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Modified function projective synchronization of complex dynamical networks with mixed time-varying and asymmetric coupling delays via new hybrid pinning adaptive control

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Abstract

This paper investigates modified function projective synchronization (MFPS) for complex dynamical networks with mixed time-varying and hybrid asymmetric coupling delays, which is composed of state coupling, time-varying delay coupling and distributed time-varying delay coupling. In contrast to previous results, the coupling configuration matrix needs not be symmetric or irreducible. The MFPS of delayed complex dynamical networks is considered via either hybrid control or hybrid pinning control with nonlinear and adaptive linear feedback control, which contains error linear term, time-varying delay error linear term and distributed time-varying delay error linear term. Based on Lyapunov stability theory, adaptive control technique, the parameter update law and the technique of dealing with some integral terms, we will show that control may be used to manipulate the scaling functions matrix such that the drive system and response networks could be synchronized up to the desired scaling function matrix. Numerical examples are given to demonstrate the effectiveness of the proposed method. The results in this article generalize and improve the corresponding results of the recent works.

Keywords: modified function projective synchronization; complex dynamical network; mixed time-varying delay; mixed coupling delays; hybrid adaptive pinning control

1 Introduction

Complex networks, as an interesting subject, have been thoroughly investigated for decades. These networks show very complicated behavior and can be used to model and explain many complex systems in nature such as computer networks, the world wide web, cellular and metabolic networks, transportation networks, communication networks, disease transmission networks, electrical power grids and so forth. Complex dynamical networks (CDNs) are prominent in describing the sophisticated collaborative dynamics in many sciences [1–5].

The time delay exists extensively in the real word networks. It is well known that the existence in a network may cause instability, poor performances and oscillations. Exam-

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ples can be found in networks such as application engineering, electrical power networks, physical networks and many other. Thus, synchronization for CDNs with time delays in the dynamical nodes and coupling has become a key and significant topic. Synchronization of a class of general CDNs with coupling delays was investigated in [6–9]. Li et al. [8], introduced some general CDNs models with time-varying delays in network couplings and time-varying delays in dynamical nodes. Song et al. [10] investigated synchronization of general CDNs with mixed time, where mixed delay appeared in the hybrid coupling term, but not in the isolate systems. Furthermore, Li [11] considered synchronization for delayed CDNs with hybrid coupling, which is made up of constant coupling, discrete delay coupling and distributed-delay coupling, but the discrete and distributed delays are not different values. Up to now, unfortunately, there have only been few papers related to the topic of synchronization of CDNs with mixed time-varying delays in the dynamical nodes and time-varying delays in the hybrid coupling, which includes constant coupling, discrete time-varying delay coupling and distributed time-varying delay coupling, simultaneously. So, it is challenging to solve this synchronization problem for CDNs.

In the past few decades, control problems for synchronization have been widely studied in CDNs. Synchronization control methods have been developed for CDNs, for instance, feedback control [12, 13], active control [14], intermittent control [15, 16], sampled-data control [17], nonlinear feedback control [18], adaptive control [19–23], hybrid adaptive control [12], impulsive control [24], active sliding mode control [25] and other control methods. CDNs have a large number of nodes. It is often impossible to realize the control goal by controlling every node. It is possible to control a few nodes to realize the same goal. In engineering, it is usually difficult to control CDNs by adding controllers to all nodes. To reduce the number of controllers, a natural approach is to control CDNs by pinning part of nodes. Thus, a pinning control is a special control method of adding controllers to part of the nodes rather than all of the nodes [6, 10, 11, 21, 26–33]. Chen et al. [31] studied the pinning control problem of the coupled networks by controlling one single node. In [32], an adaptive controller was designed to synchronize delayed CDNs with time-varying coupling strength and time-varying delay. The work in [33] studied pinning adaptive synchronization of general CDNs via pinning adaptive controllers, where the pinning nodes can be randomly selected. In [21], with the aid of the nonlinear and adaptive feedback control and adaptive pinning feedback control method, the authors considered the FPS for CNDs with asymmetric coupling. However, the adaptive feedback control with mixed time-varying delays was not considered in feedback control. Thus, in this paper, we focus on the influences of hybrid pinning feedback control method with nonlinear and adaptive linear feedback control, which contains error linear term, time-varying delay error linear term and distributed time-varying delay error linear term.

