Abbas et al. *Advances in Difference Equations* (2017) 2017:180
DOI 10.1186/s13662-017-1231-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type

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Abstract
This article deals with some existence and Ulam-Hyers-Rassias stability results for a class of functional differential equations involving the Hilfer-Hadamard fractional derivative. An application is made of a Schauder fixed point theorem for the existence of solutions. Next we prove that our problem is generalized Ulam-Hyers-Rassias stable.

MSC: 26A33; 34A08; 34K37

Keywords: functional differential equation; left-sided mixed Hadamard fractional integral; Hilfer-Hadamard fractional derivative; existence; Ulam-Hyers-Rassias stability

1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences. For some fundamental results in the theory of fractional calculus and fractional ordinary and partial differential equations, we refer the reader to the monographs of Abbas *et al.* [1, 2], Samko *et al.* [3], Kilbas *et al.* [4] and Zhou [5], the papers [6–22] and the references therein.

The stability of functional equations was originally raised by Ulam [23], next by Hyers [24]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [25] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [26], and the papers of Abbas *et al.* [6, 8, 9, 27–29], Petru *et al.* [30], Rus [31, 32], and Wang *et al.* [33, 34]. More details from historical point of view, and recent developments of such stabilities are reported in [31, 35].

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [36–42]. Motivated by the Hilfer fractional derivative (which interpolates the Riemann-Liouville derivative and the Caputo derivative), Qassim *et al.* [43, 44] considered a new type of fractional derivative (which interpolates the Hadamard derivative and its Caputo counterpart). Motivated by the above papers, in this article we discuss the existence and the Ulam stability of solutions for the following problem of Hilfer-Hadamard fractional differential equations of the form

$$\begin{cases} ({}^H D_1^{\alpha,\beta} u)(t) = f(t,u(t)); & t \in J := [1,T], \\ ({}^H I_1^{1-\gamma} u)(t)\big|_{t=1} = \phi, \end{cases} \tag{1}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $T > 1$, $\phi \in \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, ${}^H I_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^H D_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β, introduced by Hilfer in [38].

The present paper initiates the Ulam stability for differential equations involving the Hilfer-Hadamard fractional derivative.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into $\mathbb{R}$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in J} \big|v(t)\big|.$$

By $L^1(J)$, we denote the space of Lebesgue-integrable functions $v : J \to \mathbb{R}$ with the norm

$$\|v\|_1 = \int_0^T \big|v(t)\big|\,dt.$$

As usual, $\mathrm{AC}(J)$ denotes the space of absolutely continuous functions from J into $\mathbb{R}$. We denote by $\mathrm{AC}^1(J)$ the space defined by

$$\mathrm{AC}^1(J) := \left\{ w : J \to \mathbb{R} : \frac{d}{dt} w(t) \in \mathrm{AC}(J) \right\}.$$

Let

$$\delta = t\frac{d}{dt}, \qquad q > 0, \qquad n = [q] + 1,$$

where $[q]$ is the integer part of q. Define the space

$$\mathrm{AC}^n_\delta := \left\{ u : [1,T] \to E : \delta^{n-1}\big[u(t)\big] \in \mathrm{AC}(J) \right\}.$$

Let $\gamma \in (0,1]$, by $C_{\gamma,\ln}(J)$, $C_\gamma(J)$ and $C^1_\gamma(J)$, we denote the weighted spaces of continuous functions defined by

$$C_{\gamma,\ln}(J) = \left\{ w(t) : (\ln t)^{1-\gamma} w(t) \in C \right\}$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in J} \big|(\ln t)^{1-\gamma} w(t)\big|,$$

$$C_\gamma(J) = \left\{ w : (0,T] \to \mathbb{R} : t^{1-\gamma} w(t) \in C \right\}$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t\in J}\left|t^{1-\gamma} w(t)\right|,$$

and

$$C^1_\gamma(J) = \left\{w \in C : \frac{dw}{dt} \in C_\gamma\right\}$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \left\|w'\right\|_{C_\gamma}.$$

In the following, we denote $\|w\|_{C_{\gamma,\ln}}$ by $\|w\|_C$.

