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Convergence of the compensated split-step θ-method for nonlinear jump-diffusion systems

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Abstract

In this paper, our aim is to develop a compensated split-step θ (CSSθ) method for nonlinear jump-diffusion systems. First, we prove the convergence of the proposed method under a one-sided Lipschitz condition on the drift coefficient, and global Lipschitz condition on the diffusion and jump coefficients. Then we further show that the optimal strong convergence rate of CSSθ can be recovered, if the drift coefficient satisfies a polynomial growth condition. At last, a nonlinear test equation is simulated to verify the results obtained from theory. The results show that the CSSθ method is efficient for simulating the nonlinear jump-diffusion systems.

Keywords: jump-diffusion systems; nonlinear; compensated split-step θ-method; convergence rate

1 Introduction

The aim of this paper is to study the strong convergence of the CSSθ method for the following nonlinear jump-diffusion systems:

$$\mathrm{d}X(t) = f\big(X(t^-)\big)\,\mathrm{d}t + g\big(X(t^-)\big)\,\mathrm{d}W(t) + h\big(X(t^-)\big)\,\mathrm{d}N(t) \tag{1.1}$$

for $t > 0$, with $X(0^-) = X_0 \in \mathbb{R}^n$, where $X(t^-)$ denotes $\lim_{s\to t^-} X(s)$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^n$, $W(t)$ is an m-dimensional Wiener process, and $N(t)$ is a scalar Poisson process with intensity λ.

Most of the studies concerned with numerical analysis for stochastic differential equations with jumps (SDEwJs) are based on the assumption of globally Lipschitz continuous coefficients, for example, [1–6]. However, they cannot be applied to many real-world models, such as financial models [7] and biology models [8], which violate the global Lipschitz assumptions. Hence, the development of numerical methods for SDEwJs under a non-globally Lipschitz condition has become a focus point.

Firstly, we review some achievements of the numerical analysis for highly nonlinear SDEs. Here, we highlight work by Higham *et al.* [9], Hutzenthaler *et al.* [10], Szpruch and Mao [11], Mao and Szpruch [12], Huang [13], Zong *et al.* [14, 15].

However, the development of numerical methods for nonlinear jump-diffusion systems with non-globally Lipschitz continuous coefficients is not as fast as nonlinear SDEs. There are only few results on the numerical methods for nonlinear SDEwJs. For example, Higham and Kloeden proved the strong convergence and its order of the split-step backward Euler (SSBE) method and compensated split-step backward Euler (CSSBE) method for nonlinear jump-diffusion system in [16, 17]. Huang applied the split-step θ (SSθ) method to SDEwJs, but he only studied the mean-square stability of the SSθ method for SDEwJs in [13]. To the best of our knowledge, there is no result about the strong convergence of the CSSθ method for SDEwJs with non-globally Lipschitz continuous coefficients. The main difference of this paper from our previous work [5] is that we deal with SDEwJs with non-globally Lipschitz condition on the drift coefficient f.

The outline of the paper is as follows. In Section 2, we introduce some notions and assumptions for SDEwJs. In Section 3, we construct the CSSθ method for nonlinear SDEwJs. In Section 4, the strong convergence of the numerical solutions produced by the CSSθ method is investigated. The convergence rate is studied in Section 5. Finally, a nonlinear numerical experiment is given to verify the convergence and efficiency of the proposed method.

2 Conditions on the SDEwJs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which satisfies the usual conditions, *i.e.*, the filtration is continuous on the right and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\langle \cdot, \cdot \rangle$ denote the Euclidean scalar product, and $|\cdot|$ denote both the Euclidean vector norm in $\mathbb{R}^n$ and the Frobenius matrix norm in $\mathbb{R}^{n\times m}$. For simplicity, we also denote $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$.

Now, we give the following assumptions on the coefficients f, g and h.

Assumption 2.1 The functions f, g, h in (1.1) are C^1, there exist constants K, L_g and $L_h > 0$, such that the drift coefficient f satisfies a one-sided Lipschitz condition,

$$\bigl\langle x - y, f(x) - f(y)\bigr\rangle \leq K|x - y|^2, \quad \forall x, y \in \mathbb{R}^n, \tag{2.1}$$

and the diffusion and jump coefficients satisfy the global Lipschitz conditions,

$$\bigl|g(x) - g(y)\bigr|^2 \leq L_g|x - y|^2, \quad \forall x, y \in \mathbb{R}^n, \tag{2.2}$$

$$\bigl|h(x) - h(y)\bigr|^2 \leq L_h|x - y|^2, \quad \forall x, y \in \mathbb{R}^n. \tag{2.3}$$

We also assume that all moments of the initial solution are bounded, that is, for any $p \in [1, +\infty)$ there exists a positive constant C, such that

$$\mathbb{E}|Y_0|^p \leq C. \tag{2.4}$$

Lemma 2.1 *Under Assumption* 2.1, *equation* (1.1) *has a unique cadlag solution on* $[0, +\infty)$.

Proof See [16], and for a more relaxed conditions see [18]. □

From Assumption 2.1, we have the following estimates:

$$\bigl\langle x, f(x)\bigr\rangle = \bigl\langle x, f(x) - f(0) + f(0)\bigr\rangle \leq \left(K + \frac{1}{2}\right)|x|^2 + \frac{1}{2}\bigl|f(0)\bigr|^2, \tag{2.5}$$

$$\left|g(x)\right|^2 = \left|g(x) - g(0) + g(0)\right|^2 \le 2L_g|x|^2 + 2\left|g(0)\right|^2, \tag{2.6}$$

$$\left|h(x)\right|^2 = \left|h(x) - h(0) + h(0)\right|^2 \le 2L_h|x|^2 + 2\left|h(0)\right|^2. \tag{2.7}$$

It follows that

$$\langle x, f(x)\rangle \vee \left|g(x)\right|^2 \vee \left|h(x)\right|^2 \le L\left(1 + |x|^2\right), \tag{2.8}$$

where $L = \max\{(K + \frac{1}{2}), 2L_g, 2L_h, \frac{1}{2}|f(0)|^2, 2|g(0)|^2, 2|h(0)|^2\}$.

3 The compensated split-step θ-method

First defining

$$f_\lambda := f(x) + \lambda h(x),$$

we can rewrite the jump-diffusion system (1.1) in the following form:

$$\mathrm{d}X(t) = f_\lambda\left(X\left(t^-\right)\right)\mathrm{d}t + g\left(X\left(t^-\right)\right)\mathrm{d}W(t) + h\left(X\left(t^-\right)\right)\mathrm{d}\tilde{N}(t), \tag{3.1}$$

where

$$\tilde{N}(t) := N(t) - \lambda t,$$

is the compensated Poisson process.

Note that f_λ satisfies the one-sided Lipschitz condition with lager constant; that is,

$$\begin{aligned}\langle x - y, f_\lambda(x) - f_\lambda(y)\rangle &\le (K + \lambda\sqrt{L_h})|x - y|^2 \\ &:= K_\lambda|x - y|^2, \quad \forall x, y \in \mathbb{R}^n.\end{aligned} \tag{3.2}$$

Then we can get

$$\langle x, f_\lambda(x)\rangle \vee \left|g(x)\right|^2 \vee \left|h(x)\right|^2 \le L_\lambda\left(1 + |x|^2\right), \tag{3.3}$$

where $L_\lambda = \max\{(K + \lambda\sqrt{L_h} + \frac{1}{2}), 2L_g, 2L_h, \frac{1}{2}|f_\lambda(0)|^2, 2|g(0)|^2, 2|h(0)|^2\}$.