The problem of synchronization in CDNs has been extensively investigated over the past few decades. Synchronization of CDNs is one of the most important dynamical mechanisms for creating order in CDNs. Meanwhile, a number of methods developed for the synchronization of CDNs, including complete synchronization (CS) [24, 34], generalized synchronization (GS) [35], projection synchronization (PS) [36–38], outer and inner synchronization [39], module-phase synchronization [14, 23] etc., have been reported in the literature. Very recently, a new type of synchronization phenomenon in CDNs, called function projection synchronization (FPS), has emerged. FPS of CDNs was proposed in [12, 13, 21, 36], which means that the nodes of CDNs could be synchronized up to an equilibrium point or periodic orbit with a desired scaling function. Zhang et al. [40] presented the FPS in drive-response dynamical networks (DRDNs) with coupled partially linear chaotic systems by assuming that the node dynamics are identical and by using a simple control law. Furthermore, Du et al. [12] investigated the problem of FPS of CDNs with or without external disturbances using error feedback control and adaptive error feedback control. In [13], a hybrid feedback control method was proposed for achieving FPS in CDNs with constant time delay and time-varying coupling delay. Shi et al. [21] proposed a control scheme to study FPS in a complex network with asymmetric coupling via adaptive feedback control and pinning feedback control, respectively. Moreover, a new type of synchronization, called modified function projective synchronization (MFPS), where the drive and response systems could be synchronized up to a desired scale function matrix was introduced in [41, 42]. Wang et al. [43] investigated modified function projective lag synchronization (MFPLS) of dynamical complex networks with disturbance and unknown parameters. The dynamical network is a complex network model containing uncertainty and coupling delay, where delay appears in a complex network but not in the isolate systems. From the above discussions, we can see that the problem of MFPS for CDNs with mixed time-varying delays in the network hybrid coupling and time-varying delays in the dynamical nodes via hybrid adaptive control and hybrid adaptive pinning control has not been fully investigated yet and remains open.

To the best of our knowledge, this is the first time that the MFPS of complex dynamical networks with mixed time-varying and asymmetric coupling delays via new hybrid adaptive control has been studied. We will give a comprehensive study on this topic, and the main contributions of this paper lie in the following aspects. (1) The mixed time-varying delays, with discrete and distributed time-varying delays, which are considered in the dynamical nodes and in hybrid asymmetric coupling simultaneously, are different from the time-delay case in [6, 7, 10, 11, 13, 30]. (2) For the coupling matrix, we do not assume that outer coupling configuration matrix is symmetric or irreducible, which is different from coupling in [8, 16]. (3) For the control method, MFPS is studied via either hybrid adaptive control or hybrid adaptive pinning control with nonlinear and adaptive linear feedback control, which contains error linear term, time-varying delay error linear term and distributed time-varying delay error linear term. The MFPS is different from the control method in [13, 21]. In addition, the pinning nodes can be randomly selected. From the above discussions, this work is one of the first reports of such investigation to further develop the MFPS of complex dynamical networks with mixed time-varying delays in the dynamical nodes and in asymmetric coupling via hybrid adaptive control or hybrid adaptive pinning control. Based on constructing a novel Lyapunov-Krasovskii functional, the adaptive control technique, the parameter update law and the technique of dealing with some integral terms, new sufficient conditions for guaranteeing the existence of the MFPS of delayed CDNs with asymmetric coupling delays are derived. Numerical examples are included to show the effectiveness of the proposed hybrid adaptive control and hybrid adaptive pinning control scheme.

The rest of the paper is organized as follows. Section 2 provides some mathematical preliminaries and the network model. Section 3 presents MFPS of the complex dynamical network with mixed time-varying delay and hybrid asymmetric coupling by hybrid adaptive control and hybrid adaptive pinning control, respectively. In Section 4 some numerical examples illustrate given theoretical results. The paper ends with conclusions in Section 5 and cited references.

