dB_i(t). \tag{35}$$

Similarity, if we apply Itô's integration by parts formula on $V_2(t)$, then

$$\begin{aligned} d\big(V_2(y_i)\big) &= ke^{kt}\big(1+V_1(y_i)\big)^2\,dt + 2e^{kt}\big(1+V_1(y_i)\big)dV_1(y_i) + e^{kt}\big(dV_1(y_i)\big)^2 \\ &\le ke^{kt}\big(1+V_1(y_i)\big)^2\,dt \\ &\quad - 2e^{kt}V_1(y_i)\big(1+V_1(y_i)\big)\big[r_i(t) - c_i(t) - \sigma_i^2(t) - M_i a_i(t)y_i(t)\big]\,dt \\ &\quad - 2e^{kt}\sigma_i(t)\big(1+V_1(y_i)\big)V_1(y_i)\,dB_i(t) + e^{kt}\sigma_i^2(t)V_1^2(y_i)\,dt \\ &\le e^{kt}\big[-2\big(b_i(t) - \sigma_i^2(t) - 0.5k\big)V_1^2(y_i) + 2\big(M_i a_i(t) - b_i(t) + 0.5\sigma_i^2(t) + k\big)V_1(y_i) \\ &\quad + \big(2M_i^u a_i^u + k\big)\big]\,dt - 2e^{kt}\sigma_i(t)\big(1+V_1(y_i)\big)V_1(y_i)\,dB_i(t), \end{aligned}$$

which yields

$$d\big(V_2(y_i)\big) \le e^{kt}J(y_i)\,dt - 2e^{kt}\sigma_i(t)V_1(y_i)\big(1+V_1(y_i)\big)\,dB_i(t), \tag{36}$$

where

$$J(y_i) = -2\big(b_i^l - (\sigma_i^u)^2 - 0.5k\big)V_1^2(y_i) + 2\big(M_i a_i^u - b_i^l + 0.5(\sigma_i^u)^2 + k\big)V_1(y_i) + \big(2M_i^u a_i^u + k\big).$$

Since $0 < k < 2[b_i^l - (\sigma_i^u)^2]$, then $J(y_i)$ is upper bounded, and we denote $J_i := \sup_{y_i \in R^+} J(y_i) < +\infty$, then

$$d\big(V_2(y_i)\big) \le J_i e^{kt}\,dt - 2e^{kt}\sigma_i(t)V_1(y_i)\big(1 + V_1(y_i)\big)\,dB_i(t). \tag{37}$$

Integrating (37) from 0 to t, then multiplying e^{-kt} and taking expectations on each side of it, we can obtain

$$E\big[\big(1 + V_1(y_i)\big)^2\big] \le V_2(y_{i0})e^{-kt} + \frac{J_i}{k}\big(1 - e^{-kt}\big), \tag{38}$$

which yields

$$\limsup_{t\to\infty} E\big[y_i^{-2}(t)\big] \le \limsup_{t\to\infty} E\big[\big(1 + V_1(y_i)\big)^2\big] \le \limsup_{t\to\infty}\left[\frac{V_2(y_{i0})}{e^{kt}} + \frac{J_i(1 - e^{-kt})}{k}\right] = \frac{J_i}{k}. \tag{39}$$

Thus,

$$\limsup_{t\to+\infty} E\big[x_i^{-2}(t)\big] = \limsup_{t\to+\infty} E\left[\prod_{0<t_k<t}(1 + h_{ik})^{-2} y_i^{-2}(t)\right] \le \frac{J_i}{km_i^2}. \tag{40}$$

Then, for any $\xi_i > 0$, set $\eta_i = m_i\sqrt{\frac{k\xi_i}{J_i}}$, applying Chebyshev's inequality again, we have

$$\limsup_{t\to+\infty} \mathcal{P}\big\{x_i(t) < \eta_i\big\} = \limsup_{t\to+\infty} \mathcal{P}\big\{x_i^{-2}(t) > \eta_i^{-2}\big\} \le \lim_{t\to+\infty} \frac{E[x_i^{-2}(t)]}{\eta_i^{-2}} \le \xi_i. \tag{41}$$

In other words,

$$\liminf_{t\to+\infty} \mathcal{P}\big\{x_i(t) \ge \eta_i\big\} \ge 1 - \xi_i. \tag{42}$$

From (34) and (42), the stochastic permanence of system (4) is obtained. This completes the proof of this theorem. □

Remark In fact, 'persistence in the mean' in this section is not a good definition of persistence for stochastic population models. Some authors have introduced some more appropriate definitions of permanence for stochastic population models. For example, stochastic persistence in probability (see [25, 26]) or a new definition of stochastic permanence (see [27]).

