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On (p,q)-classical orthogonal polynomials and their characterization theorems

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Abstract

In this paper, we introduce a general (p,q)-Sturm-Liouville difference equation whose solutions are (p,q)-analogues of classical orthogonal polynomials leading to Jacobi, Laguerre, and Hermite polynomials as $(p,q)\to(1,1)$. In this direction, some basic characterization theorems for the introduced (p,q)-Sturm-Liouville difference equation, such as Rodrigues representation for the solution of this equation, a general three-term recurrence relation, and a structure relation for the (p,q)-classical polynomial solutions are given.

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1 Introduction

Postquantum calculus, or (p,q)-calculus, is known as an extension of quantum calculus that recovers the results as $p\to 1$. For some basic properties of (p,q)-calculus, we refer to [1–3].

In the q-case, the solutions of a q-Sturm-Liouville problem are q-orthogonal functions [4, 5], which reduce to the q-classical orthogonal polynomials, appear in a natural way [6]. Very recently [7], a new generalization of q-Sturm-Liouville problems, namely, (p,q)-Sturm-Liouville problems, has been analyzed. In this paper, we show that the (p,q)-difference equation is of hypergeometric type, that is, the (p,q)-difference of any solution of the equation is also a solution of an equation of the same type. From this fundamental property the Rodrigues formula for the solutions is derived, and the coefficients of the three-term recurrence relation, satisfied by the orthogonal polynomial solutions of the (p,q)-difference equation, are obtained.

The paper is organized as follows: In Section 2, we collect some definitions and notations of (p,q)-calculus and include some new results that will be used in this paper. In Section 3, the (p,q)-difference equations of hypergeometric type are introduced, in the sense that the (p,q)-difference of a solution of the equation is solution of an equation of the same type. In Section 4, a Rodrigues-type formula for the polynomial solutions of the (p,q)-difference equation of hypergeometric type is obtained. In Section 5, we obtain the coefficients in the three-term recurrence relation for the orthogonal polynomial solutions of the (p,q)-difference equation of hypergeometric type. A difference representation and a (p,q)-structure relation are also obtained. Finally, in Section 6, we present (p,q)-analogues of shifted Jacobi, Laguerre, and Hermite polynomials. For each of this specific families, we provide a (p,q)-difference equation of hypergeometric type, the coefficients of the three-term recurrence relation, the weight function, and the orthogonality property. Limit transitions from these (p,q)-analogues to the classical families are also given. Appell families are also studied in detail.

2 Basic definitions and notations

In this section, we summarize the basic definitions and results, which can be found in [6, 8–12] and references therein.

For $k \geq 0$, the q-shifted factorial is defined as

$$(a;q)_k = \prod_{j=0}^{k-1}\bigl(1-aq^j\bigr) \quad \text{with } (a;q)_0 = 1, \tag{1}$$

which can be generalized to the (p,q)-power as

$$\bigl((a,b);(p,q)\bigr)_k = \prod_{j=0}^{k-1}\bigl(ap^j - bq^j\bigr) \quad \text{with } \bigl((a,b);(p,q)\bigr)_0 = 1. \tag{2}$$

Moreover, for $k < 0$, we define

$$\bigl((a,b);(p,q)\bigr)_k = \frac{1}{\prod_{j=0}^{-k}(ap^{-j} - bq^{-j})}. \tag{3}$$

Hence, we have

$$\begin{aligned}
&\bigl((1,a);(1,q)\bigr)_k = (a;q)_k,\\
&\bigl((ra,rb);(p,q)\bigr)_k = r^k\bigl((a,b);(p,q)\bigr)_k,
\end{aligned}$$

and

$$(b/a;q/p)_k = a^{-k}p^{-k(k-1)/2}\bigl((a,b);(p,q)\bigr)_k.$$

Moreover,

$$(a;q)_\infty = \prod_{j=0}^{\infty}\bigl(1-aq^j\bigr) \quad \text{for } 0 < |q| < 1$$

can be generalized as

$$\bigl((a,b);(p,q)\bigr)_\infty = \prod_{j=0}^{\infty}\bigl(ap^j - bq^j\bigr) \quad \text{for } 0 < \left|\frac{q}{p}\right| < 1.$$

For any complex number λ, we also introduce

$$\bigl((a,b);(p,q)\bigr)_\lambda = \frac{((a,b);(p,q))_\infty}{((ap^\lambda, bq^\lambda);(p,q))_\infty}. \tag{4}$$

The q-numbers are defined as

$$\lim_{p\to 1}[n]_{p,q} = [n]_q = \sum_{j=0}^{n-1} q^j, \quad q \neq 1,$$

and their generalization as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = \sum_{j=0}^{n-1} q^j p^{n-1-j}, \quad n = 1, 2, \ldots, \tag{5}$$

where

$$[-1]_{p,q} = -\frac{1}{pq} \quad \text{and} \quad [0]_{p,q} = 0.$$

The (p,q)-factorial is defined by

$$[n]_{p,q}! = \prod_{j=1}^{n} [j]_{p,q}, \quad n \geq 1, \quad \text{and} \quad [0]_{p,q}! = 1. \tag{6}$$