Now we define the CSSθ method for the jump-diffusion system (1.1) by $Y_0 = X(0)$ and

$$Y_n^* = Y_n + \theta f_\lambda\left(Y_n^*\right)\Delta t, \tag{3.4}$$

$$Y_{n+1} = Y_n + f_\lambda\left(Y_n^*\right)\Delta t + g\left(Y_n^*\right)\Delta W_n + h\left(Y_n^*\right)\Delta\tilde{N}_n, \tag{3.5}$$

where $\theta \in [0,1]$, $\Delta t > 0$, Y_n is the numerical approximation of $X(t_n)$ with $t_n = n \cdot \Delta t$. Moreover, $\Delta W_n := W(t_{n+1}) - W(t_n)$, $\Delta\tilde{N}_n := \tilde{N}(t_{n+1}) - \tilde{N}(t_n)$ are independent increments of the Wiener process and Poisson process, respectively.

If we have $\theta = 1$, the CSSθ method becomes the CSSBE method in [16].

Since the CSSθ method is an implicit scheme, we need to make sure that equation (3.4) has a unique solution Y_n^* given Y_n.

In fact, under the one-sided Lipschitz condition (3.2) with $\theta \Delta t K_\lambda < 1$, equation (3.4) admits a unique solution (see [12]). Meanwhile, if $K_\lambda < 0$, then $\theta \Delta t K_\lambda < 1$ holds for any $\Delta t > 0$. Hence, we define

$$\Delta = \begin{cases} \infty, & \text{if } K_\lambda < 0, \text{ or } \theta = 0, \\ \frac{1}{\theta K_\lambda}, & \text{if } K_\lambda > 0, \theta \in (0,1]. \end{cases} \tag{3.6}$$

From now on we always assume that $\Delta t \leq \Delta$.

4 Strong convergence on finite time interval $[0, T]$

First, for $t \in [t_n, t_{n+1})$, we define the step function:

$$Y(t) = \sum_{n=0}^{N_T-1} Y_n^* I_{[n\Delta,(n+1)\Delta)}(t), \tag{4.1}$$

where N_T is the largest number such that $N_T \Delta t \leq T$, and I_A is the indicator function for the set A, *i.e.*,

$$I_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Then we define the continuous-time approximations

$$\begin{aligned} \overline{Y}(t) = {} & Y_n + f_\lambda\bigl(Y_n^*\bigr)(t - t_n) + g\bigl(Y_n^*\bigr)\bigl(W(t) - W(t_n)\bigr) \\ & + h\bigl(Y_n^*\bigr)\bigl(\tilde{N}(t) - \tilde{N}(t_n)\bigr), \quad t \in [t_n, t_{n+1}). \end{aligned} \tag{4.2}$$

Thus we can rewrite (4.2) in integral form:

$$\begin{aligned} \overline{Y}(t) = {} & Y_0 + \int_0^t f_\lambda\bigl(Y\bigl(s^-\bigr)\bigr)\,\mathrm{d}s + \int_0^t g\bigl(Y\bigl(s^-\bigr)\bigr)\,\mathrm{d}W(s) \\ & + \int_0^t h\bigl(Y\bigl(s^-\bigr)\bigr)\,\mathrm{d}\tilde{N}(s). \end{aligned} \tag{4.3}$$

It is easy to verify that $\overline{Y}(t_n) = Y_n$, that is, $\overline{Y}(t)$ is a continuous-time extension of the discrete approximation $\{Y_n\}$.

Now we will prove the strong convergence of the CSSθ method. The main technique of the following proof is based on the fundamental papers [9, 13, 16], we refer to them for a fuller description of some of the technical details.

The following two lemmas show the pth moment properties of the true solutions and numerical solutions.

Lemma 4.1 *Let Assumption* 2.1 *hold, and* $0 < \theta \leq 1, p \geq 1, 0 < \Delta t < \min\{1, \frac{1}{2\theta L_\lambda}\}$, *then there exists a positive constant A independent of* N_T *such that*

$$\mathbb{E}\Bigl(\sup_{0 \leq n\Delta t \leq T} |Y_n|^{2p}\Bigr) \vee \mathbb{E}\Bigl(\sup_{0 \leq n\Delta t \leq T} \bigl|Y_n^*\bigr|^{2p}\Bigr) < A,$$

where Y_n^* *and* Y_n *are produced by* (3.4) *and* (3.5).

Proof In the following we assume that M is a positive integer such that $n\Delta t \le M\Delta t \le T$.

Squaring both sides of (3.4), we find

$$\begin{aligned} \left|Y_n^*\right|^2 &= \left|Y_n + \theta \Delta t f_\lambda\left(Y_n^*\right)\right|^2 \\ &= |Y_n|^2 + \theta^2 \Delta t^2 \left|f_\lambda\left(Y_n^*\right)\right|^2 + 2\theta \Delta t \left\langle Y_n, f_\lambda\left(Y_n^*\right)\right\rangle \end{aligned} \tag{4.4}$$

and

$$\left\langle Y_n, f_\lambda\left(Y_n^*\right)\right\rangle = \left\langle Y_n^*, f_\lambda\left(Y_n^*\right)\right\rangle - \theta \Delta t \left\langle f_\lambda\left(Y_n^*\right), f_\lambda\left(Y_n^*\right)\right\rangle. \tag{4.5}$$

Substituting (4.5) into (4.4), we have

$$\begin{aligned} \left|Y_n^*\right|^2 &= |Y_n|^2 - \theta^2 \Delta t^2 \left\langle f_\lambda\left(Y_n^*\right), f_\lambda\left(Y_n^*\right)\right\rangle + 2\theta \Delta t \left\langle Y_n^*, f_\lambda\left(Y_n^*\right)\right\rangle \\ &\le |Y_n|^2 + 2\theta \Delta t \left\langle Y_n^*, f_\lambda\left(Y_n^*\right)\right\rangle \\ &\le |Y_n|^2 + 2\theta \Delta t L_\lambda \left(1 + \left|Y_n^*\right|^2\right), \end{aligned} \tag{4.6}$$

which gives

$$\begin{aligned} \left|Y_n^*\right|^2 &\le \frac{1}{1 - 2\theta \Delta t L_\lambda} |Y_n|^2 + \frac{2\theta \Delta t L_\lambda}{1 - 2\theta \Delta t L_\lambda} \\ &= |Y_n|^2 + \frac{2\theta \Delta t L_\lambda}{1 - 2\theta \Delta t L_\lambda} |Y_n|^2 + \frac{2\theta \Delta t L_\lambda}{1 - 2\theta \Delta t L_\lambda} \\ &= |Y_n|^2 + \alpha |Y_n|^2 + \alpha \\ &= \beta |Y_n|^2 + \alpha, \end{aligned} \tag{4.7}$$

where $\alpha = \frac{2\theta \Delta t L_\lambda}{1-2\theta \Delta t L_\lambda}$, $\beta = 1 + \alpha$. By (3.5) we have

$$\begin{aligned} |Y_{n+1}|^2 &= \left|Y_n + f_\lambda\left(Y_n^*\right)\Delta t + g\left(Y_n^*\right)\Delta W_n + h\left(Y_n^*\right)\Delta \tilde{N}_n\right|^2 \\ &= |Y_n|^2 + \left|f_\lambda\left(Y_n^*\right)\Delta t\right|^2 + \left|g\left(Y_n^*\right)\Delta W_n\right|^2 + \left|h\left(Y_n^*\right)\Delta \tilde{N}_n\right|^2 \\ &\quad + 2\left\langle Y_n, f_\lambda\left(Y_n^*\right)\Delta t\right\rangle + 2\left\langle Y_n, g\left(Y_n^*\right)\Delta W_n\right\rangle + 2\left\langle Y_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\ &\quad + 2\left\langle f_\lambda\left(Y_n^*\right)\Delta t, g\left(Y_n^*\right)\Delta W_n\right\rangle \\ &\quad + 2\left\langle f_\lambda\left(Y_n^*\right)\Delta t, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\ &\quad + 2\left\langle g\left(Y_n^*\right)\Delta W_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle. \end{aligned} \tag{4.8}$$

