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Strong global attractors for nonclassical diffusion equation with fading memory

Yubao Zhang, Xuan Wang* and Chenghua Gao

*Correspondence:
wangxuan@nwnu.edu.cn
College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, China

Abstract

Based on semigroup theory and the method of contractive function, we prove the existence of global attractors for a nonclassical diffusion equation with fading memory in the space $H^2(\Omega) \cap H_0^1(\Omega) \times L^2_\mu(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega))$, when the nonlinearity satisfies arbitrary polynomial growth.

Keywords: abstract evolution equation; memory kernel; global attractor; polynomial growth

1 Introduction

Let Ω be an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary. We consider the following nonclassical diffusion equation with fading memory and arbitrary polynomial growth nonlinearity:

$$\begin{cases} u_t - \nu\Delta u_t - \Delta u + \lambda u - \int_0^\infty k(s)\Delta u(t-s)\,ds + f(x,u) = g, & (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,t) = u_0(x,t), \quad x \in \Omega, t \le 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, t \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the memory kernel satisfies $k(s) \ge 0$, $k'(s) \le 0$, $\forall s \in \mathbb{R}^+$ and ν, λ are two positive constants.

The equation above originates from the general nonclassical diffusion equation

$$u_t - \nu\Delta u_t - \Delta u + f(u) = g, \tag{1.2}$$

which arises as a model to describe physical phenomena, such as non-Newtonian flows, soil mechanics and heat conduction theory. Equation (1.2) is known as the nonclassical diffusion equation when $\nu > 0$ and the reaction-diffusion equation when $\nu = 0$.

The long-time dynamical behaviors of the solutions to Eq. (1.2) have received considerable attention and have been studied by many researchers for both autonomous and non-autonomous dynamical systems (see [1–9] and the references therein). Let us recall some previous known results under the assumption that the nonlinearity satisfied critical exponent growth. For the autonomous case, Wu and Zhang [1] obtained the existence and the regularity of global attractor in $H_0^1(\Omega)$ when $g \in H^{-1}(\Omega)$; Xiao [2] got the existence of global attractor in $H_0^1(\Omega)$ when $g \in L^2(\Omega)$; Sun and Yang [3] proved the existence and regularity of global attractor when $g \in H^{-1}(\Omega)$. For the non-autonomous case, Sun and Yang [3] showed the existence of exponential attractor when $g(x,t)$ was translation bounded, and they also obtained the existence and structure of compact uniform attractor; Anh and Toan [4] acquired the existence and upper semicontinuity of pullback attractor in the case of non-cylindrical domains. In addition, they also proved the existence of uniform attractor with and without singularly oscillating external force, respectively, in [5, 6].

However, when the nonlinearity satisfies an arbitrary order polynomial growth condition, a few results were obtained on long-time dynamical behavior of solutions to Eq. (1.2) in the recent literature. The existence of global attractor in $H_0^1(\Omega)$ was obtained when the nonlinearity satisfied the critical exponent growth condition and the arbitrary order polynomial growth condition separately in [7]. A few years later, the existence of global attractor was proved on an unbounded domain when $g \in L^2(\mathbb{R}^n)$ in [8]. As a similar result, Xie et al. [9] obtained the existence of global attractor for the autonomous case on unbounded domain by the method of asymptotic contractive semigroup.

On account of Eq. (1.2), if we consider viscoelasticity of the conductive medium, we have to add fading memory term to Eq. (1.2), that is to be Eq. (1.1). So far, the related results for dynamical systems with fading memory are not abundant (see [10–16] and the references therein). For instance, when the nonlinearity satisfied subcritical exponent growth, Ma and Zhong [11] gained the existence of strong global attractor for the hyperbolic equations by using the semigroup approach. When the nonlinearity satisfied critical exponent growth, Sun et al. [12] established an abstract criterion for verifying the pullback asymptotic compactness, and finally obtained non-autonomous dynamical behavior of wave-type evolutionary equations with nonlinear damping. Wang et al. [13] showed the existence and regularity of global attractors for the autonomous case both in weak and strong topological spaces. Moreover, they acquired the structure and asymptotic regularity of compact uniform attractor for the non-autonomous case in [14]. Conti et al. [15, 16] studied problem (1.1) in both classical past history and minimal state framework and showed the existence of finite-dimensional regular global attractors and exponential attractors.

As far as we know, very few people study problem (1.1) because of the virtual difficulties caused by the supercritical nonlinear term and fading memory. Only in [10], Xie et al. obtained a uniform attractor with its structure when $\nu = 1$ in the weak topological space for the non-autonomous case by the method of asymptotic contractive semigroup. Comparing with [9, 10], we adjusted the computations in our paper by adding the fading memory term to the problem in [9] and the term λu to the problem in [10] and simplified the estimating process. At the same time, we suppose that $\nu > 0$ and extend the range of ν. Moreover, problem (1.1) with supercritical nonlinearity in the strong topological space $H^2(\Omega) \cap H_0^1(\Omega) \times L_\mu^2(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega))$ has not yet been studied, except by us. In order to gain the existence of strong global attractor, it is essential to testify the compactness of the semigroup in some sense. Since Eq. (1.1) contains the term $-\Delta u_t$ and the fading memory term, there are two main obstacles which are difficult to overcome. First, as to the term $-\Delta u_t$, it is different from the usual reaction-diffusion equation with fading memory virtually. The solutions of the usual reaction-diffusion equation with fading memory have higher regularity. But for Eq. (1.1), if the initial data only belong to the weak topological space, then the solution is always in the weak topological space and has no higher regularity. Consequently, we cannot use the compact Sobolev embedding to verify the key asymptotic compactness of a solution semigroup. Second, Eq. (1.1) contains the fading memory term, which is not compact, so usual methods are not in use. Thus, in this paper, we apply semigroup theory and the contractive function method to conquer the above difficulties and show the existence of global attractor in the space $H^2(\Omega)\cap H_0^1(\Omega)\times L^2_\mu(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega))$.