Now, we give some results and properties of fractional calculus.

Definition 2.1 ([2–4]; Riemann-Liouville fractional integral) The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(J)$ is defined by

$$\left(I_1^r w\right)(t) = \frac{1}{\Gamma(r)}\int_1^t (t-s)^{r-1} w(s)\,ds \quad \text{for a.e. } t \in J,$$

where $\Gamma(\cdot)$ is the (Euler's) gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1}e^{-t}\,dt; \quad \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_1^r w \in C$, and

$$\left(I_1^{r_1} I_1^{r_2} w\right)(t) = \left(I_1^{r_1+r_2} w\right)(t) \quad \text{for a.e. } t \in J.$$

Definition 2.2 ([2–4]; Riemann-Liouville fractional derivative) The Riemann-Liouville fractional derivative of order $r > 0$ of a function $w \in L^1(J)$ is defined by

$$\begin{aligned}\left(D_1^r w\right)(t) &= \left(\frac{d^n}{dt^n} I_1^{n-r} w\right)(t) \\ &= \frac{1}{\Gamma(n-r)}\frac{d^n}{dt^n}\int_1^t (t-s)^{n-r-1} w(s)\,ds \quad \text{for a.e. } t \in J,\end{aligned}$$

where $n = [r] + 1$ and $[r]$ is the integer part of r.

In particular, if $r \in (0,1]$, then

$$\begin{aligned}\left(D_1^r w\right)(t) &= \left(\frac{d}{dt} I_1^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)}\frac{d}{dt}\int_1^t (t-s)^{-r} w(s)\,ds \quad \text{for a.e. } t \in J.\end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(J)$. Then the following expression leads to the left inverse operator as follows:

$$\big(D_1^r I_1^r w\big)(t) = w(t) \quad \text{for all } t \in (1,T].$$

Moreover, if $I_1^{1-r} w \in C_{1-\gamma}^1(J)$, then the following composition is proved in [3]:

$$\big(I_1^r D_1^r w\big)(t) = w(t) - \frac{(I_1^{1-r}w)(1^+)}{\Gamma(r)} t^{r-1} \quad \text{for all } t \in (1,T].$$

Definition 2.3 ([2–4]; Caputo fractional derivative) The Caputo fractional derivative of order $r > 0$ of a function $w \in L^1(J)$ is defined by

$$\begin{aligned}\big({}^c D_1^r w\big)(t) &= \left(I_1^{n-r} \frac{d^n}{dt^n} w\right)(t) \\ &= \frac{1}{\Gamma(n-r)} \int_1^t (t-s)^{n-r-1} \frac{d^n}{ds^n} w(s)\, ds \quad \text{for a.e. } t \in J.\end{aligned}$$

In particular, if $r \in (0,1]$, then

$$\begin{aligned}\big({}^c D_1^r w\big)(t) &= \left(I_1^{1-r} \frac{d}{dt} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_1^t (t-s)^{-r} \frac{d}{ds} w(s)\, ds \quad \text{for a.e. } t \in J.\end{aligned}$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [4, 45] for a more detailed analysis.

Definition 2.4 ([4, 45]; Hadamard fractional integral) The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1(I,E)$ is defined as

$$\big({}^H I_1^q g\big)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s}\, ds,$$

provided the integral exists.

Example 2.5 Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q} \quad \text{for a.e. } t \in [0,e].$$

Set

$$\delta = x\frac{d}{dx}, \qquad q > 0, \qquad n = [q] + 1$$

and

$$\mathrm{AC}_\delta^n := \big\{u : [1,T] \to E : \delta^{n-1}\big[u(x)\big] \in \mathrm{AC}(J)\big\}.$$

Analogous to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way.