Then by (3.3), (3.4) and (4.5), we get

$$\begin{aligned} |Y_{n+1}|^2 &\le |Y_n|^2 + \left|g\left(Y_n^*\right)\Delta W_n\right|^2 + \left|h\left(Y_n^*\right)\Delta \tilde{N}_n\right|^2 + 2\left\langle Y_n^*, f_\lambda\left(Y_n^*\right)\Delta t\right\rangle \\ &\quad + 2\left\langle Y_n, g\left(Y_n^*\right)\Delta W_n\right\rangle + 2\left\langle Y_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\ &\quad + 2\left\langle \frac{Y_n^* - Y_n}{\theta}, g\left(Y_n^*\right)\Delta W_n\right\rangle \\ &\quad + 2\left\langle \frac{Y_n^* - Y_n}{\theta}, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\ &\quad + 2\left\langle g\left(Y_n^*\right)\Delta W_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \end{aligned}$$

$$\begin{aligned}
&\leq |Y_n|^2 + \left|g\left(Y_n^*\right)\Delta W_n\right|^2 + \left|h\left(Y_n^*\right)\Delta \tilde{N}_n\right|^2 + 2L_\lambda \Delta t\left(1 + \left|Y_n^*\right|^2\right) \\
&\quad + 2\left(1 - \frac{1}{\theta}\right)\left\langle Y_n, g\left(Y_n^*\right)\Delta W_n\right\rangle \\
&\quad + 2\left(1 - \frac{1}{\theta}\right)\left\langle Y_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\
&\quad + \frac{2}{\theta}\left\langle Y_n^*, g\left(Y_n^*\right)\Delta W_n\right\rangle + \frac{2}{\theta}\left\langle Y_n^*, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\
&\quad + 2\left\langle g\left(Y_n^*\right)\Delta W_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle.
\end{aligned} \tag{4.9}$$

Hence from (4.7) we have

$$\begin{aligned}
|Y_{n+1}|^2 &\leq |Y_n|^2 + 2\beta L_\lambda \Delta t|Y_n|^2 + 2(\alpha + 1)L_\lambda \Delta t \\
&\quad + \left|g\left(Y_n^*\right)\Delta W_n\right|^2 + \left|h\left(Y_n^*\right)\Delta \tilde{N}_n\right|^2 \\
&\quad + 2\left(1 - \frac{1}{\theta}\right)\left\langle Y_n, g\left(Y_n^*\right)\Delta W_n\right\rangle \\
&\quad + 2\left(1 - \frac{1}{\theta}\right)\left\langle Y_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\
&\quad + \frac{2}{\theta}\left\langle Y_n^*, g\left(Y_n^*\right)\Delta W_n\right\rangle + \frac{2}{\theta}\left\langle Y_n^*, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle \\
&\quad + 2\left\langle g\left(Y_n^*\right)\Delta W_n, h\left(Y_n^*\right)\Delta \tilde{N}_n\right\rangle.
\end{aligned} \tag{4.10}$$

By the recursive calculation, we can get

$$\begin{aligned}
|Y_n|^2 &\leq |Y_0|^2 + 2\beta L_\lambda \Delta t\sum_{j=0}^{n-1}|Y_j|^2 + 2(\alpha + 1)L_\lambda T \\
&\quad + \sum_{j=0}^{n-1}\left|g\left(Y_j^*\right)\Delta W_j\right|^2 + \sum_{j=0}^{n-1}\left|h\left(Y_j^*\right)\Delta \tilde{N}_j\right|^2 \\
&\quad + 2\left(1 - \frac{1}{\theta}\right)\sum_{j=0}^{n-1}\left\langle Y_j, g\left(Y_j^*\right)\Delta W_j\right\rangle \\
&\quad + 2\left(1 - \frac{1}{\theta}\right)\sum_{j=0}^{n-1}\left\langle Y_j, h\left(Y_j^*\right)\Delta \tilde{N}_j\right\rangle \\
&\quad + \frac{2}{\theta}\sum_{j=0}^{n-1}\left\langle Y_j^*, g\left(Y_j^*\right)\Delta W_j\right\rangle + \frac{2}{\theta}\sum_{j=0}^{n-1}\left\langle Y_j^*, h\left(Y_j^*\right)\Delta \tilde{N}_j\right\rangle \\
&\quad + 2\sum_{j=0}^{n-1}\left\langle g\left(Y_j^*\right)\Delta W_j, h\left(Y_j^*\right)\Delta \tilde{N}_j\right\rangle.
\end{aligned} \tag{4.11}$$

Raising both sides to the power p, we can obtain

$$\begin{aligned}
|Y_n|^{2p} &\leq 10^{p-1}\Bigg\{|Y_0|^{2p} + n^{p-1}(2\beta L_\lambda \Delta t)^p\sum_{j=0}^{n-1}|Y_j|^{2p} + \left(2(\alpha + 1)L_\lambda T\right)^p \\
&\quad + n^{p-1}\sum_{j=0}^{n-1}\left|g\left(Y_j^*\right)\Delta W_j\right|^{2p} + n^{p-1}\sum_{j=0}^{n-1}\left|h\left(Y_j^*\right)\Delta \tilde{N}_j\right|^{2p}
\end{aligned}$$

$$+ 2^p\left(\frac{1}{\theta}-1\right)^p\left|\sum_{j=0}^{n-1}\langle Y_j, g(Y_j^*)\Delta W_j\rangle\right|^p$$
$$+ 2^p\left(\frac{1}{\theta}-1\right)^p\left|\sum_{j=0}^{n-1}\langle Y_j, h(Y_j^*)\Delta \tilde{N}_j\rangle\right|^p$$
$$+ 2^p\left(\frac{2}{\theta}\right)^p\left|\sum_{j=0}^{n-1}\langle Y_j^*, g(Y_j^*)\Delta W_j\rangle\right|^p$$
$$+ 2^p\left(\frac{2}{\theta}\right)^p\left|\sum_{j=0}^{n-1}\langle Y_j^*, h(Y_j^*)\Delta \tilde{N}_j\rangle\right|^p$$
$$+ 2^p\left|\sum_{j=0}^{n-1}\langle g(Y_j^*)\Delta W_j, h(Y_j^*)\Delta \tilde{N}_j\rangle\right|^p\Bigg\}. \tag{4.12}$$

Notice that

$$\mathbb{E}\sup_{0\le n\le M}\left[\sum_{j=0}^{n-1}|Y_j|^{2p}\right] = \sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \tag{4.13}$$

Thus, for $0 \le M \le N_T$, we obtain

$$\mathbb{E}\sup_{0\le n\le M}|Y_n|^{2p} \le 10^{p-1}\Bigg\{|Y_0|^{2p} + n^{p-1}(2\beta L_\lambda \Delta t)^p\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p} + \big(2(\alpha+1)L_\lambda T\big)^p$$
$$+ n^{p-1}\mathbb{E}\sup_{0\le n\le M}\sum_{j=0}^{n-1}\big|g(Y_j^*)\Delta W_j\big|^{2p}$$
$$+ n^{p-1}\mathbb{E}\sup_{0\le n\le M}\sum_{j=0}^{n-1}\big|h(Y_j^*)\Delta \tilde{N}_j\big|^{2p}$$
$$+ 2^p\left(\frac{1}{\theta}-1\right)^p\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\langle Y_j, g(Y_j^*)\Delta W_j\rangle\right|^p$$
$$+ 2^p\left(\frac{1}{\theta}-1\right)^p\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\langle Y_j, h(Y_j^*)\Delta \tilde{N}_j\rangle\right|^p$$
$$+ 2^p\left(\frac{2}{\theta}\right)^p\mathbb{E}\left|\sum_{j=0}^{n-1}\langle Y_j^*, g(Y_j^*)\Delta W_j\rangle\right|^p$$
$$+ 2^p\left(\frac{2}{\theta}\right)^p\mathbb{E}\left|\sum_{j=0}^{n-1}\langle Y_j^*, h(Y_j^*)\Delta \tilde{N}_j\rangle\right|^p$$
$$+ 2^p\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\langle g(Y_j^*)\Delta W_j, h(Y_j^*)\Delta \tilde{N}_j\rangle\right|^p\Bigg\}. \tag{4.14}$$

