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# Some new formulas for the products of the Frobenius-Euler polynomials

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**Abstract**
The main purpose of this paper is, using the generating function methods and summation transform techniques, to establish some new formulas for the products of an arbitrary number of the Frobenius-Euler polynomials and give some illustrative special cases.

**MSC:** 11B83; 05A19

**Keywords:** Frobenius-Euler polynomials; Euler polynomials; summation formulas; combinational identities

## 1 Introduction

Let $\lambda$ be a complex number with $\lambda \neq 1$. Frobenius [1] introduced and studied the so-called Frobenius-Euler polynomials $H_n(x|\lambda)$, which are usually defined by the following exponential generating function:

$$\frac{1-\lambda}{e^t-\lambda}e^{xt} = \sum_{n=0}^{\infty} H_n(x|\lambda)\frac{t^n}{n!}. \tag{1.1}$$

In particular, the case $x = 0$ in (1.1) gives the Frobenius-Euler numbers $H_n(\lambda) = H_n(0|\lambda)$. It is interesting to point out that the Frobenius-Euler polynomials can be defined recursively by the Frobenius-Euler numbers as follows:

$$H_n(x|\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k(\lambda)x^{n-k} \quad (n \geq 0), \tag{1.2}$$

where, and in what follows, $\binom{a}{k}$ is the binomial coefficient defined for a complex number $a$ and a non-negative integer $k$ by

$$\binom{a}{0} = 1, \qquad \binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!} \quad (k \geq 1), \tag{1.3}$$

and the Frobenius-Euler numbers satisfy the recurrence relation

$$H_0(\lambda) = 1, \qquad \bigl(H(\lambda)+1\bigr)^n - H_n(\lambda) = \begin{cases} 1-\lambda, & n = 0, \\ 0, & n \geq 1, \end{cases} \tag{1.4}$$

with the usual convention about replacing $H^n(\lambda)$ by $H_n(\lambda)$; see, for example, [2, 3]. For some interesting arithmetic properties on the Frobenius-Euler polynomials and numbers, one is referred to [4–11].

We now turn to the Bernoulli polynomials $B_n(x)$ and the Euler polynomials $E_n(x)$, which are usually defined by the exponential generating functions

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1.5}$$

The rational numbers $B_n$ and the integers $E_n$ given by

$$B_n=B_n(0) \quad \text{and} \quad E_n=2^nE_n\left(\frac{1}{2}\right) \tag{1.6}$$

are called the Bernoulli numbers and the Euler numbers, respectively. It is easily seen from (1.1) and (1.5) that the Frobenius-Euler polynomials give the Euler polynomials when $\lambda=-1$ in (1.1), and the Bernoulli polynomials can be expressed by the Frobenius-Euler polynomials as follows:

$$m^{n-1}\sum_{k=0}^{m-1}\lambda^kB_n\left(\frac{k}{m}\right)=\frac{n}{\lambda-1}H_{n-1}\left(\frac{1}{\lambda}\right) \quad (m,n\geq 1). \tag{1.7}$$

It is well known that the Bernoulli and Euler polynomials and numbers play important roles in different areas of mathematics, and numerous interesting properties for them have been studied by many authors; see, for example, [12–14].

In the year 1963, Carlitz [15] explored some formulas of products of the Frobenius-Euler polynomials and obtained three expressions of products of the Frobenius-Euler polynomials to deduce Nielsen's [16] formulas on the Bernoulli and Euler polynomials. For example, Carlitz [15] showed that for non-negative integers $m$, $n$,

$$\begin{aligned} &H_m(x|\lambda)H_n(x|\mu)\\ &\quad=\frac{\lambda(\mu-1)}{\lambda\mu-1}\sum_{k=0}^{m}\binom{m}{k}H_k(\lambda)H_{m+n-k}(x|\lambda\mu)\\ &\qquad+\frac{\mu(\lambda-1)}{\lambda\mu-1}\sum_{k=0}^{n}\binom{n}{k}H_k(\mu)H_{m+n-k}(x|\lambda\mu)-\frac{(\lambda-1)(\mu-1)}{\lambda\mu-1}H_{m+n}(x|\lambda\mu), \end{aligned} \tag{1.8}$$

when $\lambda\mu\neq 1$. In the year 2012, Kim et al. [17] used a nice method called the Frobenius-Euler basis to establish the following new sums of products of two Frobenius-Euler polynomials:

$$\begin{aligned} &\frac{1}{n+1}\sum_{k=0}^{n}H_k(x|\lambda)H_{n-k}(x|\lambda)\\ &\quad=-\lambda\sum_{k=0}^{n-1}\binom{n}{k}\sum_{l=k}^{n}\frac{H_{l-k}(\lambda)H_{n-l}(\lambda)-2H_{n-k}(\lambda)}{n+1-k}H_k(x|\lambda)+H_n(x|\lambda), \end{aligned} \tag{1.9}$$