Definition 2.6 ([4, 45]; Hadamard fractional derivative) The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in \mathrm{AC}_\delta^n$ is defined as

$$\left({}^H D_1^q w\right)(x) = \delta^n \left({}^H I_1^{n-q} w\right)(x).$$

In particular, if $q \in (0,1]$, then

$$\left({}^H D_1^q w\right)(x) = \delta\left({}^H I_1^{1-q} w\right)(x).$$

Example 2.7 Let $0 < q < 1$. Then

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)}(\ln t)^{1-q} \quad \text{for a.e. } t \in [0,e].$$

It has been proved (see, e.g., Kilbas [46], Theorem 4.8) that in the space $L^1(J)$ the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,

$$\left({}^H D_1^q\right)\left({}^H I_1^q w\right)(x) = w(x).$$

From Theorem 2.3 of [4], we have

$$\left({}^H I_1^q\right)\left({}^H D_1^q w\right)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way.

Definition 2.8 (Caputo-Hadamard fractional derivative) The Caputo-Hadamard fractional derivative of order $q > 0$ applied to the function $w \in \mathrm{AC}_\delta^n$ is defined as

$$\left({}^{Hc} D_1^q w\right)(x) = \left({}^H I_1^{n-q} \delta^n w\right)(x).$$

In particular, if $q \in (0,1]$, then

$$\left({}^{Hc} D_1^q w\right)(x) = \left({}^H I_1^{1-q} \delta w\right)(x).$$

In [38], Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [39–41]).

Definition 2.9 (Hilfer fractional derivative) Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(J)$, $I_1^{(1-\alpha)(1-\beta)} w \in \mathrm{AC}^1(J)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$\left(D_1^{\alpha,\beta} w\right)(t) = \left(I_1^{\beta(1-\alpha)} \frac{d}{dt} I_1^{(1-\alpha)(1-\beta)} w\right)(t) \quad \text{for a.e. } t \in J. \tag{2}$$

Properties Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(J)$.

1. The operator $(D_1^{\alpha,\beta} w)(t)$ can be written as

$$\big(D_1^{\alpha,\beta} w\big)(t) = \left(I_1^{\beta(1-\alpha)} \frac{d}{dt} I_1^{1-\gamma} w\right)(t) = \big(I_1^{\beta(1-\alpha)} D_1^{\gamma} w\big)(t) \quad \text{for a.e. } t \in J.$$

   Moreover, the parameter γ satisfies

$$\gamma \in (0,1], \qquad \gamma \geq \alpha, \qquad \gamma > \beta, \qquad 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2) for $\beta = 0$ coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$D_1^{\alpha,0} = D_1^{\alpha}, \quad \text{and} \quad D_1^{\alpha,1} = {}^c D_1^{\alpha}.$$

3. If $D_1^{\beta(1-\alpha)} w$ exists and in $L^1(J)$, then

$$\big(D_1^{\alpha,\beta} I_1^{\alpha} w\big)(t) = \big(I_1^{\beta(1-\alpha)} D_1^{\beta(1-\alpha)} w\big)(t) \quad \text{for a.e. } t \in J.$$

   Furthermore, if $w \in C_\gamma(J)$ and $I_1^{1-\beta(1-\alpha)} w \in C_\gamma^1(J)$, then

$$\big(D_1^{\alpha,\beta} I_1^{\alpha} w\big)(t) = w(t) \quad \text{for a.e. } t \in J.$$

4. If $D_1^{\gamma} w$ exists and in $L^1(J)$, then

$$\big(I_1^{\alpha} D_1^{\alpha,\beta} w\big)(t) = \big(I_1^{\gamma} D_1^{\gamma} w\big)(t) = w(t) - \frac{I_1^{1-\gamma}(1^+)}{\Gamma(\gamma)} t^{\gamma-1} \quad \text{for a.e. } t \in J.$$

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [43]) is defined in the following way.

Definition 2.10 (Hilfer-Hadamard fractional derivative) Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in L^1(J)$, and ${}^H I_1^{(1-\alpha)(1-\beta)} w \in \mathrm{AC}^1(J)$. The Hilfer-Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned}\big({}^H D_1^{\alpha,\beta} w\big)(t) &= \big({}^H I_1^{\beta(1-\alpha)} \big({}^H D_1^{\gamma} w\big)\big)(t) \\ &= \big({}^H I_1^{\beta(1-\alpha)} \delta \big({}^H I_1^{1-\gamma} w\big)\big)(t) \quad \text{for a.e. } t \in J. \end{aligned} \tag{3}$$

This new fractional derivative (3) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed, for $\beta = 0$, this derivative reduces to the Hadamard fractional derivative, and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

$${}^H D_1^{\alpha,0} = {}^H D_1^{\alpha}, \quad \text{and} \quad {}^H D_1^{\alpha,1} = {}^{Hc} D_1^{\alpha}.$$

From Theorem 21 in [44], we concluded the following lemma.