To bound the fourth term on the right side of (4.14), we note that $Y_n^* \in \mathcal{F}_{t_n}$, ΔW_n is independent of $\mathcal{F}_{t_n}$ and $\mathbb{E}|\Delta W_j|^{2p} \le c_p\Delta t^p$, where c_p is a constant. Meanwhile, letting

$C = C(p, T, L_\lambda, \theta)$ be a constant that may change from line to line,

$$\begin{aligned} n^{p-1}\mathbb{E}\sup_{0\le n\le M}\sum_{j=0}^{n-1}\bigl|g(Y_j^*)\Delta W_j\bigr|^{2p} &= n^{p-1}\mathbb{E}\sum_{j=0}^{M-1}\bigl|g(Y_j^*)\Delta W_j\bigr|^{2p} \\ &\le n^{p-1}\sum_{j=0}^{M-1}\mathbb{E}\bigl|g(Y_j^*)\bigr|^{2p}\mathbb{E}|\Delta W_j|^{2p} \\ &\le n^{p-1}c_p\Delta t^p L_\lambda^p\sum_{j=0}^{M-1}\mathbb{E}\bigl[1+\bigl|Y_j^*\bigr|^2\bigr]^p \\ &\le T^{p-1}c_p\Delta t L_\lambda^p\sum_{j=0}^{M-1}\mathbb{E}\bigl[1+\beta|Y_j|^2+\alpha\bigr]^p \\ &\le 2^{p-1}T^{p-1}c_p\Delta t L_\lambda^p\sum_{j=0}^{M-1}\mathbb{E}\bigl[\beta^p+\beta^p|Y_j|^{2p}\bigr] \\ &\le (2T)^{p-1}c_p\Delta t L_\lambda^p\beta^p\left(M+\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}\right) \\ &\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \end{aligned} \tag{4.15}$$

Using a similar approach to the fifth term and noticing that $\mathbb{E}|\Delta\widetilde{N}_j|^{2p}\le c_p\Delta t^p$, we have

$$n^{p-1}\mathbb{E}\sum_{j=0}^{M-1}\bigl|h(Y_j^*)\Delta\widetilde{N}_j\bigr|^{2p}\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \tag{4.16}$$

Now we bound the sixth term in (4.14), using the Burkholder-Davis-Gundy inequality

$$\begin{aligned} \mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\bigl\langle Y_j, g(Y_j^*)\Delta W_j\bigr\rangle\right|^p &\le C\mathbb{E}\left[\sum_{j=0}^{M-1}|Y_j|^2\bigl|g(Y_j^*)\bigr|^2\Delta t\right]^{p/2} \\ &\le C\Delta t^{p/2}M^{p/2-1}L_\lambda^p\mathbb{E}\sum_{j=0}^{M-1}|Y_j|^p\bigl(1+\bigl|Y_j^*\bigr|^2\bigr)^{p/2} \\ &\le CT^{p/2-1}\Delta t\mathbb{E}\sum_{j=0}^{M-1}\frac{|Y_j|^{2p}+(1+|Y_j^*|^2)^p}{2} \\ &\le C\Delta t\mathbb{E}\sum_{j=0}^{M-1}\bigl[|Y_j|^{2p}+2^{p-1}+2^{p-1}\bigl|Y_j^*\bigr|^{2p}\bigr] \\ &\le C\Delta t\mathbb{E}\sum_{j=0}^{M-1}\bigl[|Y_j|^{2p}+2^{p-1}\bigl(\beta|Y_j|^2+\alpha\bigr)^p+2^{p-1}\bigr] \\ &\le C+C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \end{aligned} \tag{4.17}$$

Similar to the sixth term, we can bound the seventh term

$$\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\langle Y_j, h(Y_j^*)\Delta\widetilde{N}_j\rangle\right|^p \le C + C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \tag{4.18}$$

Also similar to the sixth term, we can bound the eighth term in (4.14),

$$\begin{aligned}
\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\langle Y_j^*, g(Y_j^*)\Delta W_j\rangle\right|^p &\le C\mathbb{E}\left[\sum_{j=0}^{M-1}\left|Y_j^*\right|^2\left|g(Y_j^*)\right|^2\Delta t\right]^{p/2} \\
&\le C\Delta t^{p/2}M^{p/2-1}L_\lambda^p\mathbb{E}\sum_{j=0}^{M-1}\left|Y_j^*\right|^p\left(1+\left|Y_j^*\right|^2\right)^{p/2} \\
&\le CT^{p/2-1}\Delta t\mathbb{E}\sum_{j=0}^{M-1}\frac{|Y_j^*|^{2p}+(1+|Y_j^*|^2)^p}{2} \\
&\le C\Delta t\mathbb{E}\sum_{j=0}^{M-1}\left[\left|Y_j^*\right|^{2p}+2^{p-1}+2^{p-1}\left|Y_j^*\right|^{2p}\right] \\
&\le C\Delta t\mathbb{E}\sum_{j=0}^{M-1}\left[\left(1+2^{p-1}\right)\left(\beta|Y_j|^2+\alpha\right)^p+2^{p-1}\right] \\
&\le C + C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p},
\end{aligned} \tag{4.19}$$

and the ninth term,

$$\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\langle Y_j^*, h(Y_j^*)\Delta\widetilde{N}_j\rangle\right|^p \le C + C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \tag{4.20}$$

Finally we bound the tenth term in (4.14), by (4.15)-(4.16); we have

$$\begin{aligned}
&\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\langle g(Y_j^*)\Delta W_j, h(Y_j^*)\Delta\tilde{N}_j\rangle\right|^p \\
&\quad\le 2^{-p}\mathbb{E}\sup_{0\le n\le M}\left|\sum_{j=0}^{n-1}\left(\left|g(Y_j^*)\Delta W_j\right|^2+\left|h(Y_j^*)\Delta\tilde{N}_j\right|^2\right)\right|^p \\
&\quad\le 2^{-p}M^{p-1}\mathbb{E}\sup_{0\le n\le M}\sum_{j=0}^{n-1}\left(\left|g(Y_j^*)\Delta W_j\right|^2+\left|h(Y_j^*)\Delta\tilde{N}_j\right|^2\right)^p \\
&\quad\le 2^{-1}M^{p-1}\mathbb{E}\sup_{0\le n\le M}\sum_{j=0}^{n-1}\left(\left|g(Y_j^*)\Delta W_j\right|^{2p}+\left|h(Y_j^*)\Delta\tilde{N}_j\right|^{2p}\right) \\
&\quad\le C + C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}.
\end{aligned} \tag{4.21}$$

Combining (4.15)-(4.21) into (4.14), we obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le n\le M}|Y_n|^{2p} &\le C + C\Delta t\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}\\ &\le C + C\Delta t\sum_{j=0}^{M-1}\mathbb{E}\sup_{0\le n\le j}|Y_n|^{2p}. \end{aligned}\tag{4.22}$$

Using the discrete-type Gronwall inequality and noting that $M\Delta t \le T$, we obtain

$$\mathbb{E}\sup_{0\le n\le M}|Y_n|^{2p} \le Ce^{C\Delta tM} \le Ce^{CT}. \tag{4.23}$$

By (4.7), we find that $\mathbb{E}\sup_{0\le n\le M}|Y_n^*|^{2p}$ is also bounded. □

Lemma 4.2 *Let Assumption* 2.1 *hold, and* $0 < \theta \le 1$, $p \ge 1$, $0 < \Delta t < \min\{1, \frac{1}{2\theta L_\lambda}\}$, *then the exact solution of* (3.1) *and the continuous-time extension* (4.3) *satisfy*

$$\mathbb{E}\Big(\sup_{0\le t\le T}\big|X(t)\big|^{2p}\Big) \vee \mathbb{E}\Big(\sup_{0\le t\le T}\big|\overline{Y}(t)\big|^{2p}\Big) < A_1,$$

where A_1 *is a positive constant independent of* N_T.

