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# Delay-induced oscillation phenomenon of a delayed finance model in enterprise operation

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**Abstract**

In this paper, the delayed finance model of enterprise operation is improved. The stability is investigated, and a Hopf bifurcation is demonstrated. Applying the normal form theory and the center manifold argument, some concrete expressions to judge the properties of the bifurcating periodic solutions are given. Computer simulations are performed to prove the correctness of theoretical analysis. Finally, a simple conclusion is included.

**Keywords:** finance model; stability; Hopf bifurcation; delay; periodic solution; enterprise

## 1 Introduction

In recent years, investigation on economic dynamical behaviors has become more prominent in mainstream economics in the course of enterprise running. In order to understand the highly complex dynamics of real financial and economic systems, researchers have set up several nonlinear continuous economics models to describe economics phenomena of enterprise operation, for example, Chian *et al.* [1, 2] proposed the forced van der Pol model, the IS-LM model was analyzed in [3–5], Lorenz [6] studied the Kaldorian model and Goodwin's accelerate model was discussed by Lorenz and Nusse [7]. In 2001, Ma and Chen [8, 9] reported a dynamical model of financial system which is composed of four sub-blocks: production, money, stock and labor force. By setting proper dimensions and choosing appropriate coordinates, the authors set up the following simplified three-dimensional financial model:

$$\begin{cases} \dot{u}_1(t) = u_3(t) + (u_2(t) - a)u_1(t), \\ \dot{u}_2(t) = 1 - bu_2(t) - u_1^2(t), \\ \dot{u}_3(t) = -u_1(t) - cu_3(t), \end{cases} \tag{1.1}$$

where the three state variables $u_1(t)$, $u_2(t)$ and $u_3(t)$ are the interest rate, the investment demand and the price index, respectively. $a \geq 0$ is the saving amount, $b \geq 0$ is the cost per investment, and $c \geq 0$ is the elasticity of demand of commercial markets. Ma and Chen [8, 9] investigated all the possible dynamical phenomena (including balance, stable period, fractal, Hopf bifurcation, the relationship between parameters and Hopf bifurcation and chaos *etc.*) of system (1.1) under different parameter combinations. In 2009, by adding delayed feedback to the second equation of system (1.1), Gao and Ma [10] derived the following delayed financial model:

$$\begin{cases} \dot{u}_1(t) = u_3(t) + (u_2(t) - a)u_1(t), \\ \dot{u}_2(t) = 1 - bu_2(t) - u_1^2(t) + \kappa[u_2(t) - u_2(t - \varsigma)], \\ \dot{u}_3(t) = -u_1(t) - cu_3(t), \end{cases} \tag{1.2}$$

where $\kappa$ is a real number and $\varsigma$ is time delay. They have shown that the Hopf bifurcation of system (1.2) occurs when the time delay varies.

By adding delayed feedback to the three equations of system (1.1), Chen [11] obtained the modified version of system (1.1) which takes the form

$$\begin{cases} \dot{u}_1(t) = u_3(t) + (u_2(t) - a)u_1(t) + \kappa_1[u_1(t) - u_1(t - \varsigma_1)], \\ \dot{u}_2(t) = 1 - bu_2(t) - u_1^2(t) + \kappa_2[u_2(t) - u_2(t - \varsigma_2)], \\ \dot{u}_3(t) = -u_1(t) - cu_3(t) + \kappa_3[u_1(t) - u_1(t - \kappa_3)], \end{cases} \tag{1.3}$$

where $\kappa_i$ $(i = 1, 2, 3)$ are the feedback strengths and $\varsigma_i$ $(i = 1, 2, 3)$ are the time delays. By choosing the time delays as varying parameters, Chen [11] controlled the chaotic phenomena of the unperturbed system with $a = 3$, $b = 0.1$ and $c = 1$.

Son and Park [12] further considered the dynamical behaviors of system (1.3). By local stability analysis, Son and Park [12] theoretically proved the occurrences of a Hopf bifurcation. Moreover, through numerical bifurcation analysis, they obtained the supercritical and subcritical Hopf bifurcation curves which support the theoretical predictions. Meanwhile, the folds limit cycle and Neimark-Sacker bifurcation curves were detected. Also the double Hopf bifurcation and the generalized and Hopf bifurcation codimension-2 bifurcation points were found.

Recently, Chen [13] generalized system (1.1) to the fractional order case of the form

$$\begin{cases} \frac{d^{\alpha_1} u_1}{dt^{\alpha_1}} = u_3(t) + (u_2(t) - a)u_1(t), \\ \frac{d^{\alpha_2} u_2}{dt^{\alpha_2}} = 1 - bu_2(t) - u_1^2(t), \\ \frac{d^{\alpha_3} u_3}{dt^{\alpha_3}} = -u_1(t) - cu_3(t). \end{cases} \tag{1.4}$$

Chen [13] found that system (1.4) displayed many interesting dynamical behaviors such as fixed points, periodic motions and chaos. Meanwhile, it was shown that chaos existed in fractional-order financial systems with orders less than three and period doubling and intermittency routes to chaos in the fractional-order financial system were found.

As is known to us, the time delays actually occur in the process of economic operation of enterprise. In fact, the investment demand is affected by interest and the price and has a certain time lag. Stimulated by this viewpoint and based on the former work [8–13], in this paper, we will make a discussion on the following delayed finance system:

$$\begin{cases} \dot{u}_1(t) = u_3(t) + [u_2(t - \varsigma) - a]u_1(t), \\ \dot{u}_2(t) = 1 - bu_2(t - \varsigma) - u_1^2(t), \\ \dot{u}_3(t) = -u_1(t) - cu_3(t), \end{cases} \tag{1.5}$$

where the three state variables $u_1(t)$, $u_2(t)$ and $u_3(t)$ are the interest rate, the investment demand and the price index, respectively. $a \geq 0$ is the saving amount, $b \geq 0$ is the cost per investment, and $c \geq 0$ is the elasticity of demand of commercial markets, $\varsigma$ is time delay.

In this paper, we study the stability, the local Hopf bifurcation for system (1.5). Although there are a great variety of works dealing with a Hopf bifurcation for delayed differential equations [14–24], up to now, to the best of our knowledge, few authors have considered the bifurcation behaviors of finance systems. The main contributions of this article include the three aspects: (i) some new sufficient conditions which guarantee the stability and the existence of a Hopf bifurcation of delayed finance system are established; (ii) the explicit formulas for determining the properties of the bifurcating periodic solutions are obtained; (iii) to the best of our knowledge, it is the first time to focus on the time delay effect on interest rate, the investment demand and the price index for a finance system, and the obtained results have an important guiding role to the economic operation and also complement numerous previous works.

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the equilibrium and the existence of local Hopf bifurcations. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to illustrate the validity of the main predictions. Some main conclusions are drawn in Section 5.