Since the definition of q-hypergeometric series

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \,\middle|\, q; z\right) = \sum_{j=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_j}{(b_1, \ldots, b_s; q)_j} \frac{z^j}{(q;q)_j} \left((-1)^j q^{\frac{j(j-1)}{2}}\right)^{1+s-r},$$

where

$$(a_1, \ldots, a_r; q)_j = (a_1; q)_j \cdots (a_r; q)_j,$$

is based on the symbol $(a;q)_j$ defined in (1), its generalization, known as the (p,q)-hypergeometric series, can be defined as

$$\begin{aligned} &{}_r\Phi_s\left(\begin{matrix} (a_{1p}, a_{1q}), \ldots, (a_{rp}, a_{rq}) \\ (b_{1p}, b_{1q}), \ldots, (b_{sp}, b_{sq}) \end{matrix} \,\middle|\, (p,q); z\right) \\ &\quad = \sum_{j=0}^{\infty} \frac{((a_{1p}, a_{1q}), \ldots, (a_{rp}, a_{rq}); (p,q))_j}{((b_{1p}, b_{1q}), \ldots, (b_{sp}, b_{sq}); (p,q))_j} \frac{z^j}{((p,q);(p,q))_j} \left((-1)^j (q/p)^{\frac{j(j-1)}{2}}\right)^{1+s-r}, \end{aligned} \tag{7}$$

where

$$\left((a_{1p}, a_{1q}), \ldots, (a_{rp}, a_{rq}); (p,q)\right)_j = \left((a_{1p}, a_{1q}); (p,q)\right)_j \cdots \left((a_{rp}, a_{rq}); (p,q)\right)_j,$$

and $r, s \in \mathbb{Z}_+$ and $a_{1p}, a_{1q}, \ldots, a_{rp}, a_{rq}, b_{1p}, b_{1q}, \ldots, b_{sp}, b_{sq}, z \in \mathbb{C}$.

It is clear that

$$\lim_{q\to 1} {}_r\phi_s\left(\begin{matrix} q^{a_1}, \ldots, q^{a_r} \\ q^{b_1}, \ldots, q^{b_s} \end{matrix} \,\middle|\, q; (q-1)^{1+s-r} z\right) = {}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \,\middle|\, z\right), \tag{8}$$

where

$$ {}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\ \Big|\ z\right) = \sum_{j=0}^{\infty} \frac{(a_1,\dots,a_r)_j}{(b_1,\dots,b_s)_j} \frac{z^j}{j!} $$

denotes a hypergeometric series with

$$ (a_1,\dots,a_r)_j = (a_1)_j \cdots (a_r)_j. $$

Also, when $a_{1p} = a_{2p} = \cdots = a_{rp} = b_{1p} = b_{2p} = \cdots = b_{sp} = 1$, $a_{1q} = a_1,\dots,a_{rq} = a_r$ and $b_{1q} = b_1,\dots,b_{s,q} = b_s$, we have

$$ \lim_{p\to 1} {}_r\Phi_s\left(\begin{matrix} (1,a_1),\dots,(1,a_r) \\ (1,b_1),\dots,(1,b_s) \end{matrix}\ \Big|\ (p,q);z\right) = {}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\ \Big|\ q;z\right). $$

The functions

$$ E_q(x) := \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{(q;q)_n} x^n = (-x;q)_\infty \quad \bigl(0 < |q| < 1 \text{ and } |x| < 1\bigr) \tag{9} $$

and

$$ E_{p,q}(x) := \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{((p,q);(p,q))_n} x^n = \bigl((1,-x);(p,q)\bigr)_\infty \quad \left(0 < \left|\frac{q}{p}\right| < 1 \text{ and } |x| < 1\right) \tag{10} $$

are respectively known as a q-analogue and a (p,q)-analogue of the exponential function.

The (p,q)-difference operator is defined by (see e.g. [9, 13])

$$ (\mathcal{D}_{p,q}f)(x) = \frac{\mathcal{L}_p f(x) - \mathcal{L}_q f(x)}{(p-q)x}, \quad x \neq 0, \tag{11} $$

where

$$ \mathcal{L}_a h(x) = h(ax) \tag{12} $$

and $(\mathcal{D}_{p,q}f)(0) = f'(0)$, provided that f is differentiable at 0.

The (p,q)-difference operator is a linear operator: for any constants a and b, we have

$$ \bigl(\mathcal{D}_{p,q}(af + bg)\bigr)(x) = a(\mathcal{D}_{p,q}f)(x) + b(\mathcal{D}_{p,q}g)(x). $$

Moreover, it can be proved that

$$ \begin{aligned} \bigl(\mathcal{D}_{p,q}(fg)\bigr)(x) &= f(px)(\mathcal{D}_{p,q}g)(x) + g(qx)(\mathcal{D}_{p,q}f)(x) \\ &= g(px)(\mathcal{D}_{p,q}f)(x) + f(qx)(\mathcal{D}_{p,q}g)(x). \end{aligned} \tag{13} $$

The (p,q)-integral is defined by

$$ \int_0^x f(t)\, d_{p,q}t = (p-q)x \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}} x\right). \tag{14} $$

For two nonnegative numbers a and b with $a < b$, definition (14) yields

$$\int_a^b f(x)\,d_{p,q}x = \int_0^b f(x)\,d_{p,q}x - \int_0^a f(x)\,d_{p,q}x.$$

A regular Sturm-Liouville problem of continuous type is a boundary value problem of the form

$$\frac{d}{dx}\left(r(x)\frac{dy_n(x)}{dx}\right) + \lambda_n w(x)y_n(x) = 0 \quad \bigl(r(x) > 0, w(x) > 0\bigr), \tag{15}$$

which is defined on an open interval, say (a,b), with boundary conditions

$$\alpha_1 y(a) + \beta_1 y'(a) = 0, \qquad \alpha_2 y(b) + \beta_2 y'(b) = 0, \tag{16}$$

where α_1, α_2 and β_1, β_2 are constant numbers, and $r(x)$, $r'(x)$, and $w(x)$ in (15) are assumed to be continuous for $x \in [a,b]$. In this sense, if y_n and y_m are two eigenfunctions of equation (15), then according to Sturm-Liouville theory [14], they are orthogonal with respect to the weight function $w(x)$ under the given condition (16), that is, we have

$$\int_a^b w(x)y_n(x)y_m(x)\,dx = d_n^2\delta_{mn}, \tag{17}$$

where $d_n^2 = \int_a^b w(x)y_n^2(x)\,dx$ denotes the norm square of the functions y_n, and δ_{mn} stands for the Kronecker delta.

The following result has been proved in [7].