Proof From Lemma 1 in [16], we can see that $\mathbb{E}(\sup_{0\le t\le T}|X(t)|^{2p})$ is bounded. Now we prove that $\mathbb{E}(\sup_{0\le t\le T}|\overline{Y}(t)|^{2p})$ is bounded.

From (4.2), we obtain

$$\begin{aligned}\overline{Y}(t) = {} & Y_n + f_\lambda\big(Y_n^*\big)(t-t_n) + g\big(Y_n^*\big)\big(W(t)-W(t_n)\big)\\ & + h\big(Y_n^*\big)\big(\tilde{N}(t)-\tilde{N}(t_n)\big), \quad t\in[t_n,t_{n+1}). \end{aligned}\tag{4.24}$$

Let $s\in[0,\Delta t)$, we have

$$\overline{Y}(t_n+s) = Y_n + f_\lambda\big(Y_n^*\big)s + g\big(Y_n^*\big)\Delta W_n(s) + h\big(Y_n^*\big)\Delta\widetilde{N}_n(s), \tag{4.25}$$

where

$$\begin{aligned}&\Delta W_n(s) = W(t_n+s) - W(t_n),\\ &\Delta\widetilde{N}_n(s) = \widetilde{N}(t_n+s) - \widetilde{N}(t_n).\end{aligned}$$

However, $Y_n^* = Y_n + \theta\Delta t f_\lambda(Y_n^*)$ and so, for $a = s/\Delta t$, we can rewrite equation (4.25) in the following form:

$$\overline{Y}(t_n+s) = \frac{a}{\theta}Y_n^* + \left(1-\frac{a}{\theta}\right)Y_n + g\big(Y_n^*\big)\Delta W_n(s) + h\big(Y_n^*\big)\Delta\widetilde{N}_n(s). \tag{4.26}$$

By (4.7), we have

$$\big|\overline{Y}(t_n+s)\big|^2 \le C\big[1 + |Y_n|^2 + \big|g\big(Y_n^*\big)\Delta W_n(s)\big|^2 + \big|h\big(Y_n^*\big)\Delta\widetilde{N}_n(s)\big|^2\big]. \tag{4.27}$$

Thus

$$
\begin{aligned}
&\sup_{0\le t\le T}\left|\overline{Y}(t)\right|^{2p} \\
&\quad\le \sup_{0\le n\Delta t\le T}\sup_{0\le s\le \Delta t}\left|\overline{Y}(t_n+s)\right|^{2p} \\
&\quad\le \sup_{0\le n\Delta t\le T}\sup_{0\le s\le \Delta t} C\left[1+|Y_n|^{2p}+\left|g\left(Y_n^*\right)\Delta W_n(s)\right|^{2p}+\left|h\left(Y_n^*\right)\Delta \widetilde{N}_n(s)\right|^{2p}\right] \\
&\quad\le C\Bigg[1+\sup_{0\le n\Delta t\le T}|Y_n|^{2p}+\sup_{0\le s\le \Delta t}\sum_{j=0}^{N_T}\left|g\left(Y_j^*\right)\Delta W_j(s)\right|^{2p} \\
&\qquad+\sup_{0\le s\le \Delta t}\sum_{j=0}^{N_T}\left|h\left(Y_j^*\right)\Delta \widetilde{N}_j(s)\right|^{2p}\Bigg].
\end{aligned}
$$

Now using Doob's martingale inequality, (3.3) and Lemma 4.1, we have

$$
\begin{aligned}
\mathbb{E}\sup_{0\le s\le \Delta t}\left|g\left(Y_j^*\right)\Delta W_j(s)\right|^{2p} &\le C\mathbb{E}\left|g\left(Y_j^*\right)\Delta W_j(\Delta t)\right|^{2p} \\
&\le C\mathbb{E}\left|g\left(Y_j^*\right)\right|^{2p}\mathbb{E}\left|\Delta W_j(\Delta t)\right|^{2p} \\
&\le C\left(1+\mathbb{E}\left|Y_j^*\right|^{2p}\right)\Delta t^p \\
&\le C\Delta t.
\end{aligned}
\tag{4.28}
$$

Since the $\Delta\widetilde{N}_j(s)$ is also a martingale, by a similar method, we get

$$
\mathbb{E}\sup_{0\le s\le \Delta t}\left|h\left(Y_j^*\right)\Delta \widetilde{N}_j(s)\right|^{2p}\le C\Delta t. \tag{4.29}
$$

Then by (4.28), (4.29) and Lemma 4.1, combining $N_T\Delta t\le T$, we have

$$
\mathbb{E}\left(\sup_{0\le t\le T}\left|\overline{Y}(t)\right|^{2p}\right)\le A_1.
$$

Then we get the desired results. □

Now we use the above lemmas to prove a strong convergence result.

Remark 4.1 Since $f(x)\in C^1$, *i.e.* $f'(x)$ is continuous, $|f'(x)|$ is bounded locally. Then by the mean value theorem, there exists a positive constant L_R for each $R>0$, such that

$$
\left|f(x)-f(y)\right|=\left|f'(z)\right||x-y|\le L_R|x-y|, \tag{4.30}
$$

for all $x,y,z\in\mathbb{R}^n$ with $|x|\vee|y|\le R$.

We note that the function f_λ in (3.1) automatically inherits this condition, with a larger L_R.

Theorem 4.3 *Under Assumption* 2.1, *let* $0<\theta\le 1$, $0<\Delta t<\min\{1,\frac{1}{2\theta L_\lambda}\}$, *the continuous-time approximate solution* $\overline{Y}(t)$ *defined by* (4.3) *will converge to the true solution* $X(t)$ *of*

(3.1) *in the mean-square sense, i.e.*

$$\lim_{\Delta t\to 0}\mathbb{E}\sup_{0\le t\le T}\left|\overline{Y}(t)-X(t)\right|^2=0. \tag{4.31}$$

Proof First, we define

$$\tau_d:=\inf\{t\ge 0,\left|X(t)\right|\ge d\},\qquad \sigma_d:=\inf\{t\ge 0,\left|\overline{Y}(t)\right|\ge d\},\qquad \upsilon_d=\tau_d\wedge\sigma_d,$$

and let

$$e(t)=\overline{Y}(t)-X(t).$$

Recall the Young inequality: for $\frac{1}{p}+\frac{1}{q}=1$ $(p,q>0)$, we have

$$ab=a\delta^{\frac{1}{p}}\frac{b}{\delta^{\frac{1}{p}}}\le\frac{(a\delta^{\frac{1}{p}})^p}{p}+\frac{b^q}{q\delta^{\frac{q}{p}}}=\frac{a^p\delta}{p}+\frac{b^q}{q\delta^{\frac{q}{p}}},\quad \forall a,b,\delta>0.$$

Thus, for any $\delta>0$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t)\right|^2\Big]\\
&\quad=\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t)\right|^2 I_{\{\tau_d>T\text{ and }\sigma_d>T\}}\Big]+\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t)\right|^2 I_{\{\tau_d\le T\text{ or }\sigma_d\le T\}}\Big]\\
&\quad=\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t)\right|^2 I_{\{\upsilon_d>T\}}\Big]+\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t)\right|^2 I_{\{\tau_d\le T\text{ or }\sigma_d\le T\}}\Big]\\
&\quad\le\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t\wedge\upsilon_d)\right|^2\Big]+\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t)\right|^2 I_{\{\tau_d\le T\text{ or }\sigma_d\le T\}}\Big]\\
&\quad\le\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t\wedge\upsilon_d)\right|^2\Big]+\frac{\delta}{p}\mathbb{E}\Big[\sup_{0\le t\le T}\left|e(t)\right|^{2p}\Big]\\
&\qquad+\frac{1-\frac{1}{p}}{\delta^{\frac{1}{p-1}}}\mathbb{P}\{\tau_d\le T\text{ or }\sigma_d\le T\}.
\end{aligned}\tag{4.32}$$