2 Network model and mathematic preliminaries

Notations The following notation will be used in this paper. $\mathbb{R}^n$ denotes the n-dimensional space and $\|\cdot\|$ denotes the Euclidean vector norm; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I_N denotes an N-dimensional identity matrix. For the matrix $A \in \mathbb{R}^N \times \mathbb{R}^N$, the ith row and the ith column of A is called the ith row-column pair of A. $A_l \in \mathbb{R}^{(N-l)\times(N-l)}$ is the minor matrix of $A \in \mathbb{R}^{N\times N}$ by removing arbitrary l $(1 \leq l \leq N)$ row-column pairs of A. The symbol $\otimes$ denotes the Kronecker product.

Consider a complex dynamical network consisting of N identical coupled nodes, with each node being an n-dimensional nonlinear dynamical system given by

$$\begin{aligned}
\dot{x}_i(t) = {} & f\left(x_i(t), x_i\big(t-h(t)\big), \int_{t-k(t)}^{t} x_i(s)\,ds\right) \\
& + c_1 \sum_{j=1}^{N} a_{ij} G_1 x_j(t) + c_2 \sum_{j=1}^{N} b_{ij} G_2 x_j\big(t-h(t)\big) \\
& + c_3 \sum_{j=1}^{N} c_{ij} G_3 \int_{t-k(t)}^{t} x_j(s)\,ds + \mathcal{U}_i(t), \quad t \geq 0, i = 1, 2, \ldots, N, \\
x_i(t) = {} & \phi_i(t), \quad t \in [-\tau_{\max}, 0], \tau_{\max} = \max\{h, k\},
\end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of ith node; $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear vector function which describes the local dynamics of nodes and is continuously differentiable and capable of performing abundant dynamical behaviors such as equilibrium points, periodic orbits and chaos; $\mathcal{U}_i(t) \in \mathbb{R}^m$ is the control input of the node i; the constant c_1 and $c_2, c_3 > 0$ denote the non-delayed and delayed coupling strength, respectively; $G_1, G_2, G_3 \in \mathbb{R}^{n\times n}$ are constant inner-coupling matrices, and it is assumed that G_1, G_2, G_3 are positive definite matrices; $A = (a_{ij})_{N\times N}, B = (b_{ij})_{N\times N}, C = (c_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ are the coupling configuration matrices representing the coupling weights and the topological structure for non-delayed configuration and delayed one at time t, respectively, in which a_{ij}, b_{ij} and c_{ji} are defined as follows: if there is a connection between node i and node j $(j \neq i)$, then $a_{ij} > 0$, $b_{ij} > 0$, $c_{ij} > 0$; otherwise, $a_{ij} = 0$, $b_{ij} = 0$, $c_{ij} = 0$ $(j \neq i)$, and the diagonal elements of matrices A, B and C are defined by

$$a_{ii} = -\sum_{j=1, i\neq j}^{N} a_{ij}, \qquad b_{ii} = -\sum_{j=1, i\neq j}^{N} b_{ij}, \qquad c_{ii} = -\sum_{j=1, i\neq j}^{N} c_{ij}, \quad i = 1, 2, \ldots, N. \tag{2}$$

Suppose that $C([-\tau_{\max}, 0], \mathcal{R}^n)$ is the Banach space of continuous functions with the norm

$$\big\|\phi_i(t)\big\| = \sup_{-\tau_{\max} \leq s \leq 0} \big\|\phi_i(t)\big\|.$$

The initial condition function $\phi_i(t)$ denotes a continuous vector-valued initial function of $t \in [-\tau_{\max}, 0]$. Under the initial conditions, we always assume that (1) has a unique solution.

Definition 2.1 Network (1) with time delay is said to achieve modified function projective synchronization (MFPS) if there exists a continuously differentiable scaling function matrix $\alpha(t)$ such that

$$\lim_{t\to\infty}\|e_i(t)\| = \lim_{t\to\infty}\|x_i(t)-\alpha(t)s(t)\|, \quad i=1,2,\dots,N,$$

where $\|\cdot\|$ stands for the Euclidean vector norm and $s(t)\in R^n$ can be either an equilibrium point, or a (quasi-)periodic orbit, or an orbit of a chaotic attractor, which satisfies $\dot{s}(t) = f(s(t), s(t-h(t)), \int_{t-k(t)}^{t} s(\theta)\,d\theta)$.