Lemma 2.11 *Let $f : I \times E \to E$ be such that $f(\cdot, u(\cdot)) \in C_{\gamma,\ln}(J)$ for any $u \in C_{\gamma,\ln}(J)$. Then problem* (1) *is equivalent to the problem of the solutions of the Volterra integral equation*

$$u(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \left({}^H I_1^\alpha f\left(\cdot, u(\cdot)\right)\right)(t).$$

Now, we consider the Ulam stability for problem (1). Let $\epsilon > 0$ and $\Phi : I \to [0,\infty)$ be a continuous function. We consider the following inequalities:

$$\left|\left({}^H D_1^{\alpha,\beta} u\right)(t) - f\left(t, u(t)\right)\right| \le \epsilon; \quad t \in J. \tag{4}$$

$$\left|\left({}^H D_1^{\alpha,\beta} u\right)(t) - f\left(t, u(t)\right)\right| \le \Phi(t); \quad t \in J. \tag{5}$$

$$\left|\left({}^H D_1^{\alpha,\beta} u\right)(t) - f\left(t, u(t)\right)\right| \le \epsilon\Phi(t); \quad t \in J. \tag{6}$$

Definition 2.12 ([2, 31]) Problem (1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C_{\gamma,\ln}$ of inequality (4) there exists a solution $v \in C_{\gamma,\ln}$ of (1) with

$$\left|u(t) - v(t)\right| \le \epsilon c_f; \quad t \in J.$$

Definition 2.13 ([2, 31]) Problem (1) is generalized Ulam-Hyers stable if there exists $c_f : C([0,\infty),[0,\infty))$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in C_{\gamma,\ln}$ of inequality (4) there exists a solution $v \in C_{\gamma,\ln}$ of (1) with

$$\left|u(t) - v(t)\right| \le c_f(\epsilon); \quad t \in J.$$

Definition 2.14 ([2, 31]) Problem (1) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C_{\gamma,\ln}$ of inequality (6) there exists a solution $v \in C_{\gamma,\ln}$ of (1) with

$$\left|u(t) - v(t)\right| \le \epsilon c_{f,\Phi}\Phi(t); \quad t \in J.$$

Definition 2.15 ([2, 31]) Problem (1) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each solution $u \in C_{\gamma,\ln}$ of inequality (5) there exists a solution $v \in C_{\gamma,\ln}$ of (1) with

$$\left|u(t) - v(t)\right| \le c_{f,\Phi}\Phi(t); \quad t \in J.$$

Remark 2.16 It is clear that

(i) Definition 2.12 $\Rightarrow$ Definition 2.13,
(ii) Definition 2.14 $\Rightarrow$ Definition 2.15,
(iii) Definition 2.14 for $\Phi(\cdot) = 1 \Rightarrow$ Definition 2.12.

One can have similar remarks for inequalities (4) and (6).

In the sequel we will make use of the following fixed point theorem.

Theorem 2.17 (Schauder fixed point theorem [47]) *Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E, and $N : Q \to Q$ is a compact and continuous map. Then N has at least one fixed point in Q.*

3 Existence of solutions

Let us start by defining what we mean by a solution of problem (1).

Definition 3.1 By a solution of problem (1) we mean a measurable function $u \in C_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+) = \phi$ and the equation $({}^H D_1^{\alpha,\beta} u)(t) = f(t,u(t))$ on J.

The following hypotheses will be used in the sequel.