where $n$ is a positive integer. Following the work of Carlitz and Kim et al., He and Wang [18] extended Carlitz's [15] three formulas of products of the Frobenius-Euler polynomials, by virtue of which some analogues to the summation formula (1.9) were obtained. In the year 2014, Agoh and Dilcher [19] used a generalization of the idea showed in [20] to establish the following higher-order convolution identity for the Euler polynomials:

$$\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} E_{j_1}(x)\cdots E_{j_k}(x)=\sum_{r=1}^{k}\binom{k}{r}(-2)^{r-1}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+k-1}{l_0} \times E_{l_0}(x)E_{l_1}(0)\cdots E_{l_{k-r}}(0), \tag{1.10}$$

where $n$ is a non-negative integer and $k$ is a positive integer $k$ with $2\nmid k$. In the year 2016, by using identities for difference operators, techniques of symbolic computation, and tools from the probability theory, Dilcher and Vignat [21] extended (1.10) and obtained that for a non-negative integer $n$, a positive integer $k$ with $2\nmid k$, and arbitrary real numbers $a_1,\ldots,a_k$,

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}E_{j_1}(x)\cdots E_{j_k}(x)\\&=\sum_{r=1}^{k}\sum_{|J|=r}(-2)^{r-1}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n-l_0}}\\&\quad\times E_{l_0}(x)E_{l_1}(0)\cdots E_{l_{k-r}}(0),\end{aligned} \tag{1.11}$$

where, and in what follows, $(a)_k$ is the rising factorial defined for a complex number $a$ and a non-negative integer $k$ by

$$(a)_0=1\quad\text{and}\quad(a)_k=a(a+1)(a+2)\cdots(a+k-1)\quad(k\geq 1), \tag{1.12}$$

$\binom{n}{r_1,\ldots,r_k}$ is the multinomial coefficient defined for a positive integer $k$ and non-negative integers $n$, $r_1,\ldots,r_k$ by

$$\binom{n}{r_1,\ldots,r_k}=\frac{n!}{r_1!\cdots r_k!}, \tag{1.13}$$

$|J|$ is the cardinality of a subset $J\subseteq\{1,\ldots,k\}$ and $i_{r+1},\ldots,i_k\in\bar{J}=\{1,\ldots,k\}\setminus J$.

Motivated by the work of Dilcher and Vignat [21], in this paper we establish some new summation formulas for the products of an arbitrary number of the Frobenius-Euler polynomials by making use of the generating function methods and summation transform techniques developed in [22]. It turns out that some known formulas including (1.10) and (1.11) are deduced as special cases.

## 2 The statement of results

We first state the following formula for the products of an arbitrary number of the Frobenius-Euler polynomials and the rising factorials.

**Theorem 2.1** *Let $a_1, \dots, a_k$ be arbitrary complex numbers with $k$ being a positive integer. Then, for a non-negative integer $n$,*

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\dots,j_k\geq 0}} \binom{n}{j_1,\dots,j_k} \frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n} H_{j_1}(x_1|\lambda_1)\cdots H_{j_k}(x_k|\lambda_k)\\
&\quad= \sum_{r=1}^{k} \frac{1-\lambda_r}{1-\lambda_1\cdots\lambda_k} \sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\dots,l_k\geq 0}} \binom{n}{l_1,\dots,l_k} \frac{1}{(a_1+\cdots+a_k)_{n-l_r}}\\
&\qquad\times H_{l_r}(x_r|\lambda_1\cdots\lambda_k)\prod_{i=1}^{r-1}(a_i)_{l_i}H_{l_i}(x_i-x_r+1|\lambda_i)\\
&\qquad\times \prod_{i=r+1}^{k}\lambda_i(a_i)_{l_i}H_{l_i}(x_i-x_r|\lambda_i) \quad (\lambda_1\cdots\lambda_k\neq 1). \end{aligned} \tag{2.1}$$

We next discuss some special cases of Theorem 2.1. By taking $a_1 = \cdots = a_k = 1$ in Theorem 2.1, in light of (1.12) and (1.13), we get the following result.