The plan of our work is as follows. Section 2 is devoted to some preliminaries, including the definitions and notations of function spaces involved, and some general abstract results. In Section 3, by the use of semigroup theory and contractive method, we prove the existence of absorbing sets and global attractor in the space $H^2(\Omega)\cap H_0^1(\Omega)\times L^2_\mu(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega))$.

2 Preliminaries

In this section, we recall some preliminaries and auxiliary results which will be used later.

For the sake of convenience, we denote by $|\cdot|_p$ the norm in $L^p(\Omega)$ $(1\le p\le\infty)$ thereafter. Especially, we denote by $\langle\cdot,\cdot\rangle$ the usual scalar product and by $|\cdot|$ the usual norm in $H=L^2(\Omega)$. Recall that $A=-\Delta$, the Laplacian with Dirichlet boundary conditions. It is well known that A is a positive operator on H with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$. Moreover, one can define the power A^θ of A for $\theta\in\mathbb{R}$. The space $V_\theta=D(A^{\theta/2})$ turns out to be a Hilbert space with the inner product

$$\langle u,v\rangle_\theta=\bigl\langle A^{\theta/2}u,A^{\theta/2}v\bigr\rangle.$$

We denote by $\|\cdot\|_\theta$ the norm on V_θ induced by the above inner product. In particular, $V_0=H$, $V_1=H_0^1(\Omega)$, $V_2=H^2(\Omega)\cap H_0^1(\Omega)$. The injection $V_{\theta_1}\hookrightarrow V_{\theta_2}$ is compact whenever $\theta_1>\theta_2$.

Let $L^2_\mu(\mathbb{R}^+;V_\theta)$ be the Hilbert space of V_θ-valued functions on $\mathbb{R}^+$, endowed with the following inner product and norm:

$$\langle\varphi,\psi\rangle_{\mu,\theta}=\int_0^\infty\mu(s)\langle\varphi,\psi\rangle_\theta\,ds,\qquad\|\varphi\|^2_{\mu,\theta}=\int_0^\infty\mu(s)\|\varphi\|^2_\theta\,ds.$$

Then we introduce the family of Hilbert spaces

$$\mathcal{V}_\theta=V_{\theta+1}\times L^2_\mu\bigl(\mathbb{R}^+;V_{\theta+1}\bigr)$$

with the norm

$$\|z\|^2_{\mathcal{V}_\theta}=\bigl\|\bigl(u,\eta^t\bigr)\bigr\|^2_{\mathcal{V}_\theta}=\frac{1}{2}\bigl(\|u\|^2_{\theta+1}+\bigl\|\eta^t\bigr\|^2_{\mu,\theta+1}\bigr).$$

Here, θ is omitted whenever zero, namely, $\mathcal{V}=\mathcal{V}_0$.

Define

$$\eta^t(x,s)=\int_0^s u(x,t-r)\,dr,\quad s\ge 0,$$

then we have

$$\eta^t_t(x,s)=u(x,t)-\eta^t_s(x,s),\quad s\ge 0.$$

Set $\mu(s) = -k'(s)$, which satisfies $k(\infty) = 0$, then Eq. (1.1) can be transformed into the following form:

$$\begin{cases} u_t - \nu\Delta u_t - \Delta u + \lambda u - \int_0^\infty \mu(s)\Delta\eta^t(s)\,ds + f(x,u) = g, & (x,t) \in \Omega \times \mathbb{R}^+, \\ \eta_t^t = -\eta_s^t + u, \end{cases} \tag{2.1}$$

with the corresponding initial-boundary conditions

$$\begin{cases} u(x,t) = 0, \qquad \eta^t(x,s) = 0, & x \in \partial\Omega, t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \eta^0(x,s) = \int_0^s u(x,-r)\,dr, & (x,s) \in \Omega \times \mathbb{R}^+, \end{cases} \tag{2.2}$$

where $u(\cdot)$ satisfies the condition as follows: there exist two positive constants $\mathcal{R}$ and ϱ such that

$$\int_0^\infty e^{-\varrho s}\left|\nabla u(-s)\right|^2 ds \le \mathcal{R}.$$

Assume that the memory kernel function $\mu(s)$ satisfies the following hypotheses:

(h_1) $\mu(s) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $\mu(s) \ge 0$ and $\mu'(s) \le 0$, $\forall s \in \mathbb{R}^+$;
(h_2) there is a constant $\delta \ge \varrho > 0$ such that $\mu'(s) + \delta\mu(s) \le 0$, $\forall s \in \mathbb{R}^+$.

About the nonlinearity $f(u)$, we always assume that $f(x,s) = f_1(s) + a(x)f_2(s)$ satisfies:

(g_1) $\alpha_1|s|^p - \beta_1|s|^2 \le f_1(s)s \le \gamma_1|s|^p - \delta_1|s|^2$, $f_1(s)s \ge 0$ and $f_1'(s) \ge -\kappa$;

or

(g_1)* $\alpha_1|s|^p - \beta_1|s|^2 \le f_1(s)s \le \gamma_1|s|^p - \delta_1|s|^2$, $p \ge 2$, $\beta_1 < \lambda$, and $f_1'(s) \ge -\kappa$;
(g_2) $\alpha_2|s|^p - \beta_2 \le f_2(s)s \le \gamma_2|s|^p - \delta_2$, and $f_2'(s) \ge -\kappa$;
(g_3) $a(x) \in L^1(\Omega) \cap L^\infty(\Omega)$, and $a(x) > 0$,

for any $s \in \mathbb{R}^+$, where α_i, β_i, γ_i, δ_i $(i = 1,2)$ and κ are all positive constants.

Remark 2.1 If the term λu were incorporated in the nonlinearity $f(x,u)$, the alternative assumption (g_1) or (g_1)* about nonlinearity $f(x,u)$ would be avoided, namely, the nonlinear term $f(x,u)$ would only need to satisfy assumption (g_1).