(H_1) The function $t \mapsto f(t,u)$ is measurable on I for each $u \in C_{\gamma,\ln}$, and the function $u \mapsto f(t,u)$ is continuous on $C_{\gamma,\ln}$ for a.e. $t \in J$,

(H_2) There exists a continuous function $p : I \to [0,\infty)$ such that

$$\bigl|f(t,u)\bigr| \le \frac{p(t)}{1+|u|}|u| \quad \text{for a.e. } t \in J \text{ and each } u \in \mathbb{R}.$$

Set

$$p^* = \sup_{t\in J} p(t).$$

Now, we shall prove the following theorem concerning the existence of solutions of problem (1).

Theorem 3.2 *Assume that hypotheses* (H_1) *and* (H_2) *hold. Then problem* (1) *has at least one solution defined on* J.

Proof Consider the operator $N : C_{\gamma,\ln} \to C_{\gamma,\ln}$ defined by

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s,u(s))}{s\Gamma(\alpha)}\,ds. \tag{7}$$

Clearly, the fixed points of the operator N are solution of problem (1).

For any $u \in C_{\gamma,\ln}$ and each $t \in J$, we have

$$\begin{aligned}
\bigl|(\ln t)^{1-\gamma}(Nu)(t)\bigr| &\le \frac{|\phi|}{\Gamma(\gamma)} + \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \bigl|f\bigl(s,u(s)\bigr)\bigr|\frac{ds}{s} \\
&\le \frac{|\phi|}{\Gamma(\gamma)} + \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} p(s)\frac{ds}{s} \\
&\le \frac{|\phi|}{\Gamma(\gamma)} + \frac{p^*(\ln T)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\le \frac{|\phi|}{\Gamma(\gamma)} + \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}.
\end{aligned}$$

Thus

$$\bigl\|N(u)\bigr\|_C \le \frac{|\phi|}{\Gamma(\gamma)} + \frac{p^*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} := R. \tag{8}$$

This proves that N transforms the ball $B_R := B(0,R) = \{w \in C_{\gamma,\ln} : \|w\|_C \le R\}$ into itself. We shall show that the operator $N : B_R \to B_R$ satisfies all the assumptions of Theorem 2.17. The proof will be given in several steps.

Step 1. $N : B_R \to B_R$ is continuous.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in J$, we have

$$\begin{aligned} &\left|(\ln t)^{1-\gamma}(Nu_n)(t) - (\ln t)^{1-\gamma}(Nu)(t)\right| \\ &\quad\le \frac{(\ln t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \left|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\right| \frac{ds}{s}. \end{aligned} \tag{9}$$

Since $u_n \to u$ as $n \to \infty$ and f is continuous, by the Lebesgue dominated convergence theorem, equation (9) implies

$$\left\|N(u_n) - N(u)\right\|_C \to 0 \quad \text{as } n \to \infty.$$

Step 2. $N(B_R)$ is uniformly bounded.

This is clear since $N(B_R) \subset B_R$ and B_R is bounded.

Step 3. $N(B_R)$ is equicontinuous.

Let $t_1, t_2 \in J$, $t_1 < t_2$ and let $u \in B_R$. Thus, we have

$$\begin{aligned} &\left|(\ln t_2)^{1-\gamma}(Nu)(t_2) - (\ln t_1)^{1-\gamma}(Nu)(t_1)\right| \\ &\quad\le \left|(\ln t_2)^{1-\gamma}\int_1^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s\Gamma(\alpha)}\,ds - (\ln t_1)^{1-\gamma}\int_1^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\frac{f(s,u(s))}{s\Gamma(\alpha)}\,ds\right| \\ &\quad\le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{|f(s,u(s))|}{s\Gamma(\alpha)}\,ds \\ &\qquad + \int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{|f(s,u(s))|}{s\Gamma(\alpha)}\,ds \\ &\quad\le (\ln t_2)^{1-\gamma}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\frac{p(s)}{s\Gamma(\alpha)}\,ds \\ &\qquad + \int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\frac{p(s)}{s\Gamma(\alpha)}\,ds. \end{aligned}$$

Hence, we get

$$\begin{aligned} &\left|(\ln t_2)^{1-\gamma}(Nu)(t_2) - (\ln t_1)^{1-\gamma}(Nu)(t_1)\right| \\ &\quad\le \frac{p_*(\ln T)^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}\left(\ln\frac{t_2}{t_1}\right)^{\alpha} \\ &\qquad + \frac{p_*}{\Gamma(\alpha)}\int_1^{t_1}\left|(\ln t_2)^{1-\gamma}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - (\ln t_1)^{1-\gamma}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\right|\,ds. \end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that N is continuous and compact. From an application of Schauder's theorem (Theorem 2.17), we deduce that N has at least a fixed point u which is a solution of problem (1). □

4 Ulam-Hyers-Rassias stability

Now, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (1).