By Lemma 4.2, then

$$\mathbb{P}\{\tau_d\le T\}\le\mathbb{E}\bigg[I_{\{\tau_d\le T\}}\frac{|X(\tau_d)|^{2p}}{d^{2p}}\bigg]\le\frac{1}{d^{2p}}\mathbb{E}\Big[\sup_{0\le t\le T}\left|X(t)\right|^{2p}\Big]\le\frac{A_1}{d^{2p}}.$$

Similarly, the result can be derived for σ_d

$$\mathbb{P}\{\sigma_d\le T\}=\mathbb{E}\bigg[I_{\{\sigma_d\le T\}}\frac{|\overline{Y}(\sigma_d)|^{2p}}{d^{2p}}\bigg]\le\frac{1}{d^{2p}}\mathbb{E}\Big[\sup_{0\le t\le T}\left|\overline{Y}(t)\right|^{2p}\Big]\le\frac{A_1}{d^{2p}},$$

so that

$$\mathbb{P}\{\sigma_d\le T\,\text{or}\,\nu_d\le T\}\le\mathbb{P}\{\sigma_d\le T\}+\mathbb{P}\{\nu_d\le T\}\le\frac{2A_1}{d^{2p}}.$$

Using the bounds of $X(t)$ and $\overline{Y}(t)$, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|e(t)\big|^{2p}\Big] \le 2^{2p-1}\mathbb{E}\Big[\sup_{0\le t\le T}\big(\big|X(t)\big|^{2p} + \big|\overline{Y}(t)\big|^{2p}\big)\Big] \le 2^{2p}A_1.$$

Substituting the above inequality into (4.32) leads to

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}\big|e(t)\big|^{2}\Big] \\
&\quad\le \mathbb{E}\Big[\sup_{0\le t\le T}\big|\overline{Y}(t\wedge \upsilon_d) - X(t\wedge \upsilon_d)\big|^2\Big] + \frac{2^{2p}\delta A_1}{p} + \frac{2A_1(1-\frac{1}{p})^{2p}}{d}\delta^{\frac{1}{p-1}}.
\end{aligned} \tag{4.33}$$

Now we bound the first term on the right-hand side of (4.33). By the definitions of $X(t)$ and $\overline{Y}(t)$, combining the fact that $0 < \theta \le 1$, we have

$$\begin{aligned}
&\big|\overline{Y}(t\wedge \upsilon_d) - X(t\wedge \upsilon_d)\big|^2 \\
&\quad= \bigg|\int_0^{t\wedge\upsilon_d}\big[f_\lambda\big(Y(s)\big) - f_\lambda\big(X(s)\big)\big]\,\mathrm{d}s \\
&\qquad+ \int_0^{t\wedge\upsilon_d} g\big(Y(s)\big) - g\big(X(s)\big)\,\mathrm{d}W(s) \\
&\qquad+ \int_0^{t\wedge\upsilon_d} h\big(Y(s)\big) - h\big(X(s)\big)\,\mathrm{d}\tilde{N}(s)\bigg|^2 \\
&\quad\le 3\bigg|\int_0^{t\wedge\upsilon_d}\big[f_\lambda\big(Y(s)\big) - f_\lambda\big(X(s)\big)\big]\,\mathrm{d}s\bigg|^2 \\
&\qquad+ 3\bigg|\int_0^{t\wedge\upsilon_d} g\big(Y(s)\big) - g\big(X(s)\big)\,\mathrm{d}W(s)\bigg|^2 \\
&\qquad+ 3\bigg|\int_0^{t\wedge\upsilon_d} h\big(Y(s)\big) - h\big(X(s)\big)\,\mathrm{d}\tilde{N}(s)\bigg|^2.
\end{aligned}$$

For any $\tau \in [0,T]$, using the Cauchy-Schwarz inequality and the Doob martingale inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le \tau}\big|\overline{Y}(t\wedge \upsilon_d) - X(t\wedge \upsilon_d)\big|^2\Big] \\
&\quad\le 3T\mathbb{E}\int_0^{\tau\wedge\upsilon_d}\big|f_\lambda\big(Y(s)\big) - f_\lambda\big(X(s)\big)\big|^2\,\mathrm{d}s \\
&\qquad+ 12\mathbb{E}\int_0^{\tau\wedge\upsilon_d}\big|g\big(Y(s)\big) - g\big(X(s)\big)\big|^2\,\mathrm{d}s \\
&\qquad+ 12\mathbb{E}\lambda\int_0^{\tau\wedge\upsilon_d}\big|h\big(Y(s)\big) - h\big(X(s)\big)\big|^2\,\mathrm{d}s.
\end{aligned}$$

Applying the local Lipschitz condition (4.30) and Assumption 2.1, we get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le \tau}\big|\overline{Y}(t\wedge \upsilon_d) - X(t\wedge \upsilon_d)\big|^2\Big] \\
&\quad\le (3TL_R + 12L_g + 12L_h\lambda)\mathbb{E}\int_0^{\tau\wedge\upsilon_d}\big|Y(s) - X(s)\big|^2\,\mathrm{d}s
\end{aligned}$$

$$\leq 2(3TL_R + 12L_g + 12L_h\lambda)\bigg[\mathbb{E}\int_0^{\tau\wedge\upsilon_d}\big|Y(s)-\overline{Y}(s)\big|^2\,\mathrm{d}s + \int_0^{\tau}\mathbb{E}\sup_{0\leq r\leq s}\big|\overline{Y}(r\wedge\upsilon_d)-X(r\wedge\upsilon_d)\big|^2\,\mathrm{d}s\bigg]. \tag{4.34}$$

To bound the first term inside the parentheses of (4.34), we denote by n_s the integer for which $s\in[t_{n_s},t_{n_{s+1}}]$ and note that

$$\begin{aligned}Y(s)-\overline{Y}(s) &= -f_\lambda\big(Y_{n_s}^*\big)(s-t_{n_s}) - g\big(Y_{n_s}^*\big)\big(W(s)-W(t_{n_s})\big)\\ &\quad - h\big(Y_{n_s}^*\big)\big(\tilde{N}(s)-\tilde{N}(t_{n_s})\big),\end{aligned}$$

and hence that

$$\big|Y(s)-\overline{Y}(s)\big|^2 \leq 3\big[\big|f_\lambda\big(Y_{n_s}^*\big)\Delta t\big|^2 + \big|g\big(Y_{n_s}^*\big)\Delta W_{n_s}\big|^2 + \big|h\big(Y_{n_s}^*\big)\Delta N_{n_s}\big|^2\big].$$

Note that

$$\begin{aligned}\big|f_\lambda\big(Y_{n_s}^*\big)\big|^2 &\leq 2\big[\big|f_\lambda\big(Y_{n_s}^*\big)-f_\lambda(0)\big|^2 + \big|f_\lambda(0)\big|^2\big]\\ &\leq 2L_R\big|Y_{n_s}^*\big|^2 + 2\big|f_\lambda(0)\big|^2.\end{aligned}$$

Thus by the second moments of martingale increments and the moment bound on the numerical solution Y_n^*, we can obtain

$$\mathbb{E}\int_0^{\tau\wedge\upsilon_d}\big|Y(s)-\overline{Y}(s)\big|^2\,\mathrm{d}s \leq C_1\Delta t,$$

for a constant $C_1 = C_1(R,T,A)$. Substituting this bound into (4.34) and applying the continuous Gronwall inequality gives

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\big|\overline{Y}(t\wedge\upsilon_d)-X(t\wedge\upsilon_d)\big|^2\Big] \leq C_2\Delta t e^{(3TL_R+12L_g+12L_h\lambda)T}, \tag{4.35}$$

for a constant $C_2 = C_2(R,T,A)$.