3 Numerical simulations and discussions

In this paper, a stochastic nonautonomous competitive system with impulsive perturbations is proposed and studied. We establish sufficient conditions for the extinction, nonpersistence in the mean, weak persistence, persistence in the mean and stochastic permanence of the system. Furthermore, the critical value between extinction, nonpersistence and weak persistence of at least one species in the system is obtained.

In order to verify the correctness and the feasibility of the derived conditions in the theoretical results, we will give a series of numerical examples to illustrate them by using the extension of Milstein's method (see [28]) in this section. Furthermore, we will show the effects of different white noises or impulsive perturbations to the dynamics of the system, and by the figures of corresponding simulations, one can observe the population fluctuation of the species in the competitive system more intuitively.

In the following, we choose the same initial value $(x_{10}, x_{20}) = (0.5, 0.2)$ and parameters $a_1(t) = 0.1 + 0.01\sin(t)$, $a_2(t) = 0.1 + 0.01\cos(t)$, $c_1(t) = 0.22 + 0.02\sin(t)$, $c_2(t) = 0.22 + 0.02\cos(t)$, $\Delta t = 0.01$.

Example 3.1 For system (4), we set the following choice of parameters:

$$r_1(t) = 0.2 + 0.01\sin(t), \qquad r_2(t) = 0.1 + 0.01\cos(t);$$
$$\sigma_1^2(t) = 0.5 + 0.04\sin(t), \qquad \sigma_2^2(t) = 0.4 + 0.04\cos(t);$$
$$h_{1k} = h_{2k} = e^{\frac{(-1)^k}{k^2}} - 1, \quad t_k = k, k = 1, 2, \ldots.$$

By a simple computation, we have $\bar{b}_1(t) = -0.05 - 0.01\sin(t)$, $\bar{b}_2(t) = -0.1 - 0.01\cos(t)$, then $b_1^* = -0.04 < 0$, $b_2^* = -0.09 < 0$, which satisfies the condition of Corollary 2.1, then both of the species are extinct (see Figure 1).

If we decrease the white noises of the species x_1 and let $\sigma_1^2(t) = 0.2 + 0.04\sin(t)$, while the values of other parameters are the same as above, then $b_1^* = 0.11 > 0$, at the moment the species x_2 will still be extinct, while the species x_1 can survive (see Figure 2).

Example 3.2 For system (4), if we set the following choice of parameters:

$$r_1(t) = 0.16 + 0.02\sin(t), \qquad r_2(t) = 0.24 + 0.03\cos(t);$$
$$\sigma_1^2(t) = 0.32 + 0.04\sin(t), \qquad \sigma_2^2(t) = 0.48 + 0.06\cos(t);$$
$$h_{1k} = h_{2k} = e^{\frac{(-1)^k}{k^2}} - 1, \quad t_k = k, k = 1, 2, \ldots.$$

It is easy to calculate that $\bar{b}_1(t) = \bar{b}_2(t) = 0$, then $b_1^* = b_2^* = 0$, which satisfies the condition of Corollary 2.2, then the species of system (4) is nonpersistent in the mean (see Figure 3).

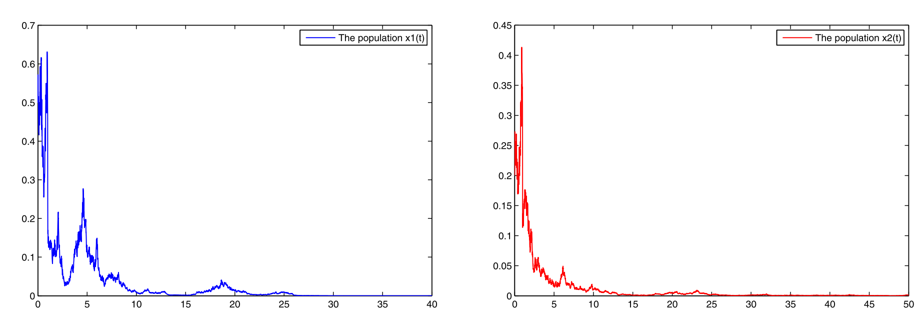

Figure 1 The time series of system (4) with $\sigma_1^2(t) = 0.5 + 0.04\sin(t)$, $\sigma_2^2(t) = 0.4 + 0.04\cos(t)$, $h_{1k} = h_{2k} = -1 + e^{\frac{(-1)^k}{k^2}}$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.