Theorem 4.1 *Assume that hypotheses* (H_1), (H_2) *and the following hypotheses hold.*

(H_3) *There exists* $\lambda_\Phi > 0$ *such that for each* $t \in J$, *we have*

$$\left({}^H I_1^\alpha \Phi\right)(t) \leq \lambda_\Phi \Phi(t);$$

(H_4) *There exists* $q \in C(J, [0, \infty))$ *such that for each* $t \in J$, *we have*

$$p(t) \leq q(t)\Phi(t).$$

Then problem (1) *is generalized Ulam-Hyers-Rassias stable.*

Proof Consider the operator $N : C_{\gamma,\ln} \to C_{\gamma,\ln}$ defined in (7). Let u be a solution of inequality (5), and let us assume that v is a solution of problem (1). Thus, we have

$$v(t) = \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, v(s))}{s\Gamma(\alpha)}\, ds.$$

From inequality (5), for each $t \in J$, we have

$$\left| u(t) - \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} - \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s\Gamma(\alpha)}\, ds \right| \leq \left({}^H I_1^\alpha \Phi\right)(t).$$

Set

$$q^* = \sup_{t\in J} q(t).$$

From hypotheses (H_3) and (H_4), for each $t \in J$, we get

$$\begin{aligned}
\left|u(t) - v(t)\right| &\leq \left| u(t) - \frac{\phi}{\Gamma(\gamma)}(\ln t)^{\gamma-1} - \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s, u(s))}{s\Gamma(\alpha)}\, ds \right| \\
&\quad + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s, u(s)) - f(s, v(s))|}{s\Gamma(\alpha)}\, ds \\
&\leq \left({}^H I_1^\alpha \Phi\right)(t) + \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{2q^*\Phi(s)}{s\Gamma(\alpha)}\, ds \\
&\leq \lambda_\phi \Phi(t) + 2q^*\left({}^H I_1^\alpha \Phi\right)(t) \\
&\leq \left[1 + 2q^*\right]\lambda_\phi \Phi(t) \\
&:= c_{f,\Phi}\Phi(t).
\end{aligned}$$

Hence, problem (1) is generalized Ulam-Hyers-Rassias stable. □

In the sequel, we will use the following theorem.

Theorem 4.2 *Let* (Ω, d) *be a generalized complete metric space and* $\Theta : \Omega \to \Omega$ *be a strictly contractive operator with a Lipschitz constant* $L < 1$. *If there exists a nonnegative integer* k *such that* $d(\Theta^{k+1}x, \Theta^k x) < \infty$ *for some* $x \in \Omega$, *then the following propositions hold true*:

(A) *The sequence* $(\Theta^k x)_{n\in N}$ *converges to a fixed point* x^* *of* Θ;
(B) x^* *is the unique fixed point of* Θ *in* $\Omega^* = \{y \in \Omega \mid d(\Theta^k x, y) < \infty\}$;
(C) *If* $y \in \Omega^*$, *then* $d(y, x^*) \leq \frac{1}{1-L} d(y, \Theta x)$.