## 2 Stability of equilibrium and local Hopf bifurcations

One can check that if

(H1) $c - b - abc > 0$

holds, then Eq. (1.5) has three equilibria $E_1(0, \frac{1}{b}, 0)$, $E_2(u_1^*, u_2^*, -u_3^*)$ and $E_3(-u_1^*, u_2^*, u_3^*)$, where

$$u_1^* = \sqrt{\frac{c-b-abc}{c}}, \qquad u_2^* = \frac{1+ac}{c}, \qquad u_3^* = \frac{1}{c}\sqrt{\frac{c-b-abc}{c}}.$$

In the following, we only focus on the existence of a local Hopf bifurcation at the equilibrium $E_2(u_1^*, u_2^*, -u_3^*)$ of system (1.5).

Let $\bar{u}_1(t) = u_1(t) - x^*$, $\bar{u}_2(t) = u_2(t) - u_2^*$, $\bar{u}_3(t) = u_3(t) + u_3^*$ and still denote $\bar{u}_1(t)$, $\bar{u}_2(t)$ and $\bar{u}_3(t)$ by $u_1(t)$, $u_2(t)$ and $u_3(t)$, respectively, then (1.5) takes the form

$$\begin{cases} \dot{u}_1(t) = (u_2^* - a)u_1(t) + u_3(t) + u_1^* u_2(t-\varsigma) + u_1(t)u_2(t-\varsigma), \\ \dot{u}_2(t) = -2u_1^* u_1(t) - bu_2(t-\varsigma) - u_1^2(t), \\ \dot{u}_3(t) = -u_1(t) - cu_3(t). \end{cases} \tag{2.1}$$

The linearization of Eq. (2.1) at $(0,0,0)$ is given by

$$\begin{cases} \dot{u}_1(t) = (u_2^* - a)u_1(t) + u_3(t) + u_1^* u_2(t-\varsigma), \\ \dot{u}_2(t) = -2u_1^* u_1(t) - bu_2(t-\varsigma), \\ \dot{u}_3(t) = -u_1(t) - cu_3(t). \end{cases} \tag{2.2}$$

The characteristic equation corresponding to the linearized equation (2.2) is given by

$$\det \begin{pmatrix} \lambda - (u_2^* - a) & -u_1^* e^{-\lambda\varsigma} & -1 \\ 2u_1^* & \lambda + be^{-\lambda\varsigma} & 0 \\ 1 & 0 & \lambda + c \end{pmatrix} = 0.$$

That is,

$$\lambda^3 + l_2\lambda^2 + l_1\lambda + l_0 + \left(m_2\lambda^2 + m_1\lambda + m_0\right)e^{-\lambda\varsigma} = 0, \tag{2.3}$$

where $l_0 = 2c(u_1^*)^2$, $l_1 = 1 - c(u_2^* - a)$, $l_2 = c + a - u_2^*$, $m_0 = b + abc - bcu_2^*$, $m_1 = bc + ab - bu_2^* + 2(u_1^*)^2$, $m_2 = b$.

For $\varsigma = 0$, (2.3) becomes

$$\lambda^3 + (l_2 + m_2)\lambda^2 + (l_1 + m_1)\lambda + l_0 + m_0 = 0. \tag{2.4}$$

In view of Routh-Hurwitz criteria, we know that all roots of (2.4) have a negative real part if the following condition

(H2) $(l_1 + m_1)(l_2 + m_2) > l_0 + m_0$, $l_0 + m_0 > 0$

is fulfilled.

For $\varpi > 0$, $i\varpi$ is a root of (2.3) if and only if

$$-i\varpi^3 - l_2\varpi^2 + il_1\varpi + l_0 + \left(-m_2\varpi^2 + im_1\varpi + m_0\right)(\cos\varpi\varsigma - i\sin\varpi\varsigma) = 0.$$

Then we have

$$\begin{cases} (m_0 - m_2\varpi^2)\cos\varpi\varsigma + m_1\varpi\sin\varpi\varsigma = l_2\varpi^2 - l_0, \\ (m_0 - m_2\varpi^2)\sin\varpi\varsigma - m_1\varpi\cos\varpi\varsigma = -\varpi^3 + l_1\varpi, \end{cases} \tag{2.5}$$

which is equivalent to

$$\left(m_0 - m_2\varpi^2\right)^2 + (m_1\varpi)^2 = \left(l_2\varpi^2 - l_0\right)^2 + \left(-\varpi^3 + l_1\varpi\right)^2,$$

namely,

$$\varpi^6 + \left(l_2^2 - 2l_1 - m_2^2\right)\varpi^4 + \left(l_1^2 - 2l_0l_2 - m_1^2 + 2m_0m_2\right)\varpi^2 + l_0^2 - m_0^2 = 0. \tag{2.6}$$

Let $z = \varpi^2$, then (2.6) becomes

$$z^3 + p_1z^2 + p_2z + p_3 = 0, \tag{2.7}$$

where $p_1 = l_2^2 - 2l_1 - m_2^2$, $p_2 = l_1^2 - 2l_0l_2 - m_1^2 + 2m_0m_2$, $p_3 = l_0^2 - m_0^2$.

Denote

$$h(z) = z^3 + p_1z^2 + p_2z + p_3. \tag{2.8}$$

Let $K = (\frac{q}{2})^2 + (\frac{r}{3})^3$, where $r = p_2 - \frac{1}{3}p_1^2$, $q = \frac{2}{27}p_1^3 - \frac{1}{3}p_1p_2 + p_3$. There are three cases for the solutions of Eq. (2.7).

(i) If $K > 0$, Eq. (2.7) has a real root and a pair of conjugate complex roots. The real root is positive and is given by

$$\nu_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{K}} + \sqrt[3]{-\frac{q}{2} - \sqrt{K}} - \frac{1}{3}p_1.$$

(ii) If $K = 0$, Eq. (2.7) has three real roots, of which two are equal. In particular, if $p_1 > 0$, there exists only one positive root, $\nu_1 = 2\sqrt[3]{-\frac{q}{2}} - \frac{p_1}{3}$; if $p_1 < 0$, there exists only one positive root, $\nu_1 = 2\sqrt[3]{-\frac{q}{2}} - \frac{p_1}{3}$ for $\sqrt[3]{-\frac{q}{2}} > -\frac{p_1}{3}$, and there exist three positive roots for $\frac{r_1}{6} < \sqrt[3]{-\frac{q}{2}} < -\frac{p_1}{3}$, $\nu_1 = 2\sqrt[3]{-\frac{q}{2}} - \frac{p_1}{3}$, $\nu_2 = \nu_3 = -\sqrt[3]{-\frac{q}{2}} - \frac{p_1}{3}$.