Theorem 2.1 *Let $\{y_n(x;p,q)\}$ be a sequence of functions satisfying the equation*

$$A(x)\bigl(\mathcal{D}_{p,q}^2 y_n\bigr)(x;p,q) + B(x)(\mathcal{D}_{p,q}y_n)(px;p,q) + \bigl(\lambda_{n,p,q}C(x) + D(x)\bigr)y_n(pqx;p,q) = 0, \tag{18}$$

where $A(x)$, $B(x)$, $C(x)$, and $D(x)$ are known functions, and $\lambda_{n,p,q}$ is a sequence of constants, then

$$\int_a^b w^*(x;p,q)y_n(x;p,q)y_m(x;p,q)\,d_{p,q}x = \left(\int_a^b w^*(x;p,q)y_n^2(x;p,q)\,d_{p,q}x\right)\delta_{n,m},$$

where

$$w^*(x;p,q) = w(x;p,q)\mathcal{L}_{pq}^{-1}C(x) = w(x;p,q)C\left(\frac{1}{pq}x\right), \tag{19}$$

and $w(x;p,q)$ is a solution of the (p,q)-Pearson difference equation

$$\bigl(\mathcal{D}_{p,q}\bigl(\mathcal{L}_p w\mathcal{L}_q^{-1}A\bigr)\bigr)(x;p,q) = B(x)\mathcal{L}_{pq}w(x;p,q), \tag{20}$$

which is equivalent to

$$\frac{w(p^2x;p,q)}{w(pqx;p,q)} = \frac{A(x) + (p-q)xB(x)}{A(pq^{-1}x)}.$$

Of course, the weight function defined in (19) *must be be positive, and*

$$w\big(q^{-1}x;p,q\big)A\big(p^{-1}q^{-2}x\big)$$

must vanish at $x = a, b$.

Remark 2.1 Let $\theta(x;p,q)$ be a known and predetermined function. The solution of the difference equation

$$\frac{w(p^2x)}{w(pqx)} = \theta(x;p,q) \tag{21}$$

can be represented as [7]

$$w(x) = \prod_{k=0}^{\infty} \theta\left(\frac{q^k}{p^{k+2}}x;p,q\right).$$

3 (p,q)-Difference equations of hypergeometric type

First, from the definition of shift operator (12) we can be verify that

$$\mathcal{D}_{p,q}\big(\mathcal{L}_q f(x)\big) = q\mathcal{L}_q\big(\mathcal{D}_{p,q} f(x)\big).$$

Let us assume in (18) that $A(x)$ and $B(x)$ are polynomials of degree at most 2 and 1, respectively, $D(x) = 0$, and $C(x) = 1$. For our purposes, it is convenient to consider a particular case of (18) as

$$\sigma(x)\big(\mathcal{D}_{p,q}^2 y\big)(x) + \tau(x)\mathcal{L}_p\big((\mathcal{D}_{p,q}y)(x)\big) + \lambda\mathcal{L}_{pq}y(x) = 0, \tag{22}$$

where

$$\sigma(x) = ax^2 + bx + c \quad \text{and} \quad \tau(x) = dx + e \tag{23}$$

with $d \neq 0$. Let $y(x)$ be a solution of (22), and let

$$v_1(x) = (\mathcal{D}_{p,q})y(x). \tag{24}$$

We prove that $v_1(x)$ is also a solution of an equation of the same type as (22).

With notation (24), we can rewrite (22) as

$$\sigma(x)(\mathcal{D}_{p,q}v_1)(x) + \tau(x)\mathcal{L}_p(v_1)(x) + \lambda\mathcal{L}_{pq}y(x) = 0. \tag{25}$$

If the (p,q)-difference operator $\mathcal{D}_{p,q}$ is applied to the latter equation, then it yields

$$\mathcal{D}_{p,q}\big(\sigma(x)(\mathcal{D}_{p,q}v_1)(x)\big) + \mathcal{D}_{p,q}\big(\tau(x)\mathcal{L}_p(v_1)(x)\big) + \mathcal{D}_{p,q}\big(\lambda\mathcal{L}_{pq}y(x)\big) = 0. \tag{26}$$

Also, since

$$\mathcal{D}_{p,q}\big(\sigma(x)(\mathcal{D}_{p,q}v_1)(x)\big) = \mathcal{L}_p\big(\mathcal{D}_{p,q}v_1(x)\big)(\mathcal{D}_{p,q}\sigma)(x) + \mathcal{L}_q\big(\sigma(x)\big)\big(D_{p,q}^2 v_1(x)\big), \tag{27}$$

$$\mathcal{D}_{p,q}\big(\tau(x)\mathcal{L}_p(\nu_1)(x)\big) = \mathcal{L}_p\tau(x)p\mathcal{L}_p\big(\mathcal{D}_{p,q}\nu_1(x)\big) + \mathcal{L}_{pq}\big(\nu_1(x)\big)\big(\mathcal{D}_{p,q}\tau(x)\big), \tag{28}$$

and

$$\mathcal{D}_{p,q}\big(\lambda\mathcal{L}_{pq}y(x)\big) = \lambda pq\mathcal{L}_{pq}\big(\nu_1(x)\big), \tag{29}$$

we obtain

$$\big(\mathcal{L}_q\sigma(x)\big)\big(\mathcal{D}^2_{p,q}\nu_1\big)(x) + \tau_1(x)\mathcal{L}_p\big((\mathcal{D}_{p,q}\nu_1)(x)\big) + \mu_1\mathcal{L}_{pq}\nu_1(x) = 0, \tag{30}$$

where

$$\tau_1(x) = p\mathcal{L}_p\big(\tau(x)\big) + \big(\mathcal{D}_{p,q}\sigma(x)\big). \tag{31}$$

Therefore, $\nu_1(x)$ defined in (24) is solution of an equation of the same type as (22).