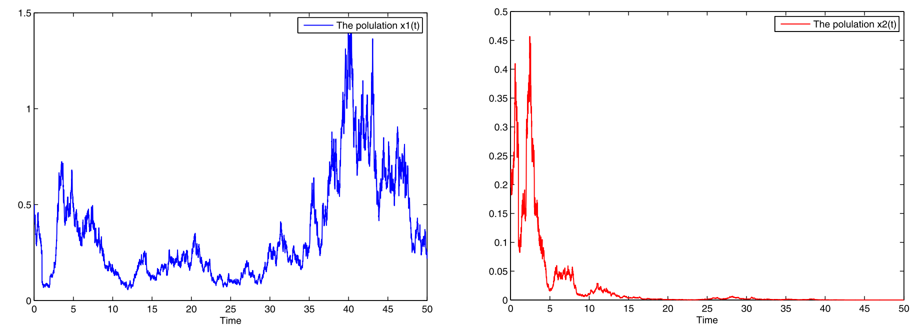

Figure 2 **The time series of system (4) with $\sigma_1^2(t) = 0.2 + 0.04\sin(t)$, $\sigma_2^2(t) = 0.4 + 0.04\cos(t)$, $h_{1k} = h_{2k} = -1 + e^{\frac{(-1)^k}{k^2}}$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.**

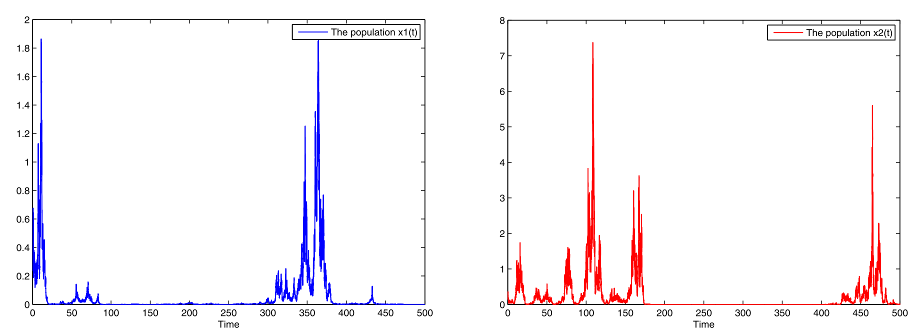

Figure 3 **The time series of system (4) with $\sigma_1^2(t) = 0.32 + 0.04\sin(t)$, $\sigma_2^2(t) = 0.48 + 0.06\cos(t)$, $h_{1k} = h_{2k} = -1 + e^{\frac{(-1)^k}{k^2}}$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.**

If we increase the intrinsic growth rate of the species as $r_1(t) = 0.18 + 0.02\sin(t)$, $r_2(t) = 0.28 + 0.03\cos(t)$, while the values of other parameters are the same as above, then one can calculate $b_1^* = 0.02 > 0$, $b_2^* = 0.04 > 0$ at this time, which satisfies the condition of Theorem 2.3. According to the theorem, at least one of the two species will be weakly persistent in the mean. On the other hand, from the stochastic simulation of this case (see Figure 4), we can observe that x_1 is weakly persistent.

If we go on increasing the intrinsic growth rate of the species as $r_1(t) = 0.40 + 0.02\sin(t)$, $r_2(t) = 0.48 + 0.03\cos(t)$, while the values of other parameters are the same as above, then one can calculate that $b_{1*} = b_{2*} = 0.01 > 0$ at this time, which satisfies the condition of Corollary 2.3, then system (4) should be persistent in the mean almost surely by this corollary. And this is also proved by our stochastic numerical simulation (see Figure 5).