(iii) If $K < 0$, there are three distinct real roots, $\nu_1 = 2\sqrt{\frac{|p|}{3}}\cos\frac{\psi}{3} - \frac{p_1}{3}$, $\nu_2 = 2\sqrt{\frac{|p|}{3}}\cos(\frac{\psi}{3} + \frac{2\pi}{3}) - \frac{p_1}{3}$, $\nu_3 = 2\sqrt{\frac{|p|}{3}}\cos(\frac{\psi}{3} + \frac{4\pi}{3}) - \frac{p_1}{3}$, where $\cos\psi = -\frac{q}{2\sqrt{(\frac{|p|}{3})^3}}$. Furthermore, if $p_1 > 0$, there exists only one positive root. Otherwise, if $p_1 < 0$, there may exist either one or three positive real roots. If there is only one positive real root, it is equal to $\max(\nu_1, \nu_2, \nu_3)$.

Obviously, the number of positive real roots of Eq. (2.7) depends on the sign of $r_1$. If $p_1 \geq 0$, Eq. (2.7) has only one positive real root. Otherwise, there may exist three positive roots. Without loss of generality, we assume that (2.7) has three positive roots, defined by $z_1, z_2, z_3$, respectively. Then Eq. (2.6) has three positive roots

$$\varpi_1 = \sqrt{z_1}, \qquad \varpi_2 = \sqrt{z_2}, \qquad \varpi_3 = \sqrt{z_3}.$$

By (2.5), we have

$$\cos\varpi\varsigma = \frac{m_1\varpi^2(\varpi^2 - l_1) - (l_2\varpi^2 - l_0)(m_2\varpi^2 - m_0)}{(m_0 - m_2\varpi^2)^2 + (m_1\varpi)^2}.$$

Thus, if we denote

$$\varsigma_k^{(j)} = \frac{1}{\varpi_k}\left\{\arccos\left[\frac{m_1\varpi^2(\varpi^2 - l_1) - (l_2\varpi^2 - l_0)(m_2\varpi^2 - m_0)}{(m_0 - m_2\varpi^2)^2 + (m_1\varpi)^2}\right] + 2j\pi\right\}, \tag{2.9}$$

where $k = 1, 2, 3$; $j = 0, 1, \ldots$, then $\pm i\varpi_k$ are a pair of purely imaginary roots of Eq. (2.4) with $\varsigma_k^{(j)}$. Define

$$\varsigma_0 = \varsigma_{k_0}^{(0)} = \min_{k\in\{1,2,3\}}\{\varsigma_k^{(0)}\}. \tag{2.10}$$

In view of [25], we have the following result.

**Lemma 2.1** *If* (H1) *and* (H2) *hold, then all roots of* (2.3) *have a negative real part when* $\varsigma \in [0, \varsigma_0)$, *and* (2.3) *admits a pair of purely imaginary roots* $\pm\varpi_k$ *when* $\varsigma = \varsigma_k^{(j)}$ ($k = 1, 2, 3$; $j = 0, 1, 2, \ldots$).

Assume that $\lambda(\varsigma) = \alpha(\varsigma) + i\varpi(\varsigma)$ is a root of (2.3) near $\varsigma = \varsigma_k^{(j)}$, and $\alpha(\varsigma_k^{(j)}) = 0$, and $\varpi(\varsigma_k^{(j)}) = \varpi_k$. In view of (2.3), one has

$$\left[\frac{d\lambda}{d\varsigma}\right]^{-1} = \frac{(3\lambda^2 + 2l_2\lambda + l_1)e^{\lambda\varsigma}}{\lambda(m_2\lambda^2 + m_1\lambda + m_0)} + \frac{2m_2\lambda + m_1}{\lambda(m_2\lambda^2 + m_1\lambda + m_0)} - \frac{\varsigma}{\lambda}. \tag{2.11}$$

Noting that

$$\left[\lambda\left(m_2\lambda^2 + m_1\lambda + m_0\right)\right]_{\varsigma=\varsigma_k^{(j)}} = m_1\varpi_k^2 + i\left(m_2\varpi_k^3 - m_0\varpi_k\right),$$

$$[2m_2\lambda + m_1]_{\varsigma=\varsigma_k^{(j)}} = m_1 + im_2\varpi_k$$

and

$$\begin{aligned}\left[\left(3\lambda^2+2l_2\lambda+l_1\right)e^{\lambda\varsigma}\right]_{\varsigma=\varsigma_k^{(j)}} &= \left[\left(l_1-3\varpi_k^2\right)\cos\varpi_k\varsigma_k^{(j)}-2l_2\varpi_k\sin\varpi_k\varsigma_k^{(j)}\right] \\ &\quad + i\left[2l_2\varpi_k\cos\varpi_k\varsigma_k^{(j)}-\left(l_1-3\varpi_k^2\right)\sin\varpi_k\varsigma_k^{(j)}\right],\end{aligned}$$

it follows from (2.5) that

$$\begin{aligned}\left[\frac{d(\operatorname{Re}\lambda(\varsigma))}{d\varsigma}\right]^{-1}_{\varsigma=\varsigma_k^{(j)}} &= \operatorname{Re}\left\{\frac{(3\lambda^2+2l_2\lambda+l_1)e^{\lambda\varsigma}}{\lambda(m_2\lambda^2+m_1\lambda+m_0)}\right\}_{\varsigma=\varsigma_k^{(j)}} \\ &\quad + \operatorname{Re}\left\{\frac{2m_2\lambda+m_1}{\lambda(m_2\lambda^2+m_1\lambda+m_0)}\right\}_{\varsigma=\varsigma_k^{(j)}} \\ &= \frac{1}{\Lambda}\Big\{-m_1\varpi_k^2\left[\left(l_1-3\varpi_k^2\right)\cos\varpi_k\varsigma_k^{(j)}-2l_2\varpi_k\sin\varpi_k\varsigma_k^{(j)}\right] \\ &\quad + \left(m_0\varpi_k-m_2\varpi_k^3\right)\left[2l_2\varpi_k\cos\varpi_k\varsigma_k^{(j)}+\left(l_1-3\varpi_k^2\right)\sin\varpi_k\varsigma_k^{(j)}\right] \\ &\quad + m_1^2\varpi_k^2+2m_2\varpi_k\left(m_0\varpi_k-m_2\varpi_k^3\right)\Big\} \\ &= \frac{1}{\Lambda}\Big\{\left(l_1-3\varpi_k^2\right)\varpi_k\left[\left(-m_0-m_2\varpi_k^2\right)\sin\varpi_k\varsigma_k^{(j)}-m_1\varpi_k\cos\varpi_k\varsigma_k^{(j)}\right] \\ &\quad + 2l_2\varpi_k^2\left[\left(m_0-m_2\varpi_k^2\right)\cos\varpi_k\varsigma_k^{(j)}+m_1\varpi_k\sin\varpi_k\varsigma_k^{(j)}\right] \\ &\quad + m_1^2\varpi_k^2+2m_2\varpi_k\left(m_0\varpi_k-m_2\varpi_k^3\right)\Big\} \\ &= \frac{1}{\Lambda}\Big[\left(3\varpi_k^6+\left(2l_2^2-4l_1+2m_2^2\right)\varpi_k^4\right. \\ &\quad \left.+\left(l_1^2-2l_0l_2+m_1^2+2m_0m_2\right)\varpi_k^2\right)\Big] \\ &= \frac{1}{\Lambda}\left(3\varpi_k^6+2r_1\varpi_k^4+r_2\varpi_k^2\right)=\frac{1}{\Lambda}\left[z_k\left(3z_k^2+2r_1z_k+r_2\right)\right]=\frac{z_k}{\Lambda}h'(z_k),\end{aligned}$$