**Corollary 2.2** *Let $k$ be a positive integer. Then, for a non-negative integer $n$,*

$$\begin{aligned}&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\dots,j_k\geq 0}} H_{j_1}(x_1|\lambda_1)\cdots H_{j_k}(x_k|\lambda_k)\\
&\quad= \sum_{r=1}^{k} \frac{1-\lambda_r}{1-\lambda_1\cdots\lambda_k} \sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\dots,l_k\geq 0}} \binom{n+k-1}{l_r} H_{l_r}(x_r|\lambda_1\cdots\lambda_k)\\
&\qquad\times \prod_{i=1}^{r-1}H_{l_i}(x_i-x_r+1|\lambda_i)\prod_{i=r+1}^{k}\lambda_i H_{l_i}(x_i-x_r|\lambda_i) \quad (\lambda_1\cdots\lambda_k\neq 1). \end{aligned} \tag{2.2}$$

The above Corollary 2.2 can be also found in [23] where it was established by using the generalized beta integral technique. In fact, Corollary 2.2 can be used to give a different expression for the new sums of products of two Frobenius-Euler polynomials appearing in (1.9). For example, taking $k = 2$ and then substituting $x$ for $x_1$, $y$ for $x_2$, $\lambda$ for $\lambda_1$, and $\mu$ for $\lambda_2$ in Corollary 2.2 gives

$$\begin{aligned}&\sum_{k=0}^{n} H_k(x|\lambda)H_{n-k}(y|\mu)\\
&\quad= \frac{\mu(1-\lambda)}{1-\lambda\mu}\sum_{k=0}^{n}\binom{n+1}{k}H_k(x|\lambda\mu)H_{n-k}(y-x|\mu)\\
&\qquad+ \frac{1-\mu}{1-\lambda\mu}\sum_{k=0}^{n}\binom{n+1}{k}H_k(y|\lambda\mu)H_{n-k}(x-y+1|\lambda) \quad (\lambda\mu\neq 1). \end{aligned} \tag{2.3}$$

Since the Frobenius-Euler polynomials satisfy the following difference equation (see, e.g., [17]):

$$H_n(x+1|\lambda) - \lambda H_n(x|\lambda) = (1-\lambda)x^n \quad (n\geq 0), \tag{2.4}$$

so by applying (2.4) to (2.3), we get

$$\begin{aligned}
&\sum_{k=0}^{n} H_k(x|\lambda)H_{n-k}(y|\mu)\\
&\quad= \frac{\mu(1-\lambda)}{1-\lambda\mu}\sum_{k=0}^{n}\binom{n+1}{k}H_k(x|\lambda\mu)H_{n-k}(y-x|\mu)\\
&\qquad+ \frac{\lambda(1-\mu)}{1-\lambda\mu}\sum_{k=0}^{n}\binom{n+1}{k}H_k(y|\lambda\mu)H_{n-k}(x-y|\lambda)\\
&\qquad+ \frac{(1-\lambda)(1-\mu)}{1-\lambda\mu}\sum_{k=0}^{n}\binom{n+1}{k}H_k(y|\lambda\mu)(x-y)^{n-k} \quad (\lambda\mu \neq 1).
\end{aligned}\tag{2.5}$$

It becomes obvious that the case $x = y$ and $\lambda = \mu$ in (2.5) gives

$$\begin{aligned}
\sum_{k=0}^{n} H_k(x|\lambda)H_{n-k}(x|\lambda) &= \frac{2\lambda}{1+\lambda}\sum_{k=0}^{n}\binom{n+1}{k}H_k\bigl(x|\lambda^2\bigr)H_{n-k}(\lambda)\\
&\quad+ \frac{1-\lambda}{1+\lambda}(n+1)H_n\bigl(x|\lambda^2\bigr) \quad (\lambda \neq -1),
\end{aligned}\tag{2.6}$$

which can be regarded as an equivalent version of (1.9). For a different proof of (2.5), see [18] for details.

On the other hand, from (2.4) and the fact (see, e.g., [24])

$$(x_1+y_1)(x_2+y_2)\cdots(x_k+y_k) = \sum_{J\subseteq\{1,\ldots,k\}}\prod_{i\in J}x_i\prod_{i\in\bar{J}}y_i \quad (k \geq 1),\tag{2.7}$$

we obtain that for a positive integer $r$,

$$\begin{aligned}
&\prod_{i=1}^{r-1} H_{j_i}(x_i - x_r + 1|\lambda_i)\\
&\quad= \sum_{J\subseteq\{1,\ldots,r-1\}}\prod_{i\in J}\lambda_i H_{j_i}(x_i - x_r|\lambda_i)\prod_{i\in\bar{J}}(1-\lambda_i)(x_i - x_r)^{j_i}.
\end{aligned}\tag{2.8}$$

Thus, by applying (2.8) to Theorem 2.1 and then taking $x_1 = \cdots = x_k = x$, we get the following result.