Example 3.3 For system (4), we set another series of parameters as follows:

$$r_1(t) = 0.80 + 0.01\sin(t), \qquad r_2(t) = 0.70 + 0.05\cos(t);$$
$$\sigma_1^2(t) = 0.12 + 0.02\sin(t), \qquad \sigma_2^2(t) = 0.10 + 0.02\cos(t);$$
$$h_{1k} = h_{2k} = e^{\frac{(-1)^k}{k^2}} - 1, \quad t_k = k, k = 1, 2, \ldots.$$

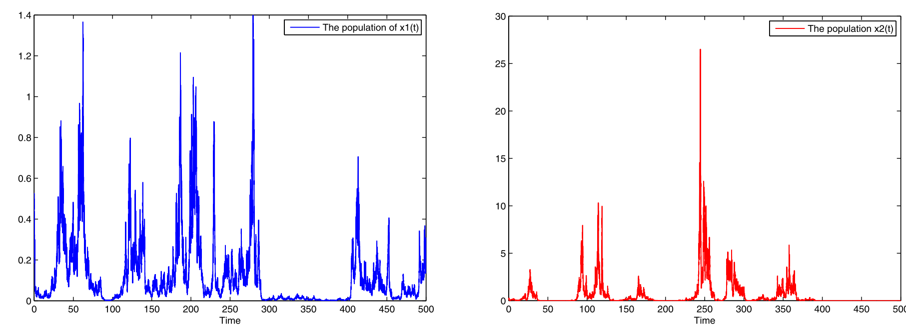

Figure 4 The time series of system (4) with $\sigma_1^2(t) = 0.32 + 0.04\sin(t)$, $\sigma_2^2(t) = 0.48 + 0.06\cos(t)$, $h_{1k} = h_{2k} = -1 + e^{\frac{(-1)^k}{k^2}}$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.

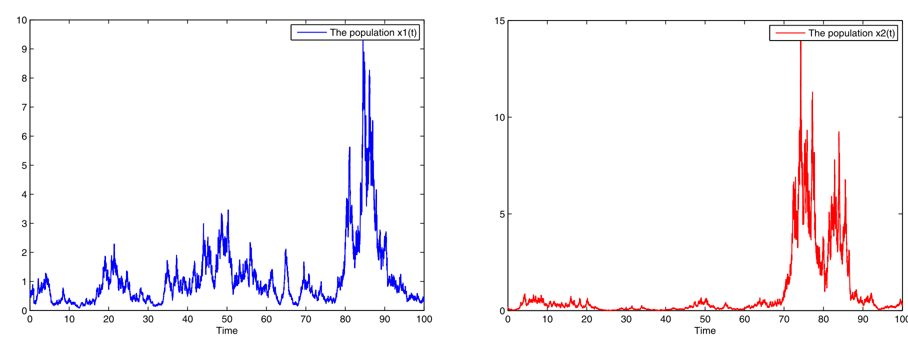

Figure 5 The time series of system (4) with $\sigma_1^2(t) = 0.32 + 0.04\sin(t)$, $\sigma_2^2(t) = 0.48 + 0.06\cos(t)$, $h_{1k} = h_{2k} = -1 + e^{\frac{(-1)^k}{k^2}}$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.

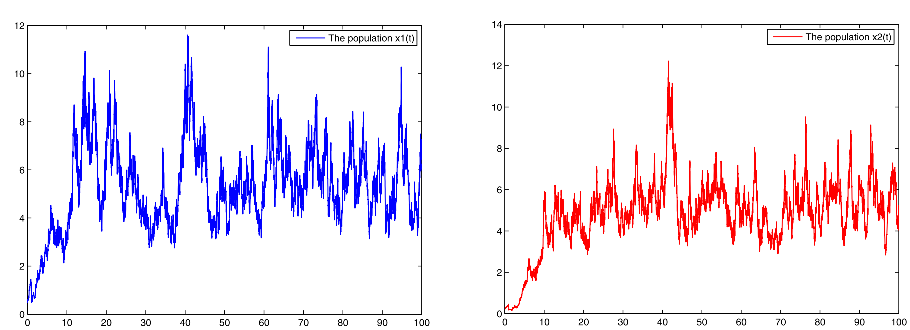

Figure 6 The time series of system (4) with $\sigma_1^2(t) = 0.12 + 0.02\sin(t)$, $\sigma_2^2(t) = 0.1 + 0.02\cos(t)$, $h_{1k} = h_{2k} = -1 + e^{\frac{(-1)^k}{k^2}}$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.

Further, by a simple computation, we can verify that $e^{-1} < \prod_{0<t_k<t}(1 + h_{ik}) < e^{-0.75}$ and $b_1(t) = r_1(t) - c_1(t) - 0.5\sigma_1^2(t) = 0.52 - 0.02\sin t$, $b_2(t) = r_2(t) - c_2(t) - 0.5\sigma_2^2(t) = 0.64 - 0.03\cos t$, which leads to $(\sigma_1^u)^2 = 0.14 < b_1^l = 0.5$, $(\sigma_2^u)^2 = 0.12 < b_2^l = 0.61$. That is to say, conditions (H1) and (H2) in Theorem 2.5 hold, then the system is stochastic permanent, and this is also proved by our stochastic numerical simulation (see Figure 6).