Let $X = X(I, \mathbb{R})$ be the metric space, with the metric

$$d(u, v) = \sup_{t\in J} \frac{\|u(t) - v(t)\|_C}{\Phi(t)}.$$

Theorem 4.3 *Assume that* (H_3) *and the following hypothesis hold.*

(H_5) *There exists* $\varphi \in C(J, [0, \infty))$ *such that for each* $t \in J$ *and all* $u, v \in \mathbb{R}$, *we have*

$$\left|f(t, u) - f(t, u)\right| \leq (\ln t)^{1-\gamma} \varphi(t)\Phi(t)|u - v|.$$

If

$$L := (\ln T)^{1-\gamma} \varphi^* \lambda_\phi < 1, \tag{10}$$

where $\varphi^* = \sup_{t\in J} \varphi(t)$, *then there exists a unique solution* u_0 *of problem* (1), *and problem* (1) *is generalized Ulam-Hyers-Rassias stable. Furthermore, we have*

$$\left|u(t) - u_0(t)\right| \leq \frac{\Phi(t)}{1 - L}.$$

Proof Let $N : C_{\gamma,\ln} \to C_{\gamma,\ln}$ be the operator defined in (7). Applying Theorem 4.2, we have

$$\begin{aligned}
\left|(Nu)(t) - (Nv)(t)\right| &\leq \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{|f(s, u(s)) - f(s, v(s))|}{s\Gamma(\alpha)}\, ds \\
&\leq \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{\varphi(s)\Phi(s)|(\ln s)^{1-\gamma} u(s) - (\ln s)^{1-\gamma} v(s)|}{s\Gamma(\alpha)}\, ds \\
&\leq \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{\varphi^*\Phi(s)\|u - v\|_C}{s\Gamma(\alpha)}\, ds \\
&\leq \varphi^* \left({}^H I_1^\alpha \Phi\right)(t)\|u - v\|_C \\
&\leq \varphi^* \lambda_\phi \Phi(t)\|u - v\|_C.
\end{aligned}$$

Thus

$$\left|(\ln t)^{1-\gamma}(Nu)(t) - (\ln t)^{1-\gamma}(Nv)(t)\right| \leq (\ln T)^{1-\gamma} \varphi^* \lambda_\phi \Phi(t)\|u - v\|_C.$$

Hence, we get

$$d\bigl(N(u), N(v)\bigr) = \sup_{t\in J} \frac{\|(Nu)(t) - (Nv)(t)\|_C}{\Phi(t)} \leq L\|u - v\|_C,$$

from which we conclude the theorem. □

5 An example

As an application of our results, we consider the following problem of Hilfer-Hadamard fractional differential equation of the form

$$\begin{cases} ({}^H D_1^{\frac{1}{2},\frac{1}{2}} u)(t) = f(t, u(t)); \quad t \in [1, e], \\ ({}^H I_1^{\frac{1}{4}} u)(t)\big|_{t=1} = 0, \end{cases} \tag{11}$$

where

$$\begin{cases} f(t, u) = \frac{(t-1)^{\frac{-1}{4}} \sin(t-1)}{64(1+\sqrt{t-1})(1+|u|)}; \quad t \in (1, e], u \in \mathbb{R}, \\ f(1, u) = 0; \quad u \in \mathbb{R}. \end{cases}$$

Clearly, the function f is continuous.

Hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \frac{(t-1)^{\frac{-1}{4}} |\sin(t-1)|}{64(1+\sqrt{t-1})}; \quad t \in (1, e], \\ p(1) = 0. \end{cases}$$

Hence, Theorem 3.2 implies that problem (11) has at least one solution defined on $[1, e]$. Also, hypothesis (H_3) is satisfied with

$$\Phi(t) = e^3, \quad \text{and} \quad \lambda_\Phi = \frac{2}{\sqrt{\pi}}.$$

Indeed, for each $t \in [1, e]$, we get

$$\begin{aligned} \left({}^H I_1^\alpha \Phi\right)(t) &\le \frac{2e^3}{\sqrt{\pi}} \\ &= \lambda_\Phi \Phi(t). \end{aligned}$$

Consequently, Theorem 4.1 implies that problem (11) is generalized Ulam-Hyers-Rassias stable.

Acknowledgements
The authors would like to thank all the anonymous reviewers and the editors for their helpful advice and hard work. The work was supported by the National Natural Science Foundation of China (No. 11671339).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SA, MB, and JEL contributed to Sections 1, 2, 3, and 4. AA and YZ contributed to Sections 1 and 5.

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Publisher's Note

Received: 28 December 2016 Accepted: 2 June 2017 Published online: 27 June 2017

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