If the above procedure is similarly iterated, then we conclude that $\nu_n(x) = \mathcal{D}^n_{p,q}y(x)$ is also a solution of the equation

$$\big(\mathcal{L}^n_q\sigma(x)\big)\big(\mathcal{D}^2_{p,q}\nu_n\big)(x) + \tau_n(x)\mathcal{L}_p\big((\mathcal{D}_{p,q}\nu_n)(x)\big) + \mu_n\mathcal{L}_{pq}\nu_n(x) = 0, \tag{32}$$

where

$$\tau_n(x) = p\mathcal{L}_p\big(\tau_{n-1}(x)\big) + \big(\mathcal{D}_{p,q}\sigma_{n-1}(x)\big). \tag{33}$$

Hence, it is proved by induction that $\nu_n(x)$ satisfies

$$\sigma_n(x)\big(\mathcal{D}^2_{p,q}\nu_n\big)(x) + \tau_n(x)\mathcal{L}_p\big((\mathcal{D}_{p,q}\nu_n)(x)\big) + \mu_n\mathcal{L}_{pq}\nu_n(x) = 0, \tag{34}$$

where

$$\sigma_n(x) = \sigma\big(q^n x\big), \qquad \mathcal{D}_{p,q}\sigma_n(x) = q^n\big(b + aq^n(p+q)x\big) \tag{35}$$

and

$$\tau_n(x) = ep^n + b[n]_{p,q} + \big(dp^{2n} + a\big(p^n + q^n\big)[n]_{p,q}\big)x. \tag{36}$$

4 Rodrigues-type representation for the polynomial solutions of equation (22)

Theorem 4.1 *The polynomial solutions of equation* (22) *satisfy the Rodrigues-type formula*

$$y_n(x) = K_n\mathcal{L}^{-n}_{pq}D^n_{p,q}\left(\mathcal{L}^n_p w(x)\prod_{k=1}^{n}\mathcal{L}^{n-k}_p\mathcal{L}^{k-2}_q\sigma(x)\right), \tag{37}$$

where

$$K_n = \frac{(-1)^n(D^n_{p,q}y_n)(x)}{(pq)^{(\frac{n^2+n-2}{2})}\prod_{k=0}^{n-1}\mu_k} \quad \textit{with } \mu_0 = \lambda.$$

Proof Let $w(z)$ and $w_n(z)$ satisfy the following (p,q)-Pearson difference equations:

$$D_{p,q}\bigl(\mathcal{L}_p w(x)\mathcal{L}_{q^{-1}}\sigma(x)\bigr) = \tau(x)\mathcal{L}_{pq}w(x)$$

and

$$D_{p,q}\bigl(\mathcal{L}_p w_n(x)\mathcal{L}_{q^{n-1}}\sigma(x)\bigr) = \tau_n(x)\mathcal{L}_{pq}w_n(x).$$

Multiplying (25) and (32) by $w(z)$ and $w_n(z)$, we can rewrite the equations in a self-adjoint form as

$$D_{p,q}\bigl(\mathcal{L}_p w(x)\mathcal{L}_{q^{-1}}\sigma(x)(D_{p,q}y)(x)\bigr) + \lambda_n\mathcal{L}_{pq}w(x)\mathcal{L}_{pq}y(x) = 0 \tag{38}$$

and

$$D_{p,q}\bigl(\mathcal{L}_p w_n(x)\mathcal{L}_{q^{n-1}}\sigma(x)(D_{p,q}v_n)(x)\bigr) + \mu_n\mathcal{L}_{pq}w_n(x)\mathcal{L}_{pq}v_n(x) = 0. \tag{39}$$

On the other hand, since

$$w_{n+1}(x) = \mathcal{L}_p w_n(x)\mathcal{L}_q^{n-1}\sigma(x) \tag{40}$$

and

$$v_{n+1}(x) = D_{p,q}v_n(x), \tag{41}$$

using (40) and (41), we can write (39) as

$$\mathcal{L}_{pq}w_n(x)\mathcal{L}_{pq}v_n(x) = -\frac{1}{\mu_n}D_{p,q}\bigl(w_{n+1}(x)v_{n+1}(x)\bigr).$$

If $y(x)$ is a polynomial of degree n, that is, $y = y_n(x)$, then

$$v_m(x) = y_n^{(m)}(x) \quad \text{and} \quad v_n(x) = y_n^{(n)}(z) = \text{const.},$$

and for $y_n^{(m)}(x)$, we obtain

$$D_{p,q}^m\bigl(\mathcal{L}_{pq}y_n(x)\bigr) = K_n'\mathcal{L}_{pq}^{-(n-m-1)}D_{p,q}^{n-m}\bigl(w_n(x)\bigr),$$

where

$$K_n' = \frac{(-1)^{n-m}(D_{p,q}^n y_n)(x)}{(pq)^{(\frac{n^2+n+2m-2}{2})}\prod_{k=m}^{n-1}\mu_k}.$$

The result follows from this expression for $m = 0$. □

5 Three-term recurrence relation for the polynomial solutions of equation (22)

First, to calculate the corresponding eigenvalues $\lambda_{n,p,q}$, since

$$D_{p,q}\big(x^n\big) = \frac{p^n x^n - q^n x^n}{(p-q)x} = [n]_{p,q} x^{n-1},$$

by equating the coefficients of x^n we obtain

$$\lambda_{n,p,q} = -\frac{[n]_{p,q}}{(pq)^n}\big(a[n-1]_{p,q} + dp^{n-1}\big). \tag{42}$$

Lemma 5.1 *For each nonnegative integer n, the uniqueness of a monic polynomial solution of equation* (22) *is equivalent to the following conditions*:

(1) *The equation in j*

$$\lambda_{j,p,q} = \lambda_{n,p,q}$$

has $j = n$ as a unique solution in $\mathbf{N}$;

(2) $\lambda_{k,p,q} \neq 0$ *for* $k = 0, 1, \ldots, n-1$.