To investigate the stability of the synchronized states (1), we set the synchronization error $e_i(t)$ in the form $e_i(t) = x_i(t)-\alpha(t)s(t)$, $i=1,\dots,N$, where $\alpha(t)$ is an n-order real diagonal matrix, i.e., $\alpha(t) = \operatorname{diag}(\alpha_1(t),\alpha_2(t),\dots,\alpha_n(t))$ and $\alpha_i(t)$ is a continuously bounded differentiable function, $\alpha_i(t)\neq 0$. Then, substituting it into complex dynamical network (1), we get the following:

$$\begin{aligned}\dot{e}_i(t) &= \dot{x}_i(t)-\dot{\alpha}(t)s(t)-\alpha(t)\dot{s}(t)\\ &= f\left(x_i(t), x_i\bigl(t-h(t)\bigr), \int_{t-k(t)}^{t} x_i(s)\,ds\right) - \alpha(t)f\left(s(t), s\bigl(t-h(t)\bigr), \int_{t-k(t)}^{t} s(\theta)\,d\theta\right)\\ &\quad + c_1\sum_{j=1}^{N} a_{ij}G_1e_j(t) + c_2\sum_{j=1}^{N} b_{ij}G_2e_j\bigl(t-h(t)\bigr) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k(t)}^{t} e_j(s)\,ds\\ &\quad - \dot{\alpha}(t)s(t) + \mathcal{U}_i(t), \quad i=1,\dots,N.\end{aligned} \tag{3}$$

Remark 1 If the scaling function matrix $\alpha(t) = \operatorname{diag}(\alpha_1(t),\alpha_2(t),\dots,\alpha_n(t))$ $(i=1,2,\dots,n)$ is the function of the time t, then the CDNs would realize modified function projective synchronization. If the scaling function matrix $\alpha_1(t)=\alpha_2(t)=\cdots=\alpha_n(t)$, then the synchronization problem will be reduced to the function projective synchronization [12, 13, 21, 36]. If the scaling function matrix $\alpha_1(t)=\alpha_1, \alpha_2(t)=\alpha_2,\dots,\alpha_n(t)=\alpha_n$, then the synchronization problem will be reduced to the projective synchronization [36, 37]. If the scaling function matrix $\alpha_1(t)=1, \alpha_2(t)=1,\dots,\alpha_n(t)=1$, then the synchronization problem will be reduced to the common synchronization [8, 16]. If the scaling function matrix $\alpha_1(t)=0, \alpha_2(t)=0,\dots,\alpha_n(t)=0$, then the synchronization problem turns into a chaos control problem [34]. Therefore, MFPS is a more general form that includes many kinds of synchronization as its special cases.

Remark 2 If $h(t)=k(t)=0$, $c_2=c_3=0$, the network model (1) turns into the complex dynamical network proposed by [12, 21, 28, 33]. If $c_3=0$, for constant delay, that is $h(t)=h$, $k(t)=0$, network (1) is translated into

$$\dot{x}_i(t) = f\bigl(x_i(t)\bigr) + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t) + c_2\sum_{j=1}^{N} b_{ij}G_2x_j(t-h), \quad i=1,2,\dots,N. \tag{4}$$

The complex dynamical network (4) was considered in [7], and if $c_1=0$, network (4) was investigated in [13]. For time-varying delay, that is, $h(t)\neq 0$, $k(t)=0$, the network model

(1) turns into the complex dynamical network presented by Wu et al. [30] as

$$\dot{x}_i(t) = f\big(x_i(t)\big) + c_1 \sum_{j=1}^{N} a_{ij} G_1 x_j(t) + c_2 \sum_{j=1}^{N} b_{ij} G_2 x_j\big(t - h(t)\big), \quad i = 1, 2, \ldots, N.$$

Hence, our network model (1) includes a previous network model, which can be regarded as a special case of the complex dynamical network (1).