Lemma 2.2 (see [14]) *Set $I = [0,T]$ for some $T > 0$. Let the memory kernel function $\mu(s)$ satisfy* (h_1)-(h_2), *then, for any $\eta^t \in C(I; L^2_\mu(\mathbb{R}^+; V_\theta))$, we have*

$$\langle \eta^t, \eta_s^t \rangle_{\mu,\theta} \ge \frac{\delta}{2}\|\eta^t\|_{\mu,\theta}^2. \tag{2.3}$$

Lemma 2.3 (see [17]) *Let X and Y be two Banach spaces such that $X \subset Y$ with a continuous injection. If a function φ belongs to $L^\infty([0,T];X)$ and is weakly continuous with values in Y, then φ is weakly continuous with values in X.*

Definition 2.4 (see [12, 18]) Let X be a Banach space and $\mathscr{B}$ be any bounded subset of X. We call the function $\psi(\cdot,\cdot)$, defined on $X \times X$, a contractive function on $\mathscr{B} \times \mathscr{B}$, if for any sequence $\{x_n\}_{n=1}^{\infty} \subset \mathscr{B}$, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty} \psi(x_{n_k}, x_{n_l}) = 0.$$

Denote by $\mathfrak{C}(\mathscr{B})$ the set of contractive functions on $\mathscr{B} \times \mathscr{B}$.

Lemma 2.5 (see [12, 17–19]) *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on the Banach space $(X, \|\cdot\|)$, with a bounded absorbing set $\mathscr{B}_0$. Moreover, if, for any $\varepsilon > 0$, there exist $T = T(\mathscr{B}_0, \varepsilon)$ and $\psi_T(\cdot,\cdot) \in \mathfrak{C}(\mathscr{B}_0)$ such that*

$$\bigl\|S(T)x - S(T)y\bigr\| \leq \varepsilon + \psi_T(x,y), \quad \forall x,y \in \mathscr{B}_0,$$

then $\{S(t)\}_{t\geq 0}$ is asymptotically compact in X.

3 Main result

3.1 Existence and uniqueness of solution

We first define the solution for Eq. (2.1) with initial-boundary condition (2.2) as follows.

Definition 3.1 Set $I = [0,T]$ for some $T > 0$. Assume that f satisfies (g_1) or $(\mathrm{g}_1)^*$, and (g_2)-(g_3), and $g \in H$. A binary form $z = (u, \eta^t)$ is said to be a strong solution for Eq. (2.1) in the time interval I, with the initial datum $z_0 = (u_0, \eta^0) \in \mathcal{V}_1$, if z satisfies (2.1), and

$$\begin{aligned} &u \in L^\infty(I; V_2), \qquad u_t \in L^2(I; V_2), \qquad \eta^t \in C\bigl(I; L^2_\mu\bigl(\mathbb{R}^+; V_2\bigr)\bigr); \\ &\eta^t_t + \eta^t_s \in L^\infty\bigl(I; L^2_\mu\bigl(\mathbb{R}^+; V_1\bigr)\bigr) \cap L^2\bigl(I; L^2_\mu\bigl(\mathbb{R}^+; V_2\bigr)\bigr). \end{aligned}$$

In order to prove the existence and uniqueness of solution for Eqs. (2.1)-(2.2) in $\mathcal{V}_1$ and the existence of bounded absorbing sets in $\mathcal{V}_1$, we first make a priori estimate of the solution for Eqs. (2.1)-(2.2) in $\mathcal{V}_0$.

Lemma 3.2 *Presume that $z(t) = (u, \eta^t)$ is a solution to Eqs.* (2.1)-(2.2). *If the nonlinearity f satisfies* (g_1) *or* $(\mathrm{g}_1)^*$, *and* (g_2)-(g_3), $g \in H^{-1}(\Omega)$, *and* (h_1)-(h_2) *hold, then for any bounded subset $\mathscr{B}$ of $\mathcal{V}_0$, there are two constants $\varrho_0 > 0$ and $t_0 = t_0(\|\mathscr{B}\|_{\mathcal{V}_0})$ such that*

$$\bigl\|z(t)\bigr\|^2_{\mathcal{V}} = \frac{1}{2}\bigl(\|u\|_1^2 + \bigl\|\eta^t\bigr\|^2_{\mu,1}\bigr) \leq \varrho_0^2, \quad t \geq t_0. \tag{3.1}$$

Proof Multiplying (2.1) by u, integrating over Ω and applying Lemma 2.2, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\bigl(|u|^2 + \nu\|u\|_1^2 + \bigl\|\eta^t\bigr\|^2_{\mu,1}\bigr) + \lambda|u|^2 + \|u\|_1^2 + \frac{\delta}{2}\bigl\|\eta^t\bigr\|^2_{\mu,1} \\ &\quad \leq -\bigl\langle f(x,u), u\bigr\rangle + \bigl\langle g(x), u\bigr\rangle. \end{aligned} \tag{3.2}$$

(i) If the nonlinearity satisfies (g_1), (g_2) and (g_3), we treat the last two terms in (3.2).

Using Hölder's and Young's inequalities, we have

$$\begin{aligned}
-\langle f(x,u),u\rangle &= -\langle f_1(u),u\rangle - \langle a(x)f_2(u),u\rangle \\
&\le \int_\Omega a(x)\bigl(-\alpha_2|u(x)|^p + \beta_2\bigr)\,dx \\
&= \beta_2|a|_1 - \alpha_2\int_\Omega a(x)|u(x)|^p\,dx,
\end{aligned}$$

and

$$\langle g(x),u\rangle \le \frac{1}{2}\|g\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|u\|_1^2.$$

So, we can see that

$$\frac{d}{dt}\bigl(|u|^2 + \nu\|u\|_1^2 + \|\eta^t\|^2_{\mu,1}\bigr) + 2\epsilon\bigl(|u|^2 + \nu\|u\|_1^2 + \|\eta^t\|^2_{\mu,1}\bigr) + 2\alpha_2\int_\Omega a(x)|u|^p\,dx \le C,$$

where $\epsilon = \min\{\lambda, \frac{1}{2\nu}, \frac{\delta}{2}\}$ and $C = 2\beta_2|a|_1 + \|g\|^2_{H^{-1}}$.