where $\Lambda=(m_1\varpi_k^2)^2+(m_0\varpi_k-m_2\varpi_k^3)^2>0$. Thus we have

$$\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda(\varsigma))}{d\varsigma}\right\}_{\varsigma=\varsigma_k^{(j)}}=\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda(\varsigma))}{d\varsigma}\right\}^{-1}_{\varsigma=\varsigma_k^{(j)}}=\operatorname{sign}\left\{\frac{z_k}{\Lambda}h'(z_k)\right\}\neq 0.$$

Since $\Lambda, z_k>0$, we can conclude that the sign of $[\frac{d(\operatorname{Re}\lambda(\varsigma))}{d\varsigma}]_{\varsigma=\varsigma_k^{(j)}}$ can be judged by that of $h'(z_k)$. The analysis above leads to the following result.

**Theorem 2.1** *Assume that $z_k=\omega_k^2$ and $h'(z_k)\neq 0$, where $h(z)$ is defined by* (2.9). *Then*

$$\left[\frac{d(\operatorname{Re}\lambda(\varsigma))}{d\tau}\right]_{\varsigma=\varsigma_k^{(j)}}\neq 0$$

*and the sign of $[\frac{d(\operatorname{Re}\lambda(\varsigma))}{d\varsigma}]_{\varsigma=\varsigma_k^{(j)}}$ is consistent with that of $h'(z_k)$.*

In the sequel, we give the following assumption:

(H3) $h'(z_k)\neq 0$.

According to the above analysis and the results of Kuang [26] and Hale [27], we have the following.

**Theorem 2.2** *Assume that* (H1) *and* (H2) *hold, then the equilibrium* $E_2(u_1^*, u_2^*, -u_3^*)$ *of system* (1.5) *is asymptotically stable for* $\tau \in [0, \tau_0)$. *Under conditions* (H1)-(H3), *system* (1.5) *undergoes a Hopf bifurcation around the equilibrium* $E_2(u_1^*, u_2^*, -u_3^*)$ *when* $\varsigma = \varsigma_k^{(j)}$, $k = 1, 2, 3; j = 0, 1, 2, \ldots$.

## 3 Direction and stability of the Hopf bifurcation

In this section, we consider the direction and stability of the Hopf bifurcation of (1.5) by using normal form and center manifold theory [28].

Let $\bar{u}_1(t) = u_1(\tau t)$, $\bar{u}_2(t) = u_2(\tau t)$, $\bar{u}_3(t) = u_3(\varsigma t)$ and $\varsigma = \varsigma_k^{(j)} + \mu$, where $\varsigma_k^{(j)}$ is defined by (2.9) and $\mu \in R$, drop the bar for the simplification of notations, then system (2.1) can be written as a functional differential equation in $C = C([-1, 0]), R^3))$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in C$ and $u_t(\theta) = u(t+\theta) = (u_1(t+\theta), u_2(t+\theta), u_3(t+\theta))^T \in C$, and $L_\mu : C \to R$, $F : R \times C \to R$ are given by

$$\begin{aligned} L_\mu \phi = {} & \left(\varsigma_k^{(j)} + \mu\right) \begin{pmatrix} u_2^* - a & 0 & 1 \\ -2u_1^* & 0 & 0 \\ -1 & 0 & -c \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} \\ & + \left(\varsigma_k^{(j)} + \mu\right) \begin{pmatrix} 0 & u_1^* & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \end{aligned} \tag{3.2}$$

and

$$f(\mu, \phi) = \left(\varsigma_k^{(j)} + \mu\right) \begin{pmatrix} \phi_1(0)\phi_2(-1) \\ -\phi_1^2(0) \\ 0 \end{pmatrix}, \tag{3.3}$$

respectively, where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$.

By the representation theorem, there is a matrix function with bounded variation components $\eta(\theta, \mu)$, $\theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta) \quad \text{for } \phi \in C. \tag{3.4}$$

In fact, we can choose

$$\begin{aligned} \eta(\theta, \mu) = {} & \left(\varsigma_k^{(j)} + \mu\right) \begin{pmatrix} u_2^* - a & 0 & 1 \\ -2u_1^* & 0 & 0 \\ -1 & 0 & -c \end{pmatrix} \delta(\theta) \\ & - \left(\varsigma_k^{(j)} + \mu\right) \begin{pmatrix} 0 & u_1^* & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1), \end{aligned} \tag{3.5}$$

where $\delta$ is the Dirac delta function. For $\phi \in C([-1,0],R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0 \end{cases} \tag{3.6}$$

and

$$R\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ f(\mu,\phi), & \theta = 0. \end{cases} \tag{3.7}$$

Then (3.1) is equivalent to

$$\dot{u_t} = A(\mu)u_t + R(\mu)u_t, \tag{3.8}$$

where $u_t(\theta) = u(t+\theta)$, $\theta \in [-1,0]$. For $\psi \in C([0,1],(R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C([-1,0],R^3)$ and $\psi \in C([0,1],(R^3)^*)$, define the bilinear form

$$\langle \psi,\phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi^T(\xi-\theta)\,d\eta(\theta)\phi(\xi)\,d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$, the $A = A(0)$ and $A^*$ are adjoint operators. By a simple computation, we can obtain

$$q(\theta) = (1,\alpha,\beta)^T e^{i\omega_k \varsigma_k^{(j)}\theta}, \qquad q^*(s) = D\big(1,\alpha^*,\beta^*\big)e^{i\varpi_k \varsigma_k^{(j)} s},$$

where

$$\alpha = \frac{i\varpi_k - u_2^* + a + 1}{u_1^* e^{-i\varpi_k \varsigma_k}(i\varpi_k + c)}, \qquad \beta = -\frac{1}{i\varpi_k + c}, \qquad \alpha^* = \frac{u_1^*}{b}, \qquad \beta^* = \frac{1}{c - i\varpi_k \varsigma_k},$$
$$D = \frac{1}{1 + \bar{\alpha}\alpha^* + \bar{\beta}\beta^* + \bar{\alpha}\varsigma_k^{(j)}(b\beta^* - u_1^*)e^{i\varpi_k \varsigma_k^{(j)}}}.$$