Proof The result can be obtained following the same steps as in the continuous case. □

Let us define a linear operator as

$$L_n\big[y(x)\big] := \big(ax^2 + bx + c\big)D^2_{p,q}y(x;p,q) + (dx+e)D_{p,q}y(px;p,q) + \lambda_n y(pqx;p,q), \tag{43}$$

where $\lambda_n = \lambda_{n,p,q}$ is defined in (42).

Lemma 5.2 *There exists a sequence $\{\beta_n\}_{n\in\mathbf{N}}$ such that the polynomial*

$$U_n(x) = L_{n+1}\big((x-\beta_n)P_n(x)\big) \tag{44}$$

has exactly degree $n-1$ for each $n \in \mathbf{N}$ and

$$\beta_n = \varpi_{1,n} + \frac{p^{-n}q^{-n}[n+1]_{p,q}(b[n]_{p,q} + ep^n)}{\lambda_{n+1} - \lambda_n}. \tag{45}$$

Moreover, $U_n(x) = \vartheta_n x^{n-1} + \cdots$ with

$$\begin{aligned}\vartheta_n = {}& \frac{1}{p^2 q}\Big(\big(-p^{1+n}q^n\lambda_{n+1}(\beta_n\varpi_{1,n} - \varpi_{2,n}) - p^n q\big(d(\beta_n\varpi_{1,n} - \varpi_{2,n})[n-1]_{p,q} \\ &+ ep(\beta_n - \varpi_{1,n})[n]_{p,q}\big) + p^2 q\big([n-1]_{p,q}\big(a(-\beta_n\varpi_{1,n} + \varpi_{2,n})[n-2]_{p,q} \\ &+ b(-\beta_n + \varpi_{1,n})[n]_{p,q}\big) + c[n]_{p,q}[n+1]_{p,q}\big)\big)\Big)\end{aligned} \tag{46}$$

and $P_n(x) = x^n + \varpi_{1,n}x^{n-1} + \cdots$.

Proof Let us expand the monic polynomial solution of equation (42):

$$y_n(x;p,q) = P_n(x) = x^n + \varpi_{1,n}x^{n-1} + \varpi_{2,n}x^{n-2} + \cdots. \tag{47}$$

Since

$$(x-\beta_n)P_n(x) = x^{n+1} + x^n(\varpi_{1,n} - \beta_n) + x^{n-1}(\varpi_{2,n} - \beta_n\varpi_{1,n}) + \cdots,$$

we have

$$\begin{aligned}
&L_{n+1}\bigl[(x-\beta_n)P_n(x)\bigr] \\
&\quad = \bigl(p^{1+n}q^{1+n}\lambda_{n+1} + \bigl(dp^n + a[n]_{p,q}\bigr)[n+1]_{p,q}\bigr)x^{n+1} \\
&\qquad + \frac{1}{p}\bigl(\bigl(p^{1+n}q^n\lambda_{n+1}(-\beta_n + \varpi_{1,n}) - dp^n\beta_n[n]_{p,q} \\
&\qquad + dp^n\varpi_{1,n}[n]_{p,q} - ap\beta_n[n-1]_{p,q}[n]_{p,q} \\
&\qquad + ap\varpi_{1,n}[n-1]_{p,q}[n]_{p,q} + ep^{1+n}[n+1]_{p,q} + bp[n]_{p,q}[n+1]_{p,q}\bigr)\bigr)x^n \\
&\qquad + \frac{1}{p^2q}\bigl(\bigl(-p^{1+n}q^n\lambda_{n+1}(\beta_n\varpi_{1,n} - \varpi_{2,n}) - p^nq\bigl(d(\beta_n\varpi_{1,n} - \varpi_{2,n})[n-1]_{p,q} \\
&\qquad + ep(\beta_n - \varpi_{1,n})[n]_{p,q}\bigr) + p^2q\bigl([n-1]_{p,q}\bigl(a(-\beta_n\varpi_{1,n} + \varpi_{2,n})[n-2]_{p,q} \\
&\qquad + b(-\beta_n + \varpi_{1,n})[n]_{p,q}\bigr) + c[n]_{p,q}[n+1]_{p,q}\bigr)\bigr)\bigr)x^{n-1}.
\end{aligned} \tag{48}$$

The coefficient in x^{n+1} in (48) is zero by noting the value of λ_n in (42). To have a polynomial of degree exactly $n-1$ in the variable x, we obtain (45) with the condition $\lambda_{n+1} \neq \lambda_n$. Finally, the coefficient of x^{n-1} is derived by (46). □

Lemma 5.3 *For each nonnegative integer n, we have*

$$L_{n-1}\bigl(U_n(x)\bigr) = 0,$$

where $U_n(x)$ is defined in (44).

By the uniqueness of the polynomial solution of (22) there exists a constant Ω_n such that

$$U_n(x) = \Omega_n P_{n-1}(x).$$

Lemma 5.4 *Let $\bar{P}_n(x)$ be the unique monic polynomial solution of degree n of* (22). *Then, there exist two sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ such that the following three-term recurrence relation holds*:

$$\bar{P}_{n+1}(x) = (x-\beta_n)\bar{P}_n(x) - \gamma_n\bar{P}_{n-1}(x). \tag{49}$$

Moreover, β_n is given in (45), *and*

$$\gamma_n = \frac{\Omega_n}{\lambda_{n-1} - \lambda_{n+1}}. \tag{50}$$

These two lemmas can be improved as follows.