Remark 3 If $h(t) = k(t)$, the network model (1) turns into the complex dynamical network proposed by [11], where discrete and distributed time-varying delays appeared in a drive-respond network. If $h(t) \neq k(t)$, the result in [11] cannot be used to decide whether the synchronization of network model (1) can be achieved.

In the rest of this paper, we need the following assumption and some lemmas.

Assumption 1 The time-varying delay functions $h(t)$ and $k(t)$ satisfy conditions that $h(t)$ is differential, $0 \leq h(t) \leq h$, $0 \leq k(t) \leq k$ and $0 \leq \dot{h}(t) \leq \beta < 1$.

Lemma 2.2 (Cauchy inequality [44]) *For any symmetric positive definite matrix $N \in M^{n\times n}$ and $x, y \in \mathbb{R}^n$, we have*

$$\pm 2x^T y \leq x^T N x + y^T N^{-1} y.$$

Lemma 2.3 ([44]) *For any constant symmetric matrix $M \in R^{m\times m}$, $M = M^T > 0$, $\gamma > 0$, the vector function $\omega : [0, \gamma] \to \mathbb{R}^m$ such that the integrations concerned are well defined*

$$\left(\int_0^{\gamma} \omega^T(s)\,ds\right)^T M \left(\int_0^{\gamma} \omega(s)\,ds\right) \leq \gamma \int_0^{\gamma} \omega^T(s) M \omega(s)\,ds.$$

Lemma 2.4 ([45]) *Let $c \in R$ and A, B, C, D be matrices with appropriate dimensions. Then*

(i) $c(A \otimes B) = (cA) \otimes B = A \otimes (cB)$,
(ii) $(A \otimes B)^T = A^T \otimes B^T$,
(iii) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,
(iv) $A \otimes B \otimes C = (A \otimes B) \otimes C = A \otimes (B \otimes C)$.

Lemma 2.5 ([46]) *Assume that A and B are the $N \times N$ Hermitian matrices. Suppose that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_N$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_N$ are eigenvalues of matrices A, B and $A + B$, respectively. Then one has $\alpha_i + \beta_N \leq \gamma_i \leq \alpha_i + \beta_1$, $i = 1, 2, \ldots, N$.*

Lemma 2.6 ([47]) *If $A = (a_{ij})_{(N\times N)}$ is irreducible and satisfies $a_{ij} = a_{ji} \geq 0$, $i \neq j$; $a_{ii} = -\sum_{j=1, i\neq j}^{N}$, $i, j = 1, 2, \ldots, N$, then, for any constant $\xi > 0$, all eigenvalues of the matrix $A - \Xi$ are negative definite, where $\Xi = \operatorname{diag}(\xi, 0, \ldots, 0)$.*

Lemma 2.7 ([48]) *For a symmetric matrix* $M \in R^{N\times N}$ *and a diagonal matrix* $D = \operatorname{diag}(d_1,\dots,d_l,\underbrace{0,\dots,0}_{N-l})$ *with* $d_i > 0$, $i = 1,2,\dots,l$ $(1 \leq l < N)$, *let*

$$M - D = \begin{bmatrix} A - \tilde{D} & B \\ B^T & M_l \end{bmatrix},$$

where M_l *is the minor matrix of* M *by removing its first* l *row-column pairs,* A *and* B *are matrices with appropriate dimensions,* $\tilde{D} = \operatorname{diag}(d_1,\dots,d_l)$. *If* $d_i > \lambda_{\max}(A - BM_l^{-1}B^T)$, $i = 1,\dots,l$, $M - D < 0$ *is equivalent to* $M_l < 0$.

3 MFPS of delayed complex dynamical networks via hybrid adaptive control and hybrid adaptive pinning control

In this section, we give some sufficient conditions for MFPS of complex dynamical networks with discrete and distributed time-varying delays and hybrid asymmetric coupling delays (1) via hybrid adaptive control and hybrid adaptive pinning control.