**Corollary 2.3** *Let $a_1,\ldots,a_k$ be arbitrary complex numbers with $k$ being a positive integer. Then, for a non-negative integer $n$,*

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}H_{j_1}(x|\lambda_1)\cdots H_{j_k}(x|\lambda_k)\\
&\quad= \sum_{r=1}^{k}\sum_{|J|=r}\frac{\lambda_J}{1-\lambda_1\cdots\lambda_k}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n}{l_0,l_1,\ldots,l_{k-r}}\frac{(a_{i_{r+1}})_{l_1}\cdots(a_{i_k})_{l_{k-r}}}{(a_1+\cdots+a_k)_{n-l_0}}\\
&\qquad\times H_{l_0}(x|\lambda_1\cdots\lambda_k)\lambda_{i_{r+1}}H_{l_1}(\lambda_{i_{r+1}})\cdots\lambda_{i_k}H_{l_{k-r}}(\lambda_{i_k}),
\end{aligned}\tag{2.9}$$

*where $\lambda_1\cdots\lambda_k \neq 1$, $\lambda_J = \prod_{j\in J}(1-\lambda_j)$ and $i_{r+1},\ldots,i_k \in \bar{J}$.*

In particular, if we take $\lambda_1 = \cdots = \lambda_k = -1$ with $2 \nmid k$ and let $a_1, \ldots, a_k$ be real numbers in Corollary 2.3, we get Dilcher and Vignat's identity (1.11) immediately. If we take $a_1 = \cdots = a_k = 1$ in Corollary 2.3, we obtain the following result.

**Corollary 2.4** *Let $n$ be a non-negative integer. Then, for a positive integer $k$,*

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} H_{j_1}(x|\lambda_1)\cdots H_{j_k}(x|\lambda_k)\\
&\quad= \sum_{r=1}^{k}\sum_{|J|=r}\frac{\lambda_J}{1-\lambda_1\cdots\lambda_k}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+k-1}{l_0}H_{l_0}(x|\lambda_1\cdots\lambda_k)\\
&\qquad\times \lambda_{i_{r+1}}H_{l_1}(\lambda_{i_{r+1}})\cdots\lambda_{i_k}H_{l_{k-r}}(\lambda_{i_k}),
\end{aligned}\tag{2.10}$$

*where $\lambda_1\cdots\lambda_k \neq 1$, $\lambda_J = \prod_{j\in J}(1-\lambda_j)$ and $i_{r+1}, \ldots, i_k \in \bar{J}$.*

The above Corollary 2.4 can be also found in [23] where it was obtained by applying (2.8) to Corollary 2.2. If we take $\lambda_1 = \cdots = \lambda_k = \lambda$ in Corollary 2.4, we obtain that for a non-negative integer $n$ and a positive integer $k$,

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} H_{j_1}(x|\lambda)\cdots H_{j_k}(x|\lambda)\\
&\quad= \sum_{r=1}^{k}\binom{k}{r}\frac{\lambda^{k-r}(1-\lambda)^r}{1-\lambda^k}\sum_{\substack{l_0+l_1+\cdots+l_{k-r}=n\\ l_0,l_1,\ldots,l_{k-r}\geq 0}}\binom{n+k-1}{l_0}H_{l_0}\bigl(x|\lambda^k\bigr)\\
&\qquad\times H_{l_1}(\lambda)\cdots H_{l_{k-r}}(\lambda).
\end{aligned}\tag{2.11}$$

Obviously, the case $\lambda = -1$ and $2 \nmid k$ in (2.11) gives Agoh and Dilcher's identity (1.10). If we take $k = 2$ in (2.11), we get that for a non-negative integer $n$,

$$\begin{aligned}
\sum_{\substack{j_1+j_2=n\\ j_1,j_2\geq 0}} H_{j_1}(x|\lambda)H_{j_2}(x|\lambda) &= \frac{2\lambda(1-\lambda)}{1-\lambda^2}\sum_{\substack{l_0+l_1=n\\ l_0,l_1\geq 0}}\binom{n+1}{l_0}H_{l_0}\bigl(x|\lambda^2\bigr)H_{l_1}(\lambda)\\
&\quad+\frac{(1-\lambda)^2}{1-\lambda^2}\binom{n+1}{n}H_n\bigl(x|\lambda^2\bigr)\quad(\lambda\neq -1),
\end{aligned}\tag{2.12}$$

which gives formula (2.6) immediately.