Applying Gronwall's lemma, we can obtain

$$|u|^2 + \nu\|u\|_1^2 + \|\eta^t\|^2_{\mu,1} \le e^{-2\epsilon t}\bigl(|u_0|^2 + \nu\|u_0\|_1^2 + \|\eta^0\|^2_{\mu,1}\bigr) + \frac{C}{2\epsilon}.$$

Taking $\sigma = \min\{1,\nu\}$, then we have the following conclusion:

$$\|z(t)\|^2_{\mathcal{V}} = \frac{1}{2}\bigl(\|u\|_1^2 + \|\eta^t\|^2_{\mu,1}\bigr) \le \frac{1}{2\sigma}e^{-2\epsilon t}\bigl(|u_0|^2 + \nu\|u_0\|_1^2 + \|\eta^0\|^2_{\mu,1}\bigr) + \frac{C}{4\epsilon\sigma} \le \varrho_0^2. \quad (3.3)$$

(ii) If the nonlinearity satisfies $(g_1)^*$, (g_2) and (g_3), we treat the last two terms in (3.2). By use of Hölder's and Young's inequalities, we obtain

$$\begin{aligned}
-\langle f(x,u),u\rangle &= -\langle f_1(u),u\rangle - \langle a(x)f_2(u),u\rangle \\
&\le \int_\Omega \bigl(-\alpha_1|u(x)|^p + \beta_1|u|^2\bigr)\,dx + \int_\Omega a(x)\bigl(-\alpha_2|u(x)|^p + \beta_2\bigr)\,dx \\
&= \beta_1|u|^2 - \alpha_1\int_\Omega |u(x)|^p\,dx + \beta_2|a|_1 - \alpha_2\int_\Omega a(x)|u(x)|^p\,dx,
\end{aligned}$$

and

$$\langle g(x),u\rangle \le \frac{1}{2}\|g\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|u\|_1^2.$$

Finally, we have

$$\begin{aligned}
&\frac{d}{dt}\bigl(|u|^2 + \nu\|u\|_1^2 + \|\eta^t\|^2_{\mu,1}\bigr) + 2\epsilon\bigl(|u|^2 + \nu\|u\|_1^2 + \|\eta^t\|^2_{\mu,1}\bigr) \\
&\quad + 2\alpha_1|u|_p^p + 2\alpha_2\int_\Omega a(x)|u(x)|^p\,dx \le C,
\end{aligned}$$

where $\epsilon = \min\{\lambda - \beta_1, \frac{1}{2\nu}, \frac{\delta}{2}\}$ and $C = 2\beta_2|a|_1 + \|g\|^2_{H^{-1}}$.

Similarly, applying Gronwall's lemma, we have the same conclusion as (3.3) in case (i). Therefore, the proof is complete. □

Then, by use of the Faedo-Galerkin approximation method (see [20]), we can get the following result about the existence and uniqueness of solution for Eqs. (2.1)-(2.2) in $\mathcal{V}_1$.

Theorem 3.3 (Existence and uniqueness of strong solution) *Suppose that f satisfies* (g_1) *or* $(\mathrm{g}_1)^*$, *and* (g_2)-(g_3), *and $g \in H$, then for any $T > 0$ and the initial datum $z_0 = (u_0, \eta^0)$, there exists a unique solution $z = (u, \eta^t)$ for Eqs.* (2.1)-(2.2) *satisfying $z \in L^\infty(I; \mathcal{V}_1)$. Moreover, the solution $z(t)$ is weakly continuous functions from $[0, T]$ to $\mathcal{V}_1$ and depends on the initial datum $z_0 = (u_0, \eta^0)$ continuously in $\mathcal{V}_1$.*

Proof Assume that λ_i, $i = 1, 2, \ldots$, are eigenvalues of operator A in $D(A)$, and they satisfy

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty, \text{as } j \to \infty.$$

Let ω_i denote the eigenvector corresponding to the eigenvalue λ_i, $i = 1, 2, 3, \ldots$. Then they form a group of canonical orthogonal bases in $D(A)$ and satisfy

$$A\omega_i = \lambda_i \omega_i, \quad \forall i \in \mathbb{N},$$

where $A = -\Delta$. At the same time, we choose an orthogonal basis $\{\zeta_j\}_{j=1}^\infty$ of $L^2_\mu(\mathbb{R}^+; V_2)$.

Firstly, we will prove the existence of solutions by use of the Faedo-Galerkin approximation method. Fix $T > 0$. Given an integer m, denote by P_m and Q_m the projections on subspaces

$$\operatorname{Span}\{\omega_1, \ldots, \omega_m\} \subset V_2 \quad \text{and} \quad \operatorname{Span}\{\zeta_1, \ldots, \zeta_m\} \subset L^2_\mu(\mathbb{R}^+; V_2),$$

respectively. By applying the basic theory of differential equation, we can gain a unique solution $z_m = (u_m, \eta^t_m)$ satisfying the following approximate equation:

$$\begin{cases} u_{mt} - \nu\Delta u_{mt} - \Delta u_m + \lambda u_m - \int_0^\infty \mu(s)\Delta\eta^t_m(s)\,ds + f(x, u_m) = P_m g(x), \\ \eta^t_{mt} = -\eta^t_{ms} + u_m, \\ u_m(0) = P_m u_0, \qquad \eta^0_m(s) = Q_m\eta^0, \end{cases} \tag{3.4}$$

where $u_m(t) = \sum_{i=1}^m u_{mi}\omega_i$, $\eta^t_m(s) = \sum_{i=1}^m \eta^t_{mi}\zeta_i$.

Now, let us show that the solution to problem (3.4) is uniformly bounded on $[0, T]$ and m. Taking the scalar product with Au_{mt} for (3.4) in H, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u_m\|_1^2 + \nu\|u_m\|_2^2 + \left\|\eta^t_m\right\|^2_{\mu,2}\right) + \lambda\|u_m\|_1^2 + \|u_m\|_2^2 + \frac{\delta}{2}\left\|\eta^t_m\right\|^2_{\mu,2} + \left\langle f(x, u_m), -\Delta u_m\right\rangle \\ &\quad \leq \langle g_m, -\Delta u_m\rangle, \end{aligned} \tag{3.5}$$

where $g_m = P_m g$.