Theorem 5.1 *Let $\bar{P}_n(x)$ be the monic polynomial solution of degree n of* (22), *where $\sigma(x)$ and $\tau(x)$ are given in* (23), *and λ_n is given in* (42). *Then, the coefficients β_n and γ_n of the three-term recurrence relation* (49) *are explicitly given by*

$$\beta_n = \varpi_{1,n} - \varpi_{1,n+1} \tag{51}$$

and

$$\gamma_n = \varpi_{2,n} - \varpi_{2,n+1} - \beta_n \varpi_{1,n}, \tag{52}$$

where

$$\varpi_{1,n} = -\frac{pq[n]_{p,q}(bp[n-1]_{p,q} + ep^n)}{[n-1]_{p,q}(ap(pq[n-2]_{p,q} - [n]_{p,q}) + dqp^n) - dp^n[n]_{p,q}} \tag{53}$$

and

$$\varpi_{2,n} = -\frac{pq^2[n-1]_{p,q}(\varpi_{1,n}(bp^2[n-2]_{p,q} + ep^n) + cp^2[n]_{p,q})}{q^2[n-2]_{p,q}(ap^3[n-3]_{p,q} + dp^n) - [n]_{p,q}(ap[n-1]_{p,q} + dp^n)}. \tag{54}$$

Next, we obtain the (p,q)-difference representation for the polynomial solutions of (22).

Theorem 5.2 *Let $P_n(x)$ be the unique monic polynomial solution of* (22). *Then, the following relation holds*:

$$P_n(px) = U_n \mathcal{D}_{p,q} P_{n+1}(x) + V_n \mathcal{D}_{p,q} P_n(x) + W_n \mathcal{D}_{p,q} P_{n-1}(x), \quad n \geq 2, \tag{55}$$

where

$$U_n = \frac{p^n}{[n+1]_{p,q}}, \tag{56}$$

$$V_n = p^n\left(\frac{\varpi_{1,n}}{p[n]_{p,q}} - \frac{\varpi_{1,n+1}}{[n+1]_{p,q}}\right), \tag{57}$$

$$W_n = p^n\left(-\frac{\varpi_{1,n}^2}{p[n]_{p,q}} + \frac{\varpi_{2,n}}{p^2[n-1]_{p,q}} + \frac{\varpi_{1,n}\varpi_{1,n+1} - \varpi_{2,n+1}}{[n+1]_{p,q}}\right), \tag{58}$$

and $\varpi_{1,n}$ and $\varpi_{2,n}$ are explicitly given in (53) *and* (54).

Proof The result follows by equating the coefficients of (55). □

Moreover, the polynomial solutions of (22) also satisfy a (p,q)-structure relation.

Theorem 5.3 *Let $P_n(x)$ be the unique monic polynomial solution of* (22). *Then, the following relation holds*:

$$\phi(x)\mathcal{D}_{p,q} P_n\left(\frac{x}{p}\right) = \hat{U}_n P_{n+1}(x) + \hat{V}_n P_n(x) + \hat{W}_n P_{n-1}(x), \quad n \geq 1, \tag{59}$$

where

$$\phi(x) = ax^2 + bpqx + cp^2q^2, \tag{60}$$

and the coefficients are explicitly given by

$$\hat{U}_n = ap^{1-n}[n]_{p,q}, \tag{61}$$

$$\hat{V}_n = p^{1-n}\big(ap[n-1]_{p,q}\varpi_{1,n} + [n]_{p,q}(bpq - a\varpi_{1,n+1})\big), \tag{62}$$

$$\begin{aligned}\hat{W}_n = p^{1-n}\big(&p\big([n-1]_{p,q}\varpi_{1,n}(bpq - a\varpi_{1,n}) + ap[n-2]_{p,q}\varpi_{2,n}\big)\\ &+ [n]_{p,q}\big(cp^2q^2 + \varpi_{1,n}(-bpq + a\varpi_{1,n+1}) - a\varpi_{2,n+1}\big)\big),\end{aligned} \tag{63}$$

where $\varpi_{1,n}$ and $\varpi_{2,n}$ are given in (53) *and* (54), *respectively.*

Proof The result follows by equating the coefficients of (59). □

6 Examples

6.1 Example 1: Appell families

If $\{P_n(x)\}_{n\in\mathbf{N}}$ is a polynomial solution of (22) such that

$$\mathcal{D}_{p,q}P_n(x) = [n]_{p,q}P_{n-1}(x), \tag{64}$$

then the solution of (64) is said to be of Appell type.

To find these families, by the (p,q)-difference representation (55) the above condition (64) is equivalent to $V_n = W_n = 0$ for all n.

By equating $V_1 = 0$, since $p \neq 0$ and $q \neq 0$, we obtain three following possibilities:

(i) $a = b = 0$, which implies that $V_n = W_n = 0$. In this case, since $d \neq 0$, we can conclude that the coefficients of the three-term recurrence relation (49) are given by

$$\beta_n = -\frac{ep^{1-n}q^{n+1}}{d} \quad \text{and} \quad \gamma_n = -\frac{cp^{3-2n}q^{n+1}}{d}[n]_{p,q}, \tag{65}$$

assuming that $p \neq q$. Notice that

$$\lim_{p\to q}\gamma_n = \lim_{p\to q} -\frac{cp^{3-2n}q^{n+1}}{d}[n]_{p,q} = -\frac{cnq^3}{d}.$$

(ii) $b = e = 0$, which implies that $V_n = 0$. In order that $W_n = 0$, we must analyze three cases,

(a) $a = 0$, which implies

$$\beta_n = 0 \quad \text{and} \quad \gamma_n = -\frac{cp^{3-2n}q^{n+1}}{d}[n]_{p,q},$$

assuming that $p \neq q$;

(b) $c = 0$, which implies $\gamma_n = 0$, and therefore we have no orthogonal polynomial sequences;

(c) $p \to q$, for which we also need $c = 0$ in order to have $W_n = 0$. Therefore we have no orthogonal polynomial sequences again.

(iii) $q = \frac{bdp-aep}{ae}$, assuming that $a \neq 0$ and $e \neq 0$, which gives no orthogonal polynomial sequence after imposing that $V_n = W_n = 0$ for $n \geq 2$.

As a consequence of this analysis, we observe that the unique possibility for having (p,q)-Appell families is $a = b = 0$, which contains as a particular case the symmetric option $a = b = e = 0$. It is possible to assume that $c = 1$ without loss of generality.