Furthermore, $\langle q^*(s),q(\theta)\rangle = 1$ and $\langle q^*(s),\bar{q}(\theta)\rangle = 0$. Next, we use the same notations as those in Hassard *et al.* [28], and we first compute the coordinates to describe the center manifold $C_0$ at $\mu = 0$. Let $u_t$ be the solution of Eq. (3.1) when $\mu = 0$. Define

$$z(t) = \langle q^*,u_t \rangle, \qquad W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\big\{z(t)q(\theta)\big\} \tag{3.9}$$

on the center manifold $C_0$, and we have

$$W(t,\theta) = W\big(z(t),\bar{z}(t),\theta\big), \tag{3.10}$$

where

$$W\big(z(t),\bar{z}(t),\theta\big) = W(z,\bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{3.11}$$

and $z$ and $\bar{z}$ are local coordinates for center manifold $C_0$ in the direction of $q^*$ and $\bar{q}^*$. Noting that $W$ is also real if $u_t$ is real, we consider only real solutions. For solutions $u_t \in C_0$ of (3.1),

$$\dot{z}(t) = i\varpi_k \tau_k^{(j)} z + \bar{q}^*(\theta) f\bigl(0, W(z,\bar{z},\theta) + 2\operatorname{Re}\{zq(\theta)\}\bigr) \stackrel{\text{def}}{=} i\varpi_k \varsigma_k^{(j)} z + \bar{q}^*(0) f_0.$$

That is,

$$\dot{z}(t) = i\varpi_k \varsigma_k^{(j)} z + g(z,\bar{z}),$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots .$$

Then we can obtain the expression of $g_{20}$, $g_{11}$, $g_{02}$ and $g_{21}$. See Appendix. Then we get

$$\begin{cases} c_1(0) = \frac{i}{2\varpi_k \varsigma_k^{(j)}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\varsigma_k^{(j)})\}}, \\ \beta_2 = 2\operatorname{Re}(c_1(0)), \\ T_2 = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\varsigma_k^{(j)})\}}{\varpi_k \varsigma_k^{(j)}}. \end{cases} \tag{3.12}$$

**Theorem 3.1** *If* $\mu_2 > 0$ ($\mu_2 < 0$), *then the periodic solution is supercritical* (*subcritical*); *if* $\beta_2 < 0$ ($\beta_2 > 0$), *then the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase* (*unstable*); *if* $T_2 > 0$ ($T_2 < 0$), *then the periods of the bifurcating periodic solutions increase* (*decrease*).

## 4 Numerical examples

Let us consider the following system:

$$\begin{cases} \dot{u}_1(t) = u_3(t) + [u_2(t-\varsigma) - 0.6]u_1(t), \\ \dot{u}_2(t) = 1 - 0.2u_2(t-\varsigma) - u_1^2(t), \\ \dot{u}_3(t) = -u_1(t) - 6u_3(t), \end{cases} \tag{4.1}$$

which has an equilibrium $E_2(0.9201, 0.7667, -0.1534)$ and satisfies the conditions indicated in Theorem 2.2. The equilibrium $E_2(0.9201, 0.7667, -0.1534)$ is asymptotically stable for $\varsigma = 0$. Using the software Matlab, we derive $\varpi_0 \approx 0.8960$, $\varsigma_0 \approx 2.6$, $\lambda'(\varsigma_0) \approx 0.9281 - 7.1128i$. Thus by algorithm (3.12) derived in Section 3, we have $c_1(0) \approx -0.7053 - 6.0952i$, $\mu_2 \approx 0.7599$, $\beta_2 \approx -1.4106$, $T_2 \approx 8.2331$. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Thus the equilibrium $E_2(0.9201, 0.7667, -0.1534)$ is stable when $\varsigma < \varsigma_0$. Figure 1 shows that the equilibrium $E_2(0.9201, 0.7667, -0.1534)$ is asymptotically stable when $\varsigma = 2.2 < \varsigma_0 \approx 2.6$. When $\varsigma$ passes through the critical value $\varsigma_0$, the equilibrium $E_2(0.9201, 0.7667, -0.1534)$ loses its stability and a Hopf bifurcation occurs. In view of $\mu_2 > 0$ and $\beta_2 < 0$, we know that the direction of the Hopf bifurcation is $\varsigma > \varsigma_0$, and these bifurcating periodic solutions from $E_2(0.9201, 0.7667, -0.1534)$ at $\varsigma_0$ are stable. Figure 2 suggests that a Hopf bifurcation occurs from the equilibrium $E_2(0.9201, 0.7667, -0.1534)$ when $\varsigma = 0.11 > \varsigma_0 \approx 0.1$.