## 3 The proof of Theorem 2.1

For convenience, we denote by $[t_1^{i_1}\cdots t_k^{i_k}]f(t_1,\ldots,t_k)$ the coefficients of $t_1^{i_1}\cdots t_k^{i_k}$ in the power series expansion of $f(t_1,\ldots,t_k)$. It is clear that for non-negative integers $i_1,\ldots,i_k$, we have

$$\left[\frac{t_1^{i_1}}{i_1!}\cdots\frac{t_k^{i_k}}{i_k!}\right]f(t_1,\ldots,t_k) = i_1!\cdots i_k!\cdot\bigl[t_1^{i_1}\cdots t_k^{i_k}\bigr]f(t_1,\ldots,t_k).\tag{3.1}$$

We now recall the famous Euler's pentagonal number theorem: for $|x| < 1$,

$$(1-x)\bigl(1-x^2\bigr)\bigl(1-x^3\bigr)\cdots = 1 + \sum_{n=1}^{\infty}(-1)^n\bigl\{x^{\frac{1}{2}n(3n-1)} + x^{\frac{1}{2}n(3n+1)}\bigr\}, \tag{3.2}$$

which can be used effectively for the calculation of the number of partitions of $n$ (see, e.g., [25]). In his original proof of (3.2), Euler used the following beautiful idea:

$$\begin{aligned}&(1+x_1)(1+x_2)(1+x_3)\cdots \\ &\quad = (1+x_1) + x_2(1+x_1) + x_3(1+x_1)(1+x_2) + \cdots.\end{aligned} \tag{3.3}$$

Obviously, the finite form of (3.3) can be expressed as (see, e.g., [26])

$$\begin{aligned}&(1+x_1)(1+x_2)\cdots(1+x_k) \\ &\quad = (1+x_1) + x_2(1+x_1) + \cdots + x_k(1+x_1)(1+x_2)\cdots(1+x_{k-1}).\end{aligned} \tag{3.4}$$

If we replace $x_r$ by $x_r - 1$ for $1 \leq r \leq k$ in (3.4), then we have

$$x_1\cdots x_k - 1 = \sum_{r=1}^{k}(x_r - 1)x_1\cdots x_{r-1}, \tag{3.5}$$

where $x_1\cdots x_{r-1}$ is considered to be equal to 1 when $r = 1$. By taking $x_r = \lambda_r e^{t_r}$ for $1 \leq r \leq k$ in (3.5), we obtain that for a positive integer $k$,

$$\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k} - 1 = \sum_{r=1}^{k}\bigl(\lambda_r e^{t_r} - 1\bigr)\prod_{i=1}^{r-1}\lambda_i e^{t_i}, \tag{3.6}$$

which implies

$$\prod_{i=1}^{k}\frac{(\lambda_i - 1)e^{x_i t_i}}{\lambda_i e^{t_i} - 1} = \sum_{r=1}^{k}\frac{\lambda_r e^{t_r} - 1}{\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k} - 1}\prod_{i=1}^{r-1}\lambda_i e^{t_i}\prod_{i=1}^{k}\frac{(\lambda_i - 1)e^{x_i t_i}}{\lambda_i e^{t_i} - 1}. \tag{3.7}$$

Observe that

$$\begin{aligned}&\bigl(\lambda_r e^{t_r} - 1\bigr)\prod_{i=1}^{r-1}\lambda_i e^{t_i}\prod_{i=1}^{k}\frac{(\lambda_i - 1)e^{x_i t_i}}{\lambda_i e^{t_i} - 1} \\ &\quad = (\lambda_r - 1)e^{x_r(t_1+\cdots+t_k)}\prod_{i=1}^{r-1}\lambda_i\frac{(\lambda_i - 1)e^{(x_i - x_r + 1)t_i}}{\lambda_i e^{t_i} - 1}\prod_{i=r+1}^{k}\frac{(\lambda_i - 1)e^{(x_i - x_r)t_i}}{\lambda_i e^{t_i} - 1}.\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\begin{aligned}\prod_{i=1}^{k}\frac{(\lambda_i - 1)e^{x_i t_i}}{\lambda_i e^{t_i} - 1} &= \sum_{r=1}^{k}\frac{(\lambda_r - 1)e^{x_r(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k} - 1} \\ &\quad \times\prod_{i=1}^{r-1}\lambda_i\frac{(\lambda_i - 1)e^{(x_i - x_r + 1)t_i}}{\lambda_i e^{t_i} - 1}\prod_{i=r+1}^{k}\frac{(\lambda_i - 1)e^{(x_i - x_r)t_i}}{\lambda_i e^{t_i} - 1}.\end{aligned} \tag{3.9}$$

It is obvious that substituting $1/\lambda$ for $\lambda$ in (1.1) gives

$$\frac{\lambda-1}{\lambda e^t-1}e^{xt}=\sum_{n=0}^{\infty}H_n\left(x\,\middle|\,\frac{1}{\lambda}\right)\frac{t^n}{n!}, \tag{3.10}$$

and from (1.3) and (1.12) we get that for a non-negative integer $k$ and a complex number $a$,

$$(a)_k=(-1)^k k!\cdot\binom{-a}{k}. \tag{3.11}$$

It follows from (1.13), (3.10) and (3.11) that for a non-negative integer $n$ and complex numbers $a_1,\ldots,a_k$,