Noting that the nonlinearity satisfies (g_1) or $(\mathrm{g}_1)^*$, and (g_2)-(g_3), we obtain

$$\left\langle f(x, u_m), -\Delta u_m\right\rangle = \int_\Omega \left(f_1'(\theta_1 u_m) + a(x)f_2'(\theta_2 u_m)\right)|\nabla u_m|^2\,dx \geq -\kappa\left(1 + |a|_\infty\right)\|u_m\|_1^2, \tag{3.6}$$

here θ_1 and θ_2 are all in $[0,1]$. By using Hölder's inequality, we have

$$\langle g_m(x), -\Delta u_m\rangle \leq \frac{1}{2}|g_m|^2 + \frac{1}{2}\|u_m\|_2^2.$$

Substituting the above estimates into (3.5), we have

$$\begin{aligned}&\frac{d}{dt}\left(\|u_m\|_1^2 + \nu\|u_m\|_2^2 + \left\|\eta_m^t\right\|_{\mu,2}^2\right) + 2\epsilon\left(\|u_m\|_1^2 + \nu\|u_m\|_2^2 + \left\|\eta_m^t\right\|_{\mu,2}^2\right)\\ &\quad\leq |g_m|^2 + 2\kappa\left(1+|a|_\infty\right)\|u_m\|_1^2,\end{aligned}\tag{3.7}$$

where $\epsilon = \min\{\lambda, \frac{1}{2\nu}, \frac{\delta}{2}\}$.

Applying Lemma 3.2 and Gronwall's lemma, we can obtain

$$\|u_m\|_1^2 + \nu\|u_m\|_2^2 + \left\|\eta_m^t\right\|_{\mu,2}^2 \leq e^{-2\epsilon t}\left(\|u_{m0}\|_1^2 + \nu\|u_{m0}\|_2^2 + \left\|\eta_m^0\right\|_{\mu,2}^2\right) + C,$$

where $C = \frac{1}{2\epsilon}(1-e^{-2\epsilon t})(|g_m|^2 + 2\kappa(1+|a|_\infty)\|u_m\|_1^2)$.

Taking $\sigma = \min\{1,\nu\}$, we have

$$\begin{aligned}\left\|z_m(t)\right\|_{\mathcal{V}_1}^2 &= \frac{1}{2}\left(\|u_m\|_2^2 + \left\|\eta_m^t\right\|_{\mu,2}^2\right)\\ &\leq \frac{1}{2\sigma}e^{-2\epsilon t}\left(\|u_{m0}\|_1^2 + \nu\|u_{m0}\|_2^2 + \left\|\eta_m^0\right\|_{\mu,2}^2\right) + \frac{C}{2\sigma}.\end{aligned}\tag{3.8}$$

Therefore, $\{z_m\}_{m=1}^\infty$ is uniformly bounded in $L^\infty([0,T];\mathcal{V}_1)$. Moreover, in view of (3.1), we know

$$u_{mt} - \nu\Delta u_{mt} = \Delta u_m - \lambda u_m + \int_0^\infty \mu(s)\Delta\eta_m^t(s)\,ds - f(x,u_m) + P_m g(x).\tag{3.9}$$

Integrating over $[0,T]$ about (3.7) and following from (3.8), we can deduce that

$$\begin{aligned}&2\epsilon\int_0^T \left\|\nabla u_m(s)\right\|^2 ds + 2\epsilon\int_0^T \left\|\Delta u_m(s)\right\|^2 ds + \delta\int_0^T \left\|\eta_m^s(r)\right\|_{\mu,\mathcal{H}_2}^2 ds\\ &\quad\leq |g_m|^2 + C, \quad t\geq 0,\end{aligned}\tag{3.10}$$

so $\Delta u_m \in L^2([0,T];L^2(\Omega))$.

From conditions (g_1) or $(g_1)^*$ and (g_2)-(g_3), we get $f(x,u_m) \in L^{\frac{p}{p-1}}([0,T];L^{\frac{p}{p-1}}(\Omega))$. Since $L^q(\Omega) \hookrightarrow H^{-\gamma}(\Omega)$, $f(x,u_m) \in L^q([0,T];H^{-\gamma}(\Omega))$, where q is a dual number of p, as $p\geq 2$, $q>1$, and $\gamma>1$.

For any $v_m \in \mathcal{H}_1$, we have

$$\begin{aligned}&\left|\int_0^\infty \mu(s)\langle\Delta\eta_m^t(s), v_m(t)\rangle\,ds\right|\\ &\quad\leq \int_0^\infty \mu(s)\left\|\nabla\eta_m^t(s)\right\|\cdot\left\|\nabla v_m(t)\right\|\,ds\\ &\quad\leq \left(\int_0^\infty \mu(s)\left\|\nabla\eta_m^t(s)\right\|^2 ds\right)^{\frac{1}{2}}\cdot\left(\int_0^\infty \mu(s)\left\|\nabla v_m(t)\right\|^2 ds\right)^{\frac{1}{2}}\\ &\quad\leq \|\nabla v_m\|\left(\int_0^\infty \mu(s)\left\|\nabla\eta_m^t(t)\right\|^2 ds\right)^{\frac{1}{2}}\cdot\left(\int_0^\infty \mu(s)\,ds\right)^{\frac{1}{2}}.\end{aligned}\tag{3.11}$$

By virtue of (3.1), we know $\int_0^\infty \mu(s)\Delta\eta_m^t(s)\,ds \in L_\mu^\infty(\mathbb{R}^+;H^{-1})$. Thus, u_{mt} is bounded in $L^q([0,T];H^{2-\gamma})$.