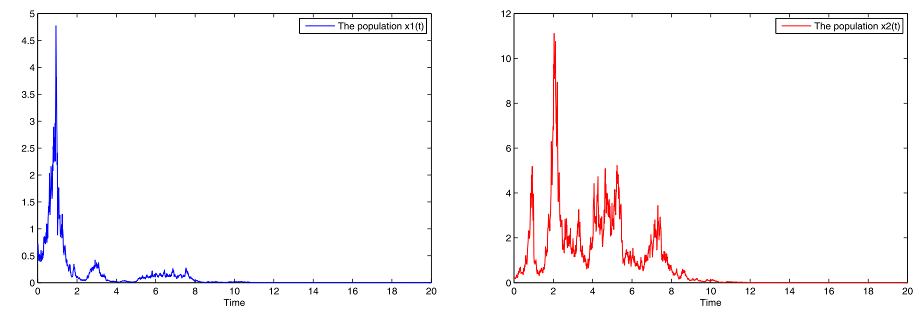

Figure 7 **The time series of system (4) with $\sigma_1^2(t) = 2.12 + 0.02\sin(t)$, $\sigma_2^2(t) = 1.92 + 0.02\cos(t)$, $h_{1k} = h_{2k} = -1 + e^{\frac{(-1)^k}{k^2}}$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.**

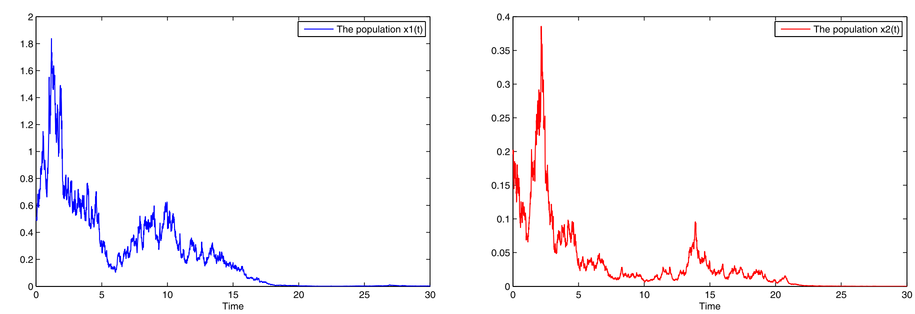

Figure 8 **The time series of system (4) with $\sigma_1^2(t) = 0.32 + 0.04\sin(t)$, $\sigma_2^2(t) = 0.48 + 0.06\cos(t)$, $h_{1k} = h_{2k} = e^{-0.2} - 1$, $\Delta t = 0.01$, $k = 1, 2, \ldots$.**

However, if we suppose that the white noises are increased as $\sigma_1^2(t) = 2.12 + 0.02\sin(t)$, $\sigma_2^2(t) = 1.92 + 0.02\cos(t)$, then it is obvious that both species of system (4) will be extinct rapidly by our stochastic simulation (see Figure 7). This means that the species in the ecological system might become extinct as the white noises increase.

On the other hand, in order to see how the impulsive perturbations will affect the system, we choose the same parameters as those in Example 3.2, but only change the intensity of the impulses to $h_{1k} = h_{2k} = e^{-0.2} - 1$, $t_k = k = 1, 2, \ldots$, then the condition $b_i^* > 0$ does not hold any more. At the moment, we can see that both species become extinct instead of being persistent in the mean by the stochastic simulation (see Figure 8).

3.1 Conclusions

From the above numerical simulations and discussions, we can conclude that both heavy intensity of environmental noises and large impulsive perturbations to the ecological system will lead to the extinction of the species. And this shows that the departments of environment protection should control the environmental noises and impulsive disturbance reasonably to protect the ecological balance.

In addition, as far as the study of population models is concerned, stability of the positive equilibrium state is one of the most interesting topics. For example, models with noise, some of the stochastic models do not keep the positive equilibrium state of the corresponding deterministic systems. And many authors have studied stability in distribution of several stochastic population models in recent years (see [29, 30] *etc.*). Thus, we could try to consider these aspects and get much more interesting results in the future investigation.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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