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\prod_{i=1}^{k}\frac{(\lambda_i-1)e^{x_i t_i}}{\lambda_i e^{t_i}-1}\right)\\
&\quad=\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}H_{j_1}\left(x_1\,\middle|\,\frac{1}{\lambda_1}\right)\cdots H_{j_k}\left(x_k\,\middle|\,\frac{1}{\lambda_k}\right)\\
&\quad=\frac{(-1)^n}{n!}\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}(a_1)_{j_1}\cdots(a_k)_{j_k}\\
&\qquad\times H_{j_1}\left(x_1\,\middle|\,\frac{1}{\lambda_1}\right)\cdots H_{j_k}\left(x_k\,\middle|\,\frac{1}{\lambda_k}\right).
\end{aligned}\tag{3.12}$$

On the other hand, since for a positive integer $k$ and a non-negative integer $N$ (see, e.g., [27]),

$$(t_1+\cdots+t_k)^N=\sum_{\substack{l_1+\cdots+l_k=N\\ l_1,\ldots,l_k\geq 0}}\binom{N}{l_1,\ldots,l_k}t_1^{l_1}\cdots t_k^{l_k}, \tag{3.13}$$

so by (3.10) and (3.13) we have

$$\frac{(\lambda_1\cdots\lambda_k-1)e^{x_r(t_1+\cdots+t_k)}}{\lambda_1\cdots\lambda_k e^{t_1+\cdots+t_k}-1}=\sum_{N=0}^{\infty}H_N\left(x_r\,\middle|\,\frac{1}{\lambda_1\cdots\lambda_k}\right)\sum_{\substack{l_1+\cdots+l_k=N\\ l_1,\ldots,l_k\geq 0}}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_k^{l_k}}{l_k!}. \tag{3.14}$$

If we multiply both sides of (3.9) by $[t_1^{j_1}\cdots t_k^{j_k}]$, with the help of (3.10) and (3.14), we discover

$$\begin{aligned}
&\left[t_1^{j_1}\cdots t_k^{j_k}\right]\left(\prod_{i=1}^{k}\frac{(\lambda_i-1)e^{x_i t_i}}{\lambda_i e^{t_i}-1}\right)\\
&\quad=\sum_{r=1}^{k}\frac{\lambda_r-1}{\lambda_1\cdots\lambda_k-1}\sum_{\substack{l_1,\ldots,l_{r-1},\\ l_{r+1},\ldots,l_k\geq 0}}\frac{H_{l_1+\cdots+l_{r-1}+j_r+l_{r+1}+\cdots+l_k}(x_r|\frac{1}{\lambda_1\cdots\lambda_k})}{l_1!\cdots l_{r-1}!\cdot j_r!\cdot l_{r+1}!\cdots l_k!}\\
&\qquad\times\prod_{i=1}^{r-1}\lambda_i\frac{H_{j_i-l_i}(x_i-x_r+1|\frac{1}{\lambda_i})}{(j_i-l_i)!}\prod_{i=r+1}^{k}\frac{H_{j_i-l_i}(x_i-x_r|\frac{1}{\lambda_i})}{(j_i-l_i)!}.
\end{aligned}\tag{3.15}$$

Hence, by replacing $l_i$ by $j_i - l_i$ for $i \neq r$ in (3.15), in light of (3.1), we obtain

$$\begin{aligned}
&\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\prod_{i=1}^{k}\frac{(\lambda_i-1)e^{x_it_i}}{\lambda_ie^{t_i}-1}\right)\\
&=\sum_{r=1}^{k}\frac{\lambda_r-1}{\lambda_1\cdots\lambda_k-1}\sum_{\substack{l_1+\cdots+l_k=j_1+\cdots+j_k\\ l_1,\ldots,l_k\geq 0}}H_{l_r}\left(x_r\middle|\frac{1}{\lambda_1\cdots\lambda_k}\right)\\
&\quad\times\prod_{i=1}^{r-1}\lambda_i\binom{j_i}{l_i}H_{l_i}\left(x_i-x_r+1\middle|\frac{1}{\lambda_i}\right)\prod_{i=r+1}^{k}\binom{j_i}{l_i}H_{l_i}\left(x_i-x_r\middle|\frac{1}{\lambda_i}\right).
\end{aligned}\tag{3.16}$$