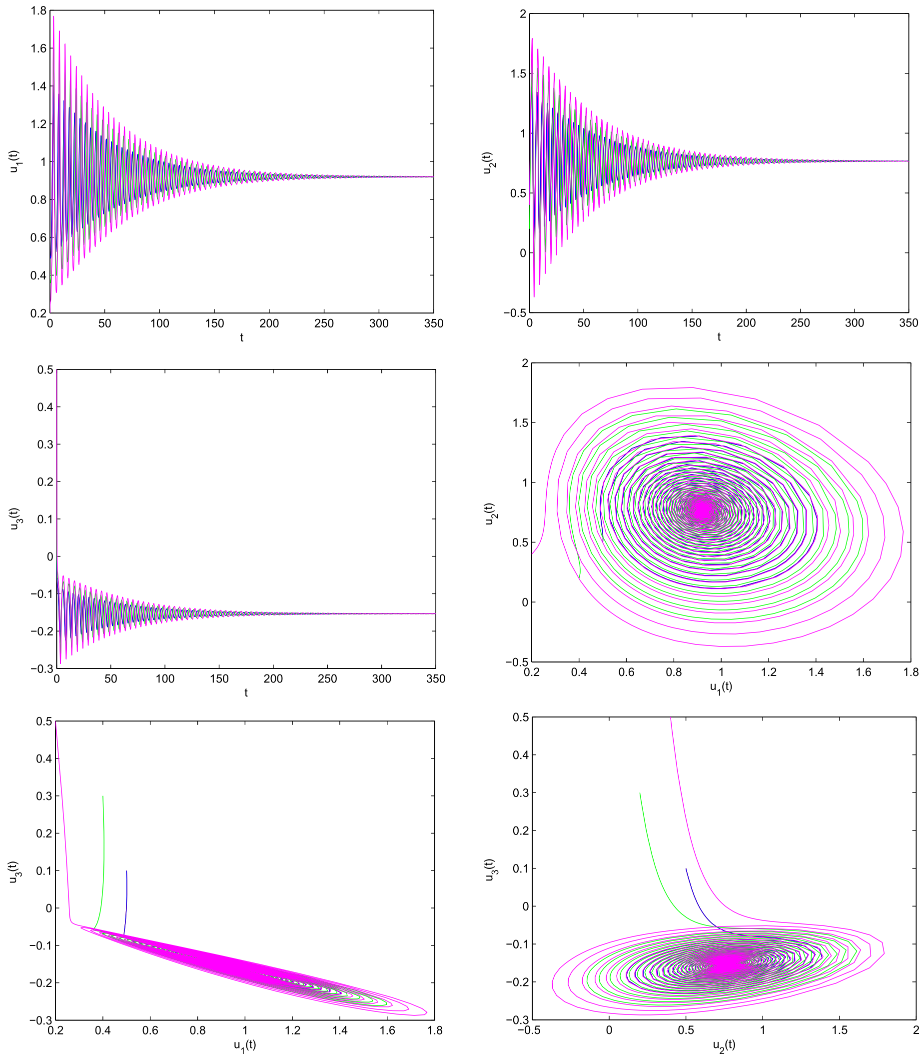


**Figure 1 Dynamics of (4.1) with $\varsigma = 0.09 < \varsigma_0 \approx 0.1$ and the initial value (0.5, 0.5, 0.1).**

## 5 Conclusions

Recently, there has been an increasing activity and interest in the study of Hopf bifurcations of the delayed differential equations. Most of them focused on the study of predator-prey models and neural network systems [14, 15, 17, 19–28]. However, to the best of our knowledge, there are few results on the properties of Hopf bifurcations for a finance model of enterprise operation. In this paper, we have investigated the qualitative behaviors of a delayed finance model of enterprise operation. The study shows that if under some conditions, finance model (1.5) is asymptotically stable, when the delay $\varsigma$ increases and crosses a threshold value $\varsigma_k$, the equilibrium loses its stability and the delayed finance system enters into a Hopf bifurcation. Thus the time delay has important effect on the stability of a finance model of enterprise operation. We also give the concrete expressions to judge the properties of the bifurcating periodic solutions. Simulation results show that theoretical analysis of this paper is correct. The obtained results are useful in applications of finance control in enterprise operation. In the real process of economic operation of enterprise,

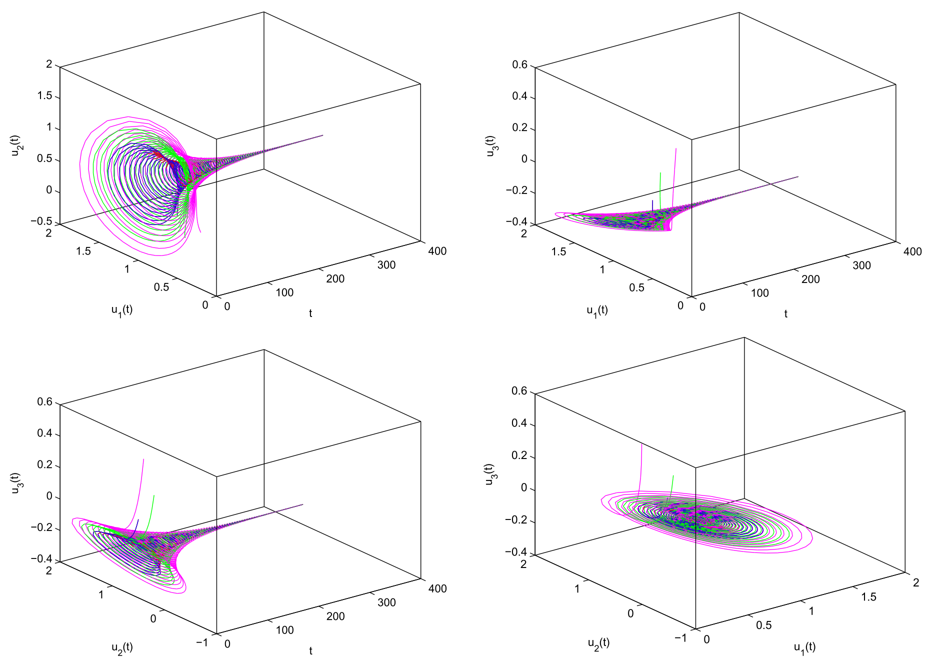

**Figure 1** Continued.

we can choose some suitable parameters and the delay of investment demand to keep the investment demand, the interest and the price a balance.

## Appendix

We give the computational process of $g_{20}$, $g_{11}$, $g_{02}$ and $g_{21}$.

$$
\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) = \bar{q}^*(0)f(0,u_t) = \tau_k^{(j)}\bar{D}\bigl(1,\bar{\alpha}^*,\bar{\beta}^*\bigr)\begin{pmatrix} u_{1t}(0)u_{2t}(-1) \\ -u_{1t}^2(0) \\ 0 \end{pmatrix} \\
&= \bar{D}\varsigma_k^{(j)}\bigl(\alpha e^{i\varpi_k\varsigma_k^{(j)}} - \bar{\alpha}^*\bigr)z^2 + \bar{D}\varsigma_k^{(j)}\bigl[2\operatorname{Re}\bigl\{\alpha e^{i\varpi_k\varsigma_k^{(j)}}\bigr\} - 2\bar{\alpha}^*\bigr]z\bar{z} \\
&\quad + \bar{D}\varsigma_k^{(j)}\bigl(\bar{\alpha}e^{i\varpi_k\varsigma_k^{(j)}} - \bar{\alpha}^*\bigr)\bar{z}^2 + \bar{D}\tau_k^{(j)}\biggl[W_{11}^{(2)}(-1) + \frac{1}{2}W_{20}^{(2)}(-1) + \alpha W_{11}^{(1)}(0)e^{-i\varpi_k\varsigma_k^{(j)}} \\
&\quad + \frac{1}{2}W_{20}^{(1)}(0)\bar{\alpha}e^{i\varpi_k\varsigma_k^{(j)}} - \bar{\alpha}^*\bigl(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\bigr)\biggr]z^2\bar{z} + \text{h.o.t.}
\end{aligned}
$$

Then

$$
\begin{aligned}
g_{20} &= 2\bar{D}\varsigma_k^{(j)}\bigl(\alpha e^{i\varpi_k\varsigma_k^{(j)}} - \bar{\alpha}^*\bigr), \\
g_{11} &= 2\bar{D}\varsigma_k^{(j)}\bigl[\operatorname{Re}\bigl\{\alpha e^{i\varpi_k\varsigma_k^{(j)}}\bigr\} - \bar{\alpha}^*\bigr], \\
g_{02} &= 2\bar{D}\varsigma_k^{(j)}\bigl(\bar{\alpha}e^{i\varpi_k\varsigma_k^{(j)}} - \bar{\alpha}^*\bigr),
\end{aligned}
$$