Theorem 6.1 *The polynomial solution of equation* (22) *in the cases $a = b = e = 0$ and $c = 1$ is explicitly given by*

$$y_n(x;p,q) = x^{\sigma_n} {}_2\Phi_1 \left(\begin{matrix} (p^{\sigma_n - n}, q^{\sigma_n - n}), (dp^{2[(n-1)/2]+1}, 0) \\ (p^{2\sigma_n+1}, q^{2\sigma_n+1}) \end{matrix} \;\middle|\; (p^2,q^2); (q-p)x^2 \right), \tag{66}$$

up to a normalizing constant, where

$$\sigma_n = \frac{1-(-1)^n}{2} = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases}$$

In this case, the Pearson-type (p,q)-difference equation reads as

$$\big(\mathcal{D}_{p,q}(\mathcal{L}_p w)\big)(x;p,q) = dx\mathcal{L}_{pq} w(x;p,q),$$

where

$$w(x;p,q) = \sum_{n=0}^{\infty} \frac{d^n q^{n(n-1)}}{p^{2n} \prod_{j=1}^{n} [2j]_{p,q}} x^{2n} = E_{p^2,q^2}\big((p-q)p^{-2} dx^2\big) \tag{67}$$

with $E_{p,q}$ defined in (10).

Remark 6.1 We emphasize that as $(p,q) \to (1,1)$, for $d = -2$, the second-order (p,q)-difference equation

$$\big(\mathcal{D}_{p,q}^2 y\big)(x) + dx\mathcal{L}_p\big((\mathcal{D}_{p,q} y)(x)\big) - \frac{dq^{-n}}{p}[n]_{p,q} \mathcal{L}_{pq} y(x) = 0 \tag{68}$$

converges formally to the differential equation of Hermite polynomials. Moreover, the polynomials $y_n(x;p,q)$ defined in (66) converge to the well-known Hermite polynomials, and the weight function $w(x;p,q)$ defined in (67) converges to $\exp(-x^2)$.

The monic polynomial solutions of (68) satisfy a three-term recurrence relation of the form

$$y_{n+1}(x;p,q) = xy_n(x;p,q) - C_n(p,q) y_{n-1}(x;p,q)$$

with

$$y_0(x;p,q) = 1, \qquad y_1(x;p,q) = x,$$

where

$$C_n(p,q) = -\frac{p^{3-2n}q^{n+1}}{d}[n]_{p,q}.$$

To have the orthogonality with respect to a positive weight function, we need to impose $d < 0$. Under this assumption, the orthogonality reads as

$$\int_{-\infty}^{\infty} y_n(x;p,q)y_m(x;p,q)E_{p^2,q^2}\big((p-q)p^{-2}dx^2\big)\,d_{p,q}x = c_0\left(\frac{-1}{d}\right)^n \frac{q^{\frac{1}{2}n(n+3)}}{p^{(n-2)n}}[n]_{p,q}!\delta_{n,m},$$

where

$$c_0 = \int_{-\infty}^{\infty} E_{p^2,q^2}\big((p-q)p^{-2}dx^2\big)\,d_{p,q}x,$$

and $[z]_{p,q}!$ is defined in (6).

6.2 Example 2: (p,q)-Laguerre polynomials

Let us now consider the second-order equation

$$x\big(\mathcal{D}_{p,q}^2 y\big)(x) + \left(\frac{p^{\alpha+1}q^{-\alpha-1}-1}{p-q} + dx\right)\mathcal{L}_p\big((\mathcal{D}_{p,q}y)(x)\big) - \frac{dq^{-n}}{p}[n]_{p,q}\mathcal{L}_{pq}y(x) = 0. \tag{69}$$

Theorem 6.2 *The polynomial solution of* (69) *is given by*

$$y_n(x;\alpha;p,q) = {}_2\Phi_1\left(\begin{matrix}(p^{-n},q^{-n}),(p^{n-1},0)\\(p^{\alpha+1},q^{\alpha+1})\end{matrix}\;\middle|\;(p,q);dq^{\alpha+1}(q-p)x\right) \tag{70}$$

up to a normalizing constant.

In this case, the Pearson-type (p,q)-difference equation reads as

$$\frac{w(p^2x;\alpha;p,q)}{w(pqx;\alpha;p,q)} = p^{\alpha}q^{-\alpha} - \frac{dq^2x}{p} + dqx,$$

in which

$$w(x;\alpha;p,q) = x^{\alpha}E_{p,q}\big(dxp^{-\alpha-3}q^{\alpha+1}(p-q)\big). \tag{71}$$

Remark 6.2 Once again, we emphasize that as $(p,q)\to(1,1)$, for $d=1$, the second-order (p,q)-difference equation (69) converges formally to the differential equation of Laguerre polynomials. Moreover, the polynomials $y_n(x;\alpha;p,q)$ defined in (70) converge to the well-known Laguerre polynomials, and the weight function $w(x;\alpha;p,q)$ defined in (71) converges to $x^{\alpha}\exp(-x)$.

The monic polynomial solutions of equation (69) satisfy a three-term recurrence relation of the form

$$y_{n+1}(x;\alpha;p,q) = \big(x-B_n(\alpha;p,q)\big)y_n(x;\alpha;p,q) - C_n(\alpha;p,q)y_{n-1}(x;\alpha;p,q)$$

with

$$y_0(x;\alpha;p,q) = 1, \qquad y_1(x;\alpha;p,q) = x - B_0(\alpha;p,q),$$

where

$$B_n(\alpha;p,q) = \frac{p^{1-2n}q^n(q^n(p+q) - p^{n+1}(p^\alpha q^{-\alpha}+1))}{d(p-q)}$$

and

$$C_n(\alpha;p,q) = \frac{p^{5-4n}q^{-\alpha+2n-1}[n]_{p,q}[\alpha+n]_{p,q}}{d^2}.$$