Now combining (4.35) with (4.33), we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\big|e(t)\big|^2\Big] \leq C_2\Delta t e^{(3TL_R+12L_g+12L_h\lambda)T} + \frac{2^{2p}\delta A_1}{p} + \frac{2A_1(1-\frac{1}{p})}{d}^{2p}\delta^{\frac{1}{p-1}}. \tag{4.36}$$

For any given $\varepsilon > 0$, we can choose δ sufficiently small for

$$\frac{2^{2p}\delta A_1}{p} \leq \frac{\varepsilon}{3},$$

and then choose d sufficient large for

$$\frac{2A_1(1-\frac{1}{p})}{d^{2p}\delta^{\frac{1}{p-1}}} < \frac{\varepsilon}{3},$$

and finally choose Δt so that

$$C_2\Delta t e^{(3TL_R+12L_g+12L_h\lambda)T} < \frac{\varepsilon}{3}.$$

Thus $\mathbb{E}[\sup_{0\le t\le T}|e(t)|^2] < \varepsilon$. The proof is completed. □

5 Convergence rate

To prove the convergence rate of the CSSθ method, we give the following assumption.

Assumption 5.1 There exist constants $D \in \mathbb{R}^+$ and $q \in \mathbb{Z}^+$ such that, for all $a, b \in \mathbb{R}^n$,

$$\left|f_\lambda(a) - f_\lambda(b)\right|^2 \le D\left(1 + |a|^q + |b|^q\right)|a-b|^2. \tag{5.1}$$

Firstly, we establish Lemma 5.1 under Assumptions 2.1 and 5.1.

Lemma 5.1 *Under Assumptions* 2.1 *and* 5.1, *let* $0 < \theta \le 1$, $0 < \Delta t < \min\{1, \frac{1}{2\theta L_\lambda}\}$, *for any given integer* $r \ge 2$, *there exists a positive constant* $E = E(r)$ *such that*

$$\mathbb{E}\sup_{0\le t\le T}\left|Y(t) - \overline{Y}(t)\right|^r \le E\Delta t^{\frac{r}{2}}. \tag{5.2}$$

Proof Since for any given $t \in [n\Delta t, (n+1)\Delta t]$, we have $Y(t) = Y_n$, and then by the continuous-time approximate solution $\overline{Y}(t)$ defined by (4.3), we can get

$$\begin{aligned}Y(t) - \overline{Y}(t) = {}&-f_\lambda\left(Y_n^*\right)(t-t_n) - g\left(Y_n^*\right)\left(W(t) - W(t_n)\right)\\&- h\left(Y_n^*\right)\left(\tilde{N}(t) - \tilde{N}(t_n)\right),\end{aligned}$$

and hence for $t - t_n \le \Delta t$

$$\begin{aligned}\left|Y(t) - \overline{Y}(t)\right|^r \le {}&3^{r-1}\left[\Delta t^r\left|f_\lambda\left(Y_n^*\right)\right|^r + \left|g\left(Y_n^*\right)\right|^r\left|W(t) - W(t_n)\right|^r\right.\\&\left.+\left|h\left(Y_n^*\right)\right|^r\left|\widetilde{N}(t) - \widetilde{N}(t_n)\right|^r\right].\end{aligned}$$

By Assumption 5.1 on f_λ, and linear growth condition (2.6)-(2.7) for g and h, we have

$$\begin{aligned}&\mathbb{E}\sup_{0\le t\le T}\left|Y(t) - \overline{Y}(t)\right|^r\\&\quad\le C_3(r)\Big[\Delta t^r\Big(1 + \sup_{0\le t\le T}\mathbb{E}\left|Y_n^*\right|^u\Big) + \Big(1 + \sup_{0\le t\le T}\mathbb{E}\left|Y_n^*\right|^u\Big)|t-t_n|^{r/2}\\&\qquad+\Big(1 + \sup_{0\le t\le T}\mathbb{E}\left|Y_n^*\right|^u\Big)|t-t_n|^{r/2}\Big],\end{aligned} \tag{5.3}$$

where $C_3(r)$ and u are both integer constants depending on q from Assumption 5.1 and r. By Lemma 4.1, we obtain

$$\mathbb{E}\sup_{0\le t\le T}\left|Y(t) - \overline{Y}(t)\right|^r \le E\Delta t^{\frac{r}{2}},$$

where $E = E(r)$ a positive constant which depends on r. □

Theorem 5.2 *Under Assumptions* 2.1 *and* 5.1, *let* $0 < \theta \leq 1$, $0 < \Delta t < \min\{1, \frac{1}{2\theta L_\lambda}\}$, *the continuous-time approximate solution* $\overline{Y}(t)$ *defined by* (4.3) *will converge to the true solution* $X(t)$ *of* (3.1) *with strong order of one half, i.e.*

$$\mathbb{E} \sup_{0\leq t\leq T} \left|\overline{Y}(t) - X(t)\right|^2 = O(\Delta t). \tag{5.4}$$

Proof Let

$$e(t) = \overline{Y}(t) - X(t).$$

From the identity

$$\begin{aligned} X(t) = X_0 &+ \int_0^t f_\lambda\bigl(X(s^-)\bigr)\,\mathrm{d}s + \int_0^t g\bigl(X(s^-)\bigr)\,\mathrm{d}W(s) \\ &+ \int_0^t h\bigl(X(s^-)\bigr)\,\mathrm{d}\tilde{N}(s), \end{aligned} \tag{5.5}$$

and (4.3), we apply the Itô formula [17] to obtain

$$\begin{aligned} \left|e(t)\right|^2 = {}& 2\int_0^t \bigl\langle f_\lambda\bigl(Y(s^-)\bigr) - f_\lambda\bigl(X(s^-)\bigr), e(s^-)\bigr\rangle\,\mathrm{d}s + \int_0^t \bigl|g\bigl(Y(s^-)\bigr) - g\bigl(X(s^-)\bigr)\bigr|^2\,\mathrm{d}s \\ &+ \lambda\int_0^t \bigl|h\bigl(Y(s^-)\bigr) - h\bigl(X(s^-)\bigr)\bigr|^2\,\mathrm{d}s \\ &+ \int_0^t 2\bigl\langle e(s^-), g\bigl(Y(s^-)\bigr) - g\bigl(X(s^-)\bigr)\,\mathrm{d}W(s)\bigr\rangle \\ &+ \int_0^t 2\bigl\langle e(s^-), h\bigl(Y(s^-)\bigr) - h\bigl(X(s^-)\bigr)\bigr\rangle\,\mathrm{d}\tilde{N}(s) \\ &+ \int_0^t \bigl|h\bigl(Y(s^-)\bigr) - h\bigl(X(s^-)\bigr)\bigr|^2\,\mathrm{d}\tilde{N}(s) \\ \leq {}& 2\int_0^t \bigl\langle f_\lambda\bigl(Y(s^-)\bigr) - f_\lambda\bigl(\overline{Y}(s^-)\bigr), e(s^-)\bigr\rangle + \bigl\langle f_\lambda\bigl(\overline{Y}(s^-)\bigr) - f_\lambda\bigl(X(s^-)\bigr), e(s^-)\bigr\rangle\,\mathrm{d}s \\ &+ \int_0^t \bigl|g\bigl(Y(s^-)\bigr) - g\bigl(X(s^-)\bigr)\bigr|^2\,\mathrm{d}s \\ &+ \lambda\int_0^t \bigl|h\bigl(Y(s^-)\bigr) - h\bigl(X(s^-)\bigr)\bigr|^2\,\mathrm{d}s \\ &+ M_1(t) + M_2(t) + M_3(t), \end{aligned}$$

where

$$\begin{aligned} M_1(t) &= \int_0^t 2\bigl\langle e(s^-), g\bigl(Y(s^-)\bigr) - g\bigl(X(s^-)\bigr)\,\mathrm{d}W(s)\bigr\rangle, \\ M_2(t) &= \int_0^t 2\bigl\langle e(s^-), h\bigl(Y(s^-)\bigr) - h\bigl(X(s^-)\bigr)\bigr\rangle\,\mathrm{d}\tilde{N}(s), \\ M_3(t) &= \int_0^t \bigl|h\bigl(Y(s^-)\bigr) - h\bigl(X(s^-)\bigr)\bigr|^2\,\mathrm{d}\tilde{N}(s). \end{aligned}$$