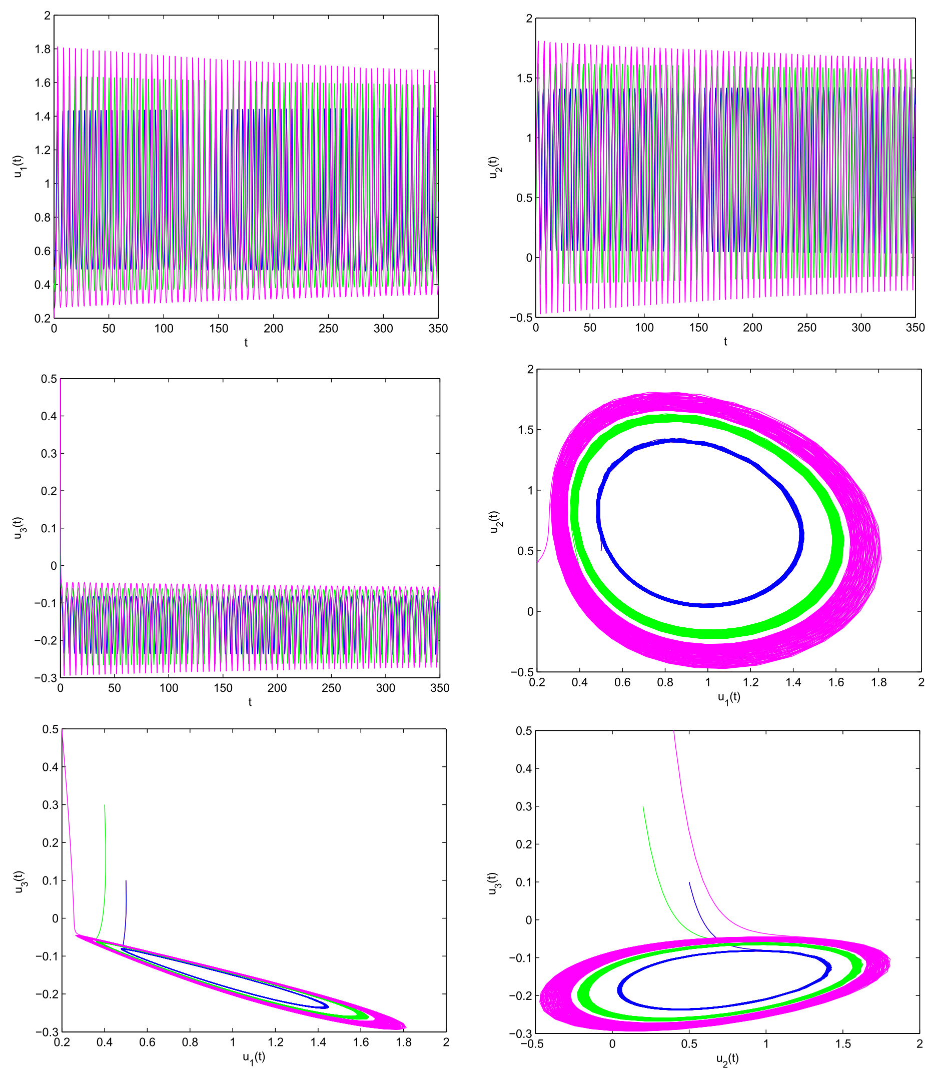


**Figure 2 Dynamics of (4.1) with $\varsigma = 0.11 > \varsigma_0 \approx 0.1$ and the initial value (0.5, 0.5, 0.1).**

$$g_{21} = 2\bar{D}\varsigma_k^{(j)}\biggl[W_{11}^{(2)}(-1) + \frac{1}{2}W_{20}^{(2)}(-1) + \alpha W_{11}^{(1)}(0)e^{-i\varpi_k\varsigma_k^{(j)}} + \frac{1}{2}W_{20}^{(1)}(0)\bar{\alpha}e^{i\varpi_k\varsigma_k^{(j)}} - \bar{\alpha}^*\bigl(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\bigr)\biggr].$$

We need to seek $W_{20}^{(i)}(0)$, $W_{11}^{(i)}(0)$, $W_{20}^{(i)}(-1)$, $W_{11}^{(i)}(-1)$ $(i = 1,2)$ in $g_{21}$. By (3.8) and (3.9), we have

$$W' = \begin{cases} AW - 2\,\mathrm{Re}\,\{\bar{q}^*(0)\bar{f}q(\theta)\}, & -1 \le \theta < 0, \\ AW - 2\,\mathrm{Re}\,\{\bar{q}^*(0)\bar{f}q(\theta)\} + \bar{f}, & \theta = 0 \end{cases} \stackrel{\text{def}}{=} AW + H(z,\bar{z},\theta), \tag{A.1}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{A.2}$$

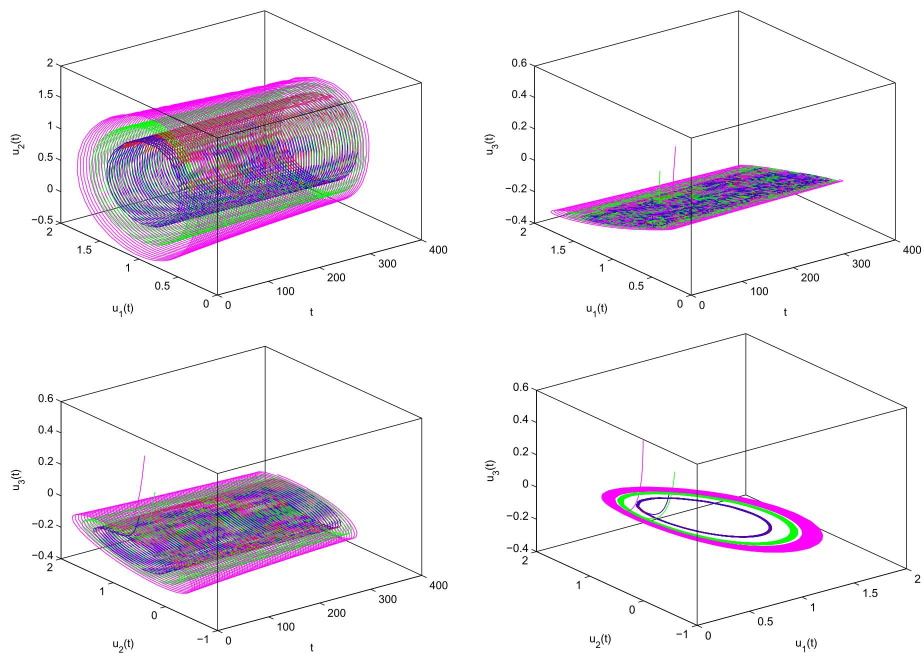


**Figure 2** Continued.

Comparing the coefficients, we obtain

$$\left(A - 2i\varsigma_k^{(j)}\varpi_k\right)W_{20} = -H_{20}(\theta), \tag{A.3}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{A.4}$$

We know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \tag{A.5}$$

By (A.5) and (A.2), one has

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{A.6}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{A.7}$$

From (A.3), (A.6) and the definition of $A$, we get

$$\dot{W}_{20}(\theta) = 2i\varpi_k\varsigma_k^{(j)}W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta). \tag{A.8}$$