Therefore, there exists a subsequence, we denote it by $\{z_m\}$ yet, as $m\to\infty$ such that

$$\begin{aligned}
&u_m \rightharpoonup u, \quad \text{star in } L^\infty\big([0,T];V_2\big);\\
&u_m \rightharpoonup u, \quad \text{in } L^2\big([0,T];V_2\big);\\
&u_{mt} \rightharpoonup u_t, \quad \text{in } L^q\big([0,T];H^{2-\gamma}\big);\\
&\eta_m^t \rightharpoonup \eta^t, \quad \text{star in } L^\infty\big([0,T];L_\mu^2\big(\mathbb{R}^+;V_2\big)\big);\\
&f(x,u_m) \rightharpoonup f(x,u), \quad \text{in } L^{\frac{p}{p-1}}\big([0,T];L^{\frac{p}{p-1}}(\Omega)\big);\\
&g_m \rightharpoonup g, \quad \text{in } L^2\big([0,T];H\big).
\end{aligned}$$

Taking limit for (3.7), we find that $z=(u,\eta^t)$ is a solution to problem (2.1)-(2.2) satisfying

$$z\in L^\infty\big([0,T],\mathcal{V}_1\big).$$

Secondly, we will prove that the solutions of problem (2.1)-(2.2) are weakly continuous from $[0,T]$ to $\mathcal{V}_1$.

From Eq. (2.1), it is easy to deduce that $z=(u,\eta^t)$ fulfills

$$u_t\in L^q\big([0,T];H^{2-\gamma}\big), \qquad u\in L^\infty\big([0,T];V_2\big),$$

which implies $u\in C([0,T];V_1)$. Moreover, applying Lemma 2.3, the weak continuity of u holds.

Now, let us show the continuity of η^t. The solution of problem (2.1)-(2.2) is given by the Duhamel integral

$$\eta^t = U(t)\eta^0 + \Phi(t),$$

where $U(t)$ is a strongly continuous semigroup of contractions on $L_\mu^2(\mathbb{R}^+;V_2)$ and dissipative,

$$\Phi(t)=\int_0^t U(t-\tau)u(\tau)\,d\tau.$$

Choose $t_0>0$, and let $0\le t<t_0$, we obtain

$$\begin{aligned}
&\big\|\Phi(t_0)-\Phi(t)\big\|_{\mu,1}\\
&\quad\le \int_0^{t_0}\big\|U(t_0-\tau)u(\tau)-U(t-\tau)u(\tau)\big\|_{\mu,2}\,d\tau+\int_t^{t_0}\big\|U(t-\tau)u(\tau)\big\|_{\mu,2}\,d\tau\\
&\quad\le \int_0^{t_0}\big\|U(t_0-\tau)u(\tau)-U(t-\tau)u(\tau)\big\|_{\mu,2}\,d\tau+\int_t^{t_0}\big\|u(\tau)\big\|_{\mu,2}\,d\tau.
\end{aligned}$$

In view of (3.8),

$$\big\|u(\tau)\big\|_{\mu,2}=\bigg(\int_0^\infty \mu(s)\big\|\Delta u(\tau)\big\|^2\,ds\bigg)^{\frac12}=\|\mu\|_{L^1(\mathbb{R}^+)}\big\|\Delta u(\tau)\big\|\in L^1\big([t,t_0]\big).$$

Simultaneously, by the strong continuity of a semigroup $U(t)$, we know

$$\lim_{t\to t_0}\|U(t_0-\tau)u(\tau)-U(t-\tau)u(\tau)\|_{\mu,2}=0$$

for a.e. $\tau\in[0,t_0]$.

Therefore, by virtue of the dominated convergence theorem,

$$\lim_{t\to t_0}\|\Phi(t_0)-\Phi(t)\|_{\mu,2}=0,$$

which implies the left-continuity of η^t. Similarly, we can obtain the right-continuity of η^t, and hence $\eta^t\in C([0,T];L^2_\mu(\mathbb{R}^+;V_2))$.

Finally, we will prove the uniqueness of solutions of problem (2.1)-(2.2).

Let $z=(u,\eta^t)$, $z^*=(u^*,\eta^{t*})$, z and z^* are two solutions to problem (2.1)-(2.2) with the initial data z_0 and z_0^*.

Define $w=u-u^*$, $\xi^t(x,s)=w(x,t)-w(x,t-s)$, we can obtain from (2.1)

$$\begin{cases} w_t-\nu\Delta w_t-\Delta w+\lambda w-\int_0^\infty\mu(s)\Delta\xi^t(s)\,ds+f(x,u)-f(x,u^*)\\ \quad=0,\quad (x,t)\in\Omega\times\mathbb{R}^+,\\ \xi_t^t=-\xi_s^t+u. \end{cases}\tag{3.12}$$

Taking the scalar product with $-\Delta w$ for (3.12) in H, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|w\|_1^2+\nu\|w\|_2^2+\|\xi^t\|_{\mu,2}^2\right)+\lambda\|w\|_1^2+\|w\|_2^2+\frac{\delta}{2}\|\xi_m^t\|_{\mu,2}^2\\ &\quad+\langle f(x,u)-f(x,u^*),-\Delta w\rangle=0.\end{aligned}\tag{3.13}$$

Notice that the nonlinearity satisfies (g_1) or (g_1)*, and (g_2)-(g_3), we obtain

$$\begin{aligned}&-\langle f(x,u)-f(x,u^*),-\Delta w\rangle\\ &\quad=\int_\Omega-\left(f_1'(u^*+\theta_1w)+a(x)f_2'(u^*+\theta_2w)\right)|\nabla w|^2\,dx\\ &\quad\le\kappa\left(1+|a|_\infty\right)\|w\|_1^2,\end{aligned}\tag{3.14}$$

where θ_1, θ_2 are all in $[0,1]$.

Thus, combining (3.14) with (3.13), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|_1^2+\nu\|w\|_2^2+\|\xi^t\|_{\mu,2}^2\right)\le C\left(\|w\|_1^2+\nu\|w\|_2^2+\|\xi^t\|_{\mu,2}^2\right).$$

By Gronwall's lemma, we conclude that

$$\begin{aligned}\|z(t)-z^*(t)\|_{\mathcal{V}_1}^2&=\frac{1}{2}\left(\|w(t)\|_2^2+\|\xi^t(s)\|_{\mu,2}^2\right)\\ &\le C_1\left(\|w(t)\|_1^2+\nu\|w(t)\|_2^2+\|\xi^t(s)\|_{\mu,2}^2\right)\\ &\le C_1\left(\|w(0)\|_1^2+\nu\|w(0)\|_2^2+\|\xi^0\|_{\mu,2}^2\right)e^{Ct}\end{aligned}$$

$$\le C_2\big(\|w(0)\|_2^2 + \|\xi^0\|_{\mu,2}^2\big)e^{CT}$$
$$\le C_2\|z_0 - z_0^*\|_{\mathcal{V}_1}^2 e^{CT}, \tag{3.15}$$

where $t \in [0, T]$, and C_1, C_2, C are all positive.