It follows from (3.16) that

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\prod_{i=1}^{k}\frac{(\lambda_i-1)e^{x_it_i}}{\lambda_ie^{t_i}-1}\right)\\
&=\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\sum_{r=1}^{k}\frac{\lambda_r-1}{\lambda_1\cdots\lambda_k-1}\sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}}\binom{-a_r}{j_r}H_{l_r}\left(x_r\middle|\frac{1}{\lambda_1\cdots\lambda_k}\right)\\
&\quad\times\prod_{i=1}^{r-1}\lambda_i\binom{-a_i}{j_i}\binom{j_i}{l_i}H_{l_i}\left(x_i-x_r+1\middle|\frac{1}{\lambda_i}\right)\\
&\quad\times\prod_{i=r+1}^{k}\binom{-a_i}{j_i}\binom{j_i}{l_i}H_{l_i}\left(x_i-x_r\middle|\frac{1}{\lambda_i}\right).
\end{aligned}\tag{3.17}$$

It is clear from (1.3) that for non-negative integers $k$, $n$ and a complex number $a$,

$$\binom{a}{n}\binom{n}{k}=\binom{a}{k}\binom{a-k}{n-k},\tag{3.18}$$

which together with the famous Chu-Vandermonde convolution identity showed in [28] yields that for non-negative integers $l_1,\ldots,l_k$ with $l_1+\cdots+l_k=n$,

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_r}{j_r}\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i}{j_i}\binom{j_i}{l_i}\\
&=\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i}{l_i}\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{-a_r}{j_r}\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i-l_i}{j_i-l_i}\\
&=\prod_{\substack{i=1\\ i\neq r}}^{k}\binom{-a_i}{l_i}\binom{-(a_1+\cdots+a_k)-(n-l_r)}{n-(n-l_r)}.
\end{aligned}\tag{3.19}$$

By applying (3.19) to (3.17), in view of (3.11), we obtain

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}} \binom{-a_1}{j_1}\cdots\binom{-a_k}{j_k}\left[\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_k^{j_k}}{j_k!}\right]\left(\prod_{i=1}^{k}\frac{(\lambda_i-1)e^{x_i t_i}}{\lambda_i e^{t_i}-1}\right)\\
&=\frac{(-1)^n}{n!}\sum_{r=1}^{k}\frac{\lambda_r-1}{\lambda_1\cdots\lambda_k-1}\sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}}\binom{n}{l_1,\ldots,l_k}(a_1+\cdots+a_k+n-l_r)_{l_r}\\
&\quad\times H_{l_r}\left(x_r\,\middle|\,\frac{1}{\lambda_1\cdots\lambda_k}\right)\prod_{i=1}^{r-1}\lambda_i(a_i)_{l_i}H_{l_i}\left(x_i-x_r+1\,\middle|\,\frac{1}{\lambda_i}\right)\\
&\quad\times\prod_{i=r+1}^{k}(a_i)_{l_i}H_{l_i}\left(x_i-x_r\,\middle|\,\frac{1}{\lambda_i}\right).
\end{aligned}\tag{3.20}$$

Observe that

$$(a_1+\cdots+a_k+n-l_r)_{l_r}\cdot(a_1+\cdots+a_k)_{n-l_r}=(a_1+\cdots+a_k)_n.\tag{3.21}$$

By equating (3.12) and (3.20), in light of (3.21), we get

$$\begin{aligned}
&\sum_{\substack{j_1+\cdots+j_k=n\\ j_1,\ldots,j_k\geq 0}}\binom{n}{j_1,\ldots,j_k}\frac{(a_1)_{j_1}\cdots(a_k)_{j_k}}{(a_1+\cdots+a_k)_n}H_{j_1}\left(x_1\,\middle|\,\frac{1}{\lambda_1}\right)\cdots H_{j_k}\left(x_k\,\middle|\,\frac{1}{\lambda_k}\right)\\
&=\sum_{r=1}^{k}\frac{\lambda_r-1}{\lambda_1\cdots\lambda_k-1}\sum_{\substack{l_1+\cdots+l_k=n\\ l_1,\ldots,l_k\geq 0}}\binom{n}{l_1,\ldots,l_k}\frac{1}{(a_1+\cdots+a_k)_{n-l_r}}\\
&\quad\times H_{l_r}\left(x_r\,\middle|\,\frac{1}{\lambda_1\cdots\lambda_k}\right)\prod_{i=1}^{r-1}\lambda_i(a_i)_{l_i}H_{l_i}\left(x_i-x_r+1\,\middle|\,\frac{1}{\lambda_i}\right)\\
&\quad\times\prod_{i=r+1}^{k}(a_i)_{l_i}H_{l_i}\left(x_i-x_r\,\middle|\,\frac{1}{\lambda_i}\right).
\end{aligned}\tag{3.22}$$

Thus, by replacing $\lambda_i$ by $1/\lambda_i$ for $1\leq i\leq k$ in (3.22), the desired result follows immediately. This completes the proof of Theorem 2.1.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors participated in drafting, revising, and commenting on the manuscript. All authors read and approved the final manuscript.