Using the Assumptions 2.1 and 5.1, and (3.2) we have

$$
\begin{aligned}
|e(t)|^2 &\le \int_0^t 2\langle f_\lambda(Y(s^-)) - f_\lambda(\overline{Y}(s^-)), e(s^-)\rangle + 2K_\lambda |e(s^-)|^2 \,\mathrm{d}s \\
&\quad + \int_0^t (L_g + \lambda L_h)|Y(s^-) - X(s^-)|^2 \,\mathrm{d}s \\
&\quad + M_1(t) + M_2(t) + M_3(t) \\
&\le \int_0^t |f_\lambda(Y(s^-)) - f_\lambda(\overline{Y}(s^-))|^2 + |e(s^-)|^2 \,\mathrm{d}s + 2K_\lambda \int_0^t |e(s^-)|^2 \,\mathrm{d}s \\
&\quad + 2(L_g + \lambda L_h) \int_0^t |e(s^-)|^2 + |Y(s^-) - \overline{Y}(s^-)|^2 \,\mathrm{d}s \\
&\quad + M_1(t) + M_2(t) + M_3(t) \\
&\le [1 + 2(K_\lambda + L_g + \lambda L_h)] \int_0^t |e(s^-)|^2 \,\mathrm{d}s \\
&\quad + D \int_0^t (1 + |Y(s^-)|^q + |\overline{Y}(s^-)|^q)|Y(s^-) - \overline{Y}(s^-)|^2 \,\mathrm{d}s \\
&\quad + 2(L_g + \lambda L_h) \int_0^t |Y(s^-) - \overline{Y}(s^-)|^2 \,\mathrm{d}s \\
&\quad + M_1(t) + M_2(t) + M_3(t) \\
&\le [1 + 2(K_\lambda + L_g + \lambda L_h)] \int_0^t |e(s^-)|^2 \,\mathrm{d}s \\
&\quad + D_1 \Bigl(\sup_{0\le s\le t} |Y(s^-) - \overline{Y}(s^-)|^2 \Bigr) \int_0^t (1 + |Y(s^-)|^q + |\overline{Y}(s^-)|^q) \,\mathrm{d}s \\
&\quad + M_1(t) + M_2(t) + M_3(t),
\end{aligned}
$$

where we use D_1 to denote a generic constant (independent of Δt) that may change from line to line.

Using the Lemma 4.1, Lemma 4.2 and Lemma 5.1, we have

$$
\begin{aligned}
\mathbb{E} \sup_{0\le s\le t} |e(s)|^2 &\le D_1 \int_0^t \mathbb{E}|e(s^-)|^2 \,\mathrm{d}s + D_1 \Delta t \\
&\quad + \mathbb{E} \sup_{0\le s\le t} M_1(s) + \mathbb{E} \sup_{0\le s\le t} M_2(s) + \mathbb{E} \sup_{0\le s\le t} M_3(s).
\end{aligned} \tag{5.6}
$$

Now, as in the proof of [9], the Burkholder-Davis-Gundy inequality can be used to get the estimate

$$
\mathbb{E} \sup_{0\le s\le t} M_i(s) \le \frac{1}{2} \mathbb{E} \sup_{0\le s\le t} |e(t)|^2 + D_1 \int_0^t \mathbb{E}|e(s^-)|^2 \,\mathrm{d}s + D_1 \Delta t, \quad i = 1, 2, 3.
$$

Using this in (5.6), we obtain

$$
\mathbb{E} \sup_{0\le s\le t} |e(s)|^2 \le D_1 \int_0^t \mathbb{E} \sup_{0\le s\le t} |e(s^-)|^2 \,\mathrm{d}s + D_1 \Delta t.
$$

The result follows from the continuous Gronwall inequality. □

6 Numerical experiments

We consider the following nonlinear stochastic different equation with jumps from [16]:

$$\begin{cases} \mathrm{d}X(t) = (-4X(t^-) - X^3(t^-))\,\mathrm{d}t + X(t^-)\,\mathrm{d}W(t) + X(t^-)\,\mathrm{d}N(t), \\ X(0) = 1. \end{cases} \tag{6.1}$$

Define $f(x(t)) = -4x(t) - x^3(t)$, $g(x(t)) = x(t)$, $h(x(t)) = x(t)$. It is easy to compute that

$$\begin{aligned} \langle x-y, f(x)-f(y)\rangle &= \langle x-y, -4(x-y) - (x^3-y^3)\rangle \\ &= -4|x-y|^2(1+x^2+xy+y^2) \\ &\le -4|x-y|^2, \end{aligned}$$

which implies that $f(x)$ satisfies the one-sided Lipschitz condition, $g(x)$ and $h(x)$ satisfy the global Lipschitz condition, then the Assumptions of Theorem 5.2 hold. That is to say, the numerical solution by our method will converge to the true solution of system (6.1).

To show the convergence of the CSSθ method for system (6.1), we fix $\Delta t = 2^{-14}$, $T = 2$, $\lambda = 1$, $\theta = 0.7$. Noting that the exact solution of nonlinear jump-diffusion system (6.1) is not available, we use the numerical solution by the SSBE method with step size $\Delta t = 2^{-14}$ as the 'referenced exact solution' (Theorem 2 in [16] can guarantee its strong convergence) in Figure 1.

In Figure 1, we show the numerical solution by the CSSθ method with step size $\Delta t = 2^{-10}$ and the 'referenced exact solution'. we can easy to find that the CSSθ approximation and the 'referenced exact solution' make no major difference between both paths. That is to say the CSSθ method converges to the 'referenced exact solution' well. Hence our method is efficient for the nonlinear jump-diffusion systems.

To show the strong convergence order of the CSSθ method with different parameter θ, we fix $T = 2$, $\lambda = 1$ and change $\theta = 0.5, 0.7, 0.9$, respectively. The 'referenced exact solution' of system (6.1) is also used by the SSBE method with step size $\Delta t = 2^{-14}$. We simulate the numerical solutions with five different step sizes $h = 2^{p-1}\Delta t$ for $1 \le p \le 5$, $\Delta t = 2^{-14}$. The mean-square errors $\varepsilon = 1/5{,}000 \sum_{i=1}^{5{,}000} |Y_n(\omega_i) - X(T)|^2$, all measured at time $T = 2$, are estimated by trajectory averaging. We plot our approximation to $\sqrt{\epsilon}$ against Δt on a log-log scale. For reference a dashed line of slope one-half is added.

In Figure 2, we see that the slopes of the four curves appear to match well. Therefore, the results verify the strong convergence order of the proposed method.

Figure 1 CSSθ solution and 'referenced exact solution' for system (6.1).

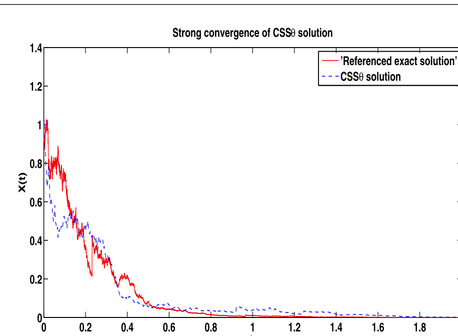
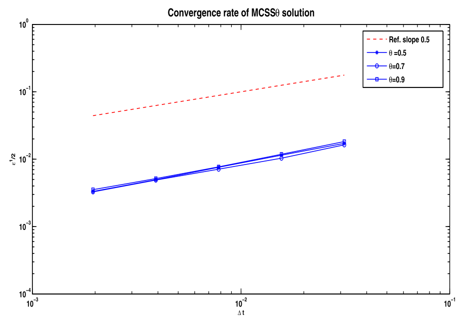


Figure 2 Errors simulation by CSSθ method with different θ.



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Competing interests
The authors declare that they have no competing interests.

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