We complete the proof. □

According to Theorem 3.3 above, we can define the solution operator $S(t) : z_0 \to z(t)$, $\forall t \in \mathbb{R}^+$, which reflects $\mathcal{V}_1$ on itself. Obviously, $\{S(t)\}_{t\ge 0}$ is a norm-to-weak continuous semigroup, thus we denote by $\{S(t)\}_{t\ge 0}$ the solution semigroup for Eqs. (2.1)-(2.2) hereafter.

3.2 Bounded absorbing sets in $\mathcal{V}_1$

We will make a priori estimate to the solutions to problem (2.1)-(2.2) in $\mathcal{V}_1$.

Lemma 3.4 *Assume that* $z(t) = (u, \eta^t)$ *is a strong solution to Eqs.* (2.1)-(2.2) *with respect to the inial value* $z_0 = (u_0, \eta^0)$. *If the nonlinearity* f *satisfies* (g_1) *or* $(g_1)^*$, *and* (g_2)-(g_3), $g \in H$, *and* (h_1)-(h_2) *hold, then for any bounded subset* $\mathscr{B} \subset \mathcal{V}_1$, *there exist two positive constants* ϱ_1 *and* $t_0 = t(\|\mathscr{B}\|_{\mathcal{V}_1})$ *such that*

$$\|z(t)\|_{\mathcal{V}_1}^2 = \frac{1}{2}\big(\|u\|_2^2 + \|\eta^t\|_{\mu,2}^2\big) \le \varrho_1^2, \quad t \ge t_0.$$

Proof Multiplying (2.1) by $-\Delta u$, integrating over Ω, and then applying Lemma 2.2, we have

$$\frac{1}{2}\frac{d}{dt}\big(\|u\|_1^2 + \nu\|u\|_2^2 + \|\eta^t\|_{\mu,2}^2\big) + \lambda\|u\|_1^2 + \|u\|_2^2 + \frac{\delta}{2}\|\eta^t\|_{\mu,2}^2 + \big\langle f(x,u), -\Delta u\big\rangle$$
$$\le \langle g, -\Delta u\rangle. \tag{3.16}$$

No matter that the nonlinearity satisfies (g_1) or $(g_1)^*$, and (g_2)-(g_3) in (3.16), we have

$$\big\langle f(x,u), -\Delta u\big\rangle = \int_\Omega \big(f_1'(\theta_1 u) + a(x)f_2'(\theta_2 u)\big)|\nabla u|^2\,dx \ge -\kappa\big(1 + |a|_\infty\big)\|u\|_1^2, \tag{3.17}$$

here, θ_1, θ_2 are all in $[0, 1]$. And using Hölder's inequality, we get

$$\big\langle g(x), -\Delta u\big\rangle \le \frac{1}{2}|g|^2 + \frac{1}{2}\|u\|_2^2.$$

By replacing the above estimates into (3.16), it follows that

$$\frac{d}{dt}\big(\|u\|_1^2 + \nu\|u\|_2^2 + \|\eta^t\|_{\mu,2}^2\big) + 2\epsilon\big(\|u\|_1^2 + \nu\|u\|_2^2 + \|\eta^t\|_{\mu,2}^2\big) \le |g|^2 + 2\kappa\big(1 + |a|_\infty\big)\|u\|_1^2,$$

where $\epsilon = \min\{\lambda, \frac{1}{2\nu}, \frac{\delta}{2}\}$.

Combining the above estimates with Lemma 3.2 and applying Gronwall's lemma, we can obtain

$$\|u\|_1^2 + \nu\|u\|_2^2 + \|\eta^t\|_{\mu,2}^2 \le e^{-2\epsilon t}\big(\|u_0\|_1^2 + \nu\|u_0\|_2^2 + \|\eta^0\|_{\mu,2}^2\big) + C,$$

where $C = \frac{1}{2\epsilon}(1 - e^{-2\epsilon t})(|g|^2 + 2\kappa(1 + |a|_\infty)\|u\|_1^2)$.

Taking $\sigma = \min\{1, \nu\}$, then we deduce the following result:

$$\begin{aligned}\|z(t)\|_{\mathcal{V}_1}^2 &= \frac{1}{2}\left(\|u\|_2^2 + \|\eta^t\|_{\mu,2}^2\right)\\ &\le \frac{1}{2\sigma}e^{-2\epsilon t}\left(\|u_0\|_1^2 + \nu\|u_0\|_2^2 + \|\eta^0\|_{\mu,2}^2\right) + \frac{C}{2\sigma} \le \varrho_1^2.\end{aligned}\tag{3.18}$$

The proof is complete. □

According to Lemma 3.4, we conclude the existence of bounded absorbing sets as follows.

Theorem 3.5 *Under the same assumptions as in Lemma* 3.4, *the solution semigroup* $\{S(t)\}_{t\ge0}$ *has a bounded absorbing set* $\mathscr{B}_0$ *for Eqs*. (2.1)-(2.2). *That is, for any bounded sets* $\mathscr{B} \subset \mathcal{V}_1$, *there exists a constant* $t_0 = t(\|\mathscr{B}\|_{\mathcal{V}_1})$ *such that* $S(t)\mathscr{B} \subset \mathscr{B}_0$ *for* $t \ge t_0$.

3.3 The existence of global attractor in $\mathcal{V}_1$

In order to prove the existence of global attractor, it is necessary to verify the asymptotic compactness for the solution semigroup $\{S(t)\}_{t\ge0}$.

Lemma 3.6 *Under the same assumptions as in Lemma* 3.4, *the solution semigroup* $\{S(t)\}_{t\ge0}$ *for Eqs*. (2.1)-(2.2) *is asymptotically compact on* $\mathcal{V}_1$.