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**References**


1. Frobenius, FG: Über die Bernoullischen Zahlen und die Eulerischen Polynome. Sitzungsber. K. Preußischen Akad. Wissenschaft, Berlin (1910)
2. Araci, S, Acikgoz, M: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math. **22**, 399-406 (2012)
3. Kim, T: Symmetry of power sum polynomials and multivariate fermionic $p$-adic invariant integral on $\mathbb{Z}_p$. Russ. J. Math. Phys. **16**, 93-96 (2009)
4. Araci, S, Acikgoz, M: On the von Staudt-Clausen's theorem related to $q$-Frobenius-Euler numbers. J. Number Theory **159**, 329-339 (2016)
5. Kim, DS, Kim, T, Seo, J-J, Komatsu, T: Barnes' multiple Frobenius-Euler and poly-Bernoulli mixed-type polynomials. Adv. Differ. Equ. **2014**, Article ID 92 (2014)
6. Kim, T: An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic $p$-adic invariant $q$-integrals on $\mathbb{Z}_p$. Rocky Mt. J. Math. **41**, 239-247 (2011)
7. Kim, T: Identities involving Frobenius-Euler polynomials arising from non-linear differential equations. J. Number Theory **132**, 2854-2865 (2012)
8. Kim, T, Lee, B, Lee, S-H, Rim, S-H: Some identities for the Frobenius-Euler numbers and polynomials. J. Comput. Anal. Appl. **15**, 544-551 (2013)
9. Kim, T, Mansour, T: Umbral calculus associated with Frobenius-type Eulerian polynomials. Russ. J. Math. Phys. **21**, 484-493 (2014)
10. Simsek, Y, Bayad, A, Lokesha, V: $q$-Bernstein polynomials related to $q$-Frobenius-Euler polynomials, $l$-functions, and $q$-Stirling numbers. Math. Methods Appl. Sci. **35**, 877-884 (2012)
11. Simsek, Y: Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their applications. Fixed Point Theory Appl. **2013**, Article ID 87 (2013)
12. Cohen, H: Number Theory, Volume II, Analytic and Modern Tools. Graduate Texts in Math., vol. 240. Springer, New York (2007)
13. Nörlund, NE: Vorlesungen über Differenzenrechnung. Springer, Berlin (1924)
14. Srivastava, HM, Choi, J: Zeta and $q$-Zeta Functions and Associated Series and Integrals. Elsevier, Amsterdam (2012)
15. Carlitz, L: The product of two Eulerian polynomials. Math. Mag. **36**, 37-41 (1963)
16. Nielsen, N: Traité élémentaire des nombres de Bernoulli. Gauthier-Villars, Paris (1923)
17. Kim, DS, Kim, T: Some new identities of Frobenius-Euler numbers and polynomials. J. Inequal. Appl. **2012**, Article ID 307 (2012)
18. He, Y, Wang, SJ: New formulae of products of the Frobenius-Euler polynomials. J. Inequal. Appl. **2014**, Article ID 261 (2014)
19. Agoh, T, Dilcher, K: Higher-order convolutions for Bernoulli and Euler polynomials. J. Math. Anal. Appl. **419**, 1235-1247 (2014)
20. Agoh, T: Convolution identities for Bernoulli and Genocchi polynomials. Electron. J. Comb. **21**, Article ID P1.65 (2014)
21. Dilcher, K, Vignat, C: General convolution identities for Bernoulli and Euler polynomials. J. Math. Anal. Appl. **435**, 1478-1498 (2016)
22. He, Y, Kim, T: General convolution identities for Apostol-Bernoulli, Euler and Genocchi polynomials. J. Nonlinear Sci. Appl. **9**, 4780-4797 (2016)
23. He, Y, Araci, S, Srivastava, HM: Summation formulas for the products of the Frobenius-Euler polynomials. Ramanujan J. (2016). doi:10.1007/s11139-016-9802-4
24. Stanley, RP: Enumerative Combinatorics, vol. 1. Cambridge University Press, Cambridge (1997)
25. Hardy, GH, Wright, EM: An Introduction to the Theory of Numbers, 5th edn. Oxford University Press, New York (1979)
26. Bhatnagar, G: In praise of an elementary identity of Euler. Electron. J. Comb. **18**, Article ID P13 (2011)
27. Comtet, L: Advanced Combinatorics, the Art of Finite and Infinite Expansions. Reidel, Dordrecht (1974)
28. Gould, HW, Srivastava, HM: Some combinatorial identities associated with the Vandermonde convolution. Appl. Math. Comput. **84**, 97-102 (1997)