To have orthogonality with respect to a positive weight function, we need to impose $\alpha > -1$. Under this assumption, the orthogonality reads as

$$\begin{aligned}&\int_0^\infty y_n(x;\alpha;p,q)y_m(x;\alpha;p,q)x^\alpha E_{p,q}\bigl(dxp^{-\alpha-3}q^{\alpha+1}(p-q)\bigr)\,d_{p,q}x\\&\quad= c_0(\alpha)\frac{p^{(3-2n)n}q^{n(n-\alpha)}}{d^2n}[n]_{p,q}![n+\alpha]_{p,q}!\delta_{n,m},\end{aligned}$$

where

$$c_0(\alpha) = \int_0^\infty x^\alpha E_{p,q}\bigl(dxp^{-\alpha-3}q^{\alpha+1}(p-q)\bigr)\,d_{p,q}x.$$

6.3 Example 3: (p,q)-shifted Jacobi polynomials

Consider the second-order (p,q)-difference equation

$$\begin{aligned}&\frac{qx(qx-p)}{p^2}\bigl(\mathcal{D}^2_{p,q}y\bigr)(x) + \left(\frac{xp^{\alpha+\beta+2}q^{-\alpha-\beta} - p^{\beta+2}q^{-\beta} + pq - q^2x}{p^2(p-q)}\right)\mathcal{L}_p\bigl((\mathcal{D}_{p,q}y)(x)\bigr)\\&\quad+ [n]_{p,q}\left(\frac{qp^{-n-2} - p^{\alpha+\beta-1}q^{-\alpha-\beta-n}}{p-q}\right)\mathcal{L}_{pq}y(x) = 0.\end{aligned} \tag{72}$$

Theorem 6.3 *The polynomial solution of* (72) *is given by*

$$y_n(x;\alpha,\beta;p,q) = {}_2\Phi_1\left(\begin{matrix}(p^{-n},q^{-n}),(p^{\alpha+\beta+n+1},q^{\alpha+\beta+n+1})\\(p^{\beta+1},q^{\beta+1})\end{matrix}\;\middle|\;(p,q);\frac{xq^{-\alpha}}{p}\right) \tag{73}$$

up to a normalizing constant.

In this case, the Pearson-type (p,q)-difference equation reads as

$$\frac{w(p^2x;\alpha;p,q)}{w(pqx;\alpha;p,q)} = \frac{p^\beta q^{-\alpha-\beta}(xp^\alpha - q^\alpha)}{x-1},$$

where

$$w(x;\alpha;p,q) = \frac{x^\beta}{((1,xp^{-2});(p,q))_{-\alpha}}, \tag{74}$$

and $((a,b);(p,q))_\lambda$ is defined in (4).

Remark 6.3 It is straightforward to check that as $(p,q) \to (1,1)$, the second-order (p,q)-difference equation (72) converges formally to the differential equation of shifted Jacobi polynomials. Moreover, the polynomials $y_n(x;\alpha,\beta;p,q)$ defined in (73) converge to the well-known shifted Jacobi polynomials, and the weight function $w(x;\alpha,\beta;p,q)$ defined in (74) converges to $x^\alpha(1-x)^\beta$.

The monic polynomial solutions of equation (72) satisfy a three-term recurrence relation of the form

$$y_{n+1}(x;\alpha,\beta;p,q) = \bigl(x - B_n(\alpha,\beta;p,q)\bigr)y_n(x;\alpha,\beta;p,q) - C_n(\alpha,\beta;p,q)y_{n-1}(x;\alpha,\beta;p,q)$$

with

$$y_0(x;\alpha,\beta;p,q) = 1, \qquad y_1(x;\alpha,\beta;p,q) = x - B_0(\alpha,\beta;p,q),$$

where

$$\begin{aligned} B_n(\alpha,\beta;p,q) &= \frac{p^{n+2}q^{\alpha+n+1}}{(p-q)^2[\alpha+\beta+2n]_{p,q}[\alpha+\beta+2n+2]_{p,q}} \\ &\quad \times \bigl(\bigl(p^\beta+q^\beta\bigr)q^{\alpha+\beta+2n+1} - (p+q)\bigl(p^\alpha+q^\alpha\bigr)p^{\beta+n}q^{\beta+n} \\ &\quad + \bigl(p^\beta+q^\beta\bigr)p^{\alpha+\beta+2n+1}\bigr), \\ C_n(\alpha,\beta;p,q) &= \frac{p^{\beta+2n+3}q^{2\alpha+\beta+2n+1}[n]_{p,q}[\alpha+n]_{p,q}[\beta+n]_{p,q}[\alpha+\beta+n]_{p,q}}{[\alpha+\beta+2n-1]_{p,q}([\alpha+\beta+2n]_{p,q})^2[\alpha+\beta+2n+1]_{p,q}}. \end{aligned}$$

To have the orthogonality with respect to a positive weight function, we need to impose $\alpha,\beta > -1$. Under these assumptions, the orthogonality reads as

$$\begin{aligned} &\int_0^{p/q} y_n(x;\alpha,\beta;p,q)y_m(x;\alpha,\beta;p,q)\frac{x^\beta}{((1,xp^{-2});(p,q))_{-\alpha}}\,d_{p,q}x \\ &\quad = c_0(\alpha,\beta)\frac{p^{n(\beta+n+4)}q^{n(2\alpha+\beta+n+2)}[n]_{p,q}![\alpha+n]_{p,q}![\beta+n]_{p,q}![\alpha+\beta+n]_{p,q}!}{[\alpha+\beta+2n-1]_{p,q}!([\alpha+\beta+2n]_{p,q}!)^2[\alpha+\beta+2n+1]_{p,q}!}\delta_{n,m}, \end{aligned}$$

where

$$c_0(\alpha,\beta) = \int_0^{p/q} \frac{x^\beta}{((1,xp^{-2});(p,q))_{-\alpha}}\,d_{p,q}x.$$

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Authors' contributions
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