Proof Let $z_n = (u_n, \eta_n^t)$ and $z_m = (u_m, \eta_m^t)$ be two solutions for Eqs. (2.1)-(2.2), with the corresponding initial data $z_n^0 = (u_n^0, \eta_n^0)$ and $z_m^0 = (u_m^0, \eta_m^0)$. Set $\omega(t) = u_n(t) - u_m(t)$, $\xi^t = \eta_n^t - \eta_m^t$, then we infer from Eq. (2.1) that

$$\begin{cases}\omega_t - \nu\Delta\omega_t - \Delta\omega + \lambda\omega - \int_0^\infty \mu(s)\Delta\xi^t\,ds + f(x,u_n) - f(x,u_m) = 0,\\ \xi_t^t = \omega - \xi_s^t,\end{cases}\tag{3.19}$$

with the corresponding initial data $\omega_0 = u_n^0 - u_m^0$, $\xi^0 = \eta_n^0 - \eta_m^0$.

Multiplying Eq. (3.19) by $-\Delta\omega$, integrating over Ω and combining with Lemma 2.2, we easily deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_1^2 + \nu\|\omega\|_2^2 + \|\xi^t\|_{\mu,2}^2\right) + \lambda\|\omega\|_1^2 + \|\omega\|_2^2 + \frac{\delta}{2}\|\xi^t\|_{\mu,2}^2\\ &\quad\le -\left\langle f(x,u_n) - f(x,u_m), -\Delta\omega\right\rangle.\end{aligned}\tag{3.20}$$

Then, from (g_1), (g_2), (g_3) or $(g_1)^*$, (g_2), (g_3), we infer

$$\begin{aligned}&-\left\langle f(x,u_n) - f(x,u_m), -\Delta\omega\right\rangle\\ &\quad= -\left\langle f_1(u_n) - f_1(u_m), -\Delta\omega\right\rangle - \left\langle a(x)\left(f_2(u_n) - f_2(u_m)\right), -\Delta\omega\right\rangle\\ &\quad\le \left\langle -f_1'\left(u_m + \vartheta(u_n - u_m)\right)(u_n - u_m), -\Delta\omega\right\rangle\\ &\qquad+ \left\langle a(x)\left(-f_2'\left(u_m + \vartheta(u_n - u_m)\right)(u_n - u_m)\right), -\Delta\omega\right\rangle\\ &\quad\le \kappa\left(1 + |a|_\infty\right)\|\omega\|_1^2,\end{aligned}\tag{3.21}$$

here, $\vartheta \in [0,1]$.

Substituting (3.21) into (3.20), we can obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_1^2+\nu\|\omega\|_2^2+\left\|\xi^t\right\|_{\mu,2}^2\right)+\lambda\|\omega\|_1^2+\|\omega\|_2^2+\frac{\delta}{2}\left\|\xi^t\right\|_{\mu,2}^2\\&\quad\le\kappa\left(1+|a|_\infty\right)\|\omega\|_1^2.\end{aligned}$$

Taking $\epsilon=\min\{\lambda,\frac{1}{\nu},\frac{\delta}{2}\}$, then we have

$$\begin{aligned}&\frac{d}{dt}\left(\|\omega\|_1^2+\nu\|\omega\|_2^2+\left\|\xi^t\right\|_{\mu,2}^2\right)+2\epsilon\left(\|\omega\|_1^2+\nu\|\omega\|_2^2+\left\|\xi^t\right\|_{\mu,2}^2\right)\\&\quad\le 2\kappa\left(1+|a|_\infty\right)\|\omega\|_1^2.\end{aligned}\tag{3.22}$$

Multiplying (3.22) by $e^{2\epsilon t}$ and integrating over $[0,T]$, we get

$$\begin{aligned}&\left\|\omega(T)\right\|_1^2+\nu\left\|\omega(T)\right\|_2^2+\left\|\xi^T\right\|_{\mu,2}^2\\&\quad\le e^{-2\epsilon T}\left(\|\omega_0\|_1^2+\nu\|\omega_0\|_2^2+\left\|\xi^0\right\|_{\mu,2}^2\right)+2\kappa\left(1+|a|_\infty\right)\int_0^T\|\omega\|_1^2\,ds.\end{aligned}\tag{3.23}$$

Set $\psi(z_n,z_m)=2\kappa(1+|a|_\infty)\int_0^T\|\omega\|_1^2\,ds$. Suppose that $\psi(z_n,z_m)\in\mathfrak{C}(\mathscr{B}_0)$, by use of Lemma 2.5, we can easily get the asymptotic compactness of the solution semigroup $\{S(t)\}_{t\ge0}$. So, we need verify that $\psi(z_n,z_m)$ is a contractive function.

Thanks to the compact embedding $V_2\hookrightarrow V_1$, the boundedness of u_n in V_2 and $u_n\in C(I;V_2)$, there is a subsequence u_{n_k} of u_n, which is convergent in V_1, that is,

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T\left\|u_{n_k}(s)-u_{n_l}(s)\right\|_1^2\,ds=0.\tag{3.24}$$

Therefore, combining (3.23) with (3.24), we can get $\{S(t)\}_{t\ge0}$ is asymptotically compact.

The proof is complete. □

Due to the general theorem about global attractor for infinite-dimensional dynamical systems (see Temam [17], Zhong [21]), we can finally acquire the existence of global attractor in $\mathcal{V}_1$ as follows.

Theorem 3.7 (The existence of global attractor) *Under the same assumptions as in Lemma* 3.4. *If the solution semigroup* $\{S(t)\}_{t\ge0}$ *has a bounded absorbing set* $\mathscr{B}_0$ *for Eqs.* (2.1)-(2.2) *and is asymptotically compact in* $\mathcal{V}_1$, *then* $\{S(t)\}_{t\ge0}$ *has a global attractor* $\mathscr{A}$ *in* $\mathcal{V}_1$, *which is compact, invariant in* $\mathcal{V}_1$ *and attracts any bounded subsets of* $\mathcal{V}_1$ *with respect to the norm* $\|\cdot\|_{\mathcal{V}_1}$.

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