Noting that $q(\theta) = q(0)e^{i\varpi_k\varsigma_k^{(j)}\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\varpi_k\varsigma_k^{(j)}}q(0)e^{i\varpi_k\varsigma_k^{(j)}\theta} + \frac{i\bar{g}_{02}}{3\varpi_k\varsigma_k^{(j)}}\bar{q}(0)e^{-i\varpi_k\varsigma_k^{(j)}\theta} + G_1 e^{2i\varpi_k\varsigma_k^{(j)}\theta}, \tag{A.9}$$

where $G_1 = (G_1^{(1)}, G_1^{(2)}, G_1^{(3)})^T \in R^3$ is a constant vector. In view of (A.4), (A.7) and the definition of $A$, we get

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \tag{A.10}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\varpi_k \varsigma_k^{(j)}} q(0)e^{i\varpi_k \varsigma_k^{(j)}\theta} + \frac{i\bar{g}_{11}}{\varpi_k \varsigma_k^{(j)}} \bar{q}(0)e^{-i\varpi_k \varsigma_k^{(j)}\theta} + H_2, \tag{A.11}$$

where $H_2 = (H_2^{(1)}, H_2^{(2)}, H_2^{(3)})^T \in R^3$ is a constant vector. Now we shall compute $H_1$, $H_2$ in (A.9), (A.11), respectively. It follows from the definition of $A$ and (A.6), (A.7) that

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\varpi_k \varsigma_k^{(j)} W_{20}(0) - H_{20}(0) \tag{A.12}$$

and

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{A.13}$$

where $\eta(\theta) = \eta(0,\theta)$. From (A3), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\varsigma_k^{(j)} \begin{pmatrix} \alpha e^{i\varpi_k \varsigma_k^{(j)}} \\ -1 \\ 0 \end{pmatrix}, \tag{A.14}$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}(0)\bar{q}(0) + 2\varsigma_k^{(j)} \begin{pmatrix} \mathrm{Re}\{\alpha e^{i\varpi_k \varsigma_k^{(j)}}\} \\ -1 \\ 0 \end{pmatrix}. \tag{A.15}$$

Considering that

$$\left(i\varpi_k \varsigma_k^{(j)} I - \int_{-1}^{0} e^{i\varpi_k \varsigma_k^{(j)}\theta}\, d\eta(\theta)\right) q(0) = 0,$$
$$\left(-i\varpi_k \varsigma_k^{(j)} I - \int_{-1}^{0} e^{-i\varpi_k \varsigma_k^{(j)}\theta}\, d\eta(\theta)\right) \bar{q}(0) = 0$$

and substituting (A.9) and (A.14) into (A.12), we have

$$\left(2i\varpi_k \varsigma_k^{(j)} I - \int_{-1}^{0} e^{2i\varpi_k \varsigma_k^{(j)}\theta}\, d\eta(\theta)\right) E_1 = 2\varsigma_k^{(j)} \begin{pmatrix} \alpha e^{i\varpi_k \varsigma_k^{(j)}} \\ -1 \\ 0 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 2i\varpi_k - u_2^* + a & -u_1^* e^{-2i\varpi_k \varsigma_k^{(j)}} & -1 \\ 2u_1^* & 2i\varpi_k + be^{-2i\varpi_k \varsigma_k^{(j)}} & 0 \\ 1 & 0 & 2i\varpi_k + c \end{pmatrix} \begin{pmatrix} G_1^{(1)} \\ G_1^{(2)} \\ G_1^{(3)} \end{pmatrix} = 2 \begin{pmatrix} \alpha e^{i\varpi_k \varsigma_k^{(j)}} \\ -1 \\ 0 \end{pmatrix}.$$

It follows that

$$G_1^{(1)} = \frac{\Theta_{11}}{\Theta_1}, \qquad G_1^{(2)} = \frac{\Theta_{12}}{\Theta_1}, \qquad G_1^{(3)} = \frac{\Theta_{13}}{\Theta_1},$$

where

$$\begin{aligned}
\Theta_1 &= \bigl(2i\varpi_k - u_2^* + a\bigr)\bigl(2i\varpi_k + be^{-2i\varpi_k\varsigma_k^{(j)}}\bigr)(2i\varpi_k + c) \\
&\quad + 2i\varpi_k + be^{-2i\varpi_k\varsigma_k^{(j)}} + 2\bigl(u_1^*\bigr)^2(2i\varpi_k + c)e^{-2i\varpi_k\varsigma_k^{(j)}}, \\
\Theta_{11} &= 2\bigl(2i\varpi_k + be^{-2i\varpi_k\varsigma_k^{(j)}}\bigr)(2i\varpi_k + c)\alpha e^{-2i\varpi_k\varsigma_k^{(j)}} - 2(2i\varpi_k + c)u_1^* e^{-2i\varpi_k\varsigma_k^{(j)}}, \\
\Theta_{12} &= -2\bigl(2i\varpi_k - u_2^* + a\bigr)(2i\varpi_k + c) - 4u_1^*\alpha(2i\varpi_k + c)e^{-2i\varpi_k\varsigma_k^{(j)}} - 2, \\
\Theta_{13} &= 2u_1^* e^{-2i\varpi_k\varsigma_k^{(j)}} - 2\bigl(2i\varpi_k + be^{-2i\varpi_k\varsigma_k^{(j)}}\bigr)\alpha e^{i\varpi_k\varsigma_k^{(j)}}.
\end{aligned}$$

Similarly, by (A.10), (A.15) and (A.13), we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 = 2\varsigma_k^{(j)} \begin{pmatrix} \mathrm{Re}\{\alpha e^{i\varpi_k\varsigma_k^{(j)}}\} \\ -1 \\ 0 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} u-2^*-a & u_1^* & 0 \\ -2u_1^* & -b & 0 \\ -1 & 0 & -c \end{pmatrix} \begin{pmatrix} H_2^{(1)} \\ H_2^{(2)} \\ H_2^{(3)} \end{pmatrix} = 2 \begin{pmatrix} -\mathrm{Re}\{\alpha e^{i\varpi_k\varsigma_k^{(j)}}\} \\ 1 \\ 0 \end{pmatrix}.$$

It follows that

$$\begin{aligned}
H_2^{(1)} &= \frac{2bc\,\mathrm{Re}\{\alpha e^{i\varpi_k\varsigma_k^{(j)}}\} + 2cu_1^*}{bcu_2^* - abc - 2c(u_1^*)^2}, \\
H_2^{(2)} &= \frac{4cu_1^*\,\mathrm{Re}\{\alpha e^{i\varpi_k\varsigma_k^{(j)}}\} - 2cu_2^* + 2ac}{bcu_2^* - abc - 2c(u_1^*)^2}, \\
H_2^{(3)} &= \frac{2b\,\mathrm{Re}\{\alpha e^{i\varpi_k\varsigma_k^{(j)}}\} - 2cu_1^*}{bcu_2^* - abc - 2c(u_1^*)^2}.
\end{aligned}$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors have made the same contribution. All authors read and approved the final manuscript.

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## Publisher's Note



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