3.1 MFPS under hybrid adaptive control

We first stabilize the origin of delayed complex dynamical network (1) by means of the hybrid adaptive control $\mathcal{U}_i(t)$ such as

$$\mathcal{U}_i(t) = u_{i1}(t) + u_{i2}(t), \quad i = 1,2,\dots,N, \tag{5}$$

where

$$\begin{aligned} u_{i1}(t) &= \dot{\alpha}(t)s(t), \\ u_{i2}(t) &= -c_1 d_{i1}(t)G_1 e_i(t) - c_2 d_{i2}(t)G_2 e_i\bigl(t-h(t)\bigr) - c_3 d_{i3}(t)G_3 \int_{t-k(t)}^{t} e_i(s)\,ds, \end{aligned}$$

and the updating laws are

$$\begin{aligned} \dot{d}_{i1}(t) &= q_{i1}e_i^T(t)G_1 e_i(t), \\ \dot{d}_{i2}(t) &= q_{i2}e_i^T(t)G_2 e_i\bigl(t-h(t)\bigr), \\ \dot{d}_{i3}(t) &= q_{i3}e_i^T(t)G_3 \left[\int_{t-k(t)}^{t} e_i(s)\,ds\right], \end{aligned} \tag{6}$$

where q_{i1}, q_{i2} and q_{i3} are positive constants and $s(t)$ is a solution of an isolated node, satisfying $\dot{s}(t) = f(s(t), s(t-h(t)), \int_{t-k(t)}^{t} s(\theta)\,d\theta)$. The controller in (5), $u_{i1}(t)$ is the nonlinear control and $u_{i2}(t)$ is the hybrid adaptive linear feedback control. Then, substituting it into complex dynamical network (3), we get the following:

$$\begin{aligned} \dot{e}_i(t) &= \dot{x}_i(t) - \dot{\alpha}(t)s(t) - \alpha(t)\dot{s}(t) \\ &= f\left(x_i(t), x_i\bigl(t-h(t)\bigr), \int_{t-k(t)}^{t} x_i(s)\,ds\right) - \alpha(t)f\left(s(t), s\bigl(t-h(t)\bigr), \int_{t-k(t)}^{t} s(\theta)\,d\theta\right) \\ &\quad + c_1\sum_{j=1}^{N} a_{ij}G_1 e_j(t) + c_2\sum_{j=1}^{N} b_{ij}G_2 e_j\bigl(t-h(t)\bigr) + c_3\sum_{j=1}^{N} c_{ij}G_3 \int_{t-k(t)}^{t} e_j(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&\quad - c_1 d_{i1}(t) G_1 e_i(t) - c_2 d_{i2}(t) G_2 e_i\bigl(t-h(t)\bigr) \\
&\quad - c_3 d_{i3}(t) G_3 \int_{t-k(t)}^{t} e_i(s)\,ds, \quad i = 1,\dots,N, \qquad (7)\\
&\dot{d}_{i1}(t) = q_{i1} e_i^T(t) G_1 e_i(t), \quad i = 1,\dots,N, \\
&\dot{d}_{i2}(t) = q_{i2} e_i^T(t) G_2 e_i\bigl(t-h(t)\bigr), \quad i = 1,\dots,N, \\
&\dot{d}_{i3}(t) = q_{i3} e_i^T(t) G_3 \biggl[\int_{t-k(t)}^{t} e_i(s)\,ds\biggr], \quad i = 1,\dots,N.
\end{aligned}
$$

Let us set

1. $J(t) = f'(s(t), s(t-h(t)), \int_{t-k(t)}^{t} s(\theta)\,d\theta) \in R^{n\times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k(t)}^{t} x(s)\,ds)$ at $s(t)$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k(t)}^{t} x(s)\,ds)$ with respect to $x(t)$,
2. $J_h(t) = f'(s(t), s(t-h(t)), \int_{t-k(t)}^{t} s(\theta)\,d\theta) \in R^{n\times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k(t)}^{t} x(s)\,ds)$ at $s(t-h(t))$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k(t)}^{t} x(s)\,ds)$ with respect to $x(t-h(t))$,
3. $J_k(t) = f'(s(t), s(t-h(t)), \int_{t-k(t)}^{t} s(\theta)\,d\theta) \in R^{n\times n}$ is the Jacobian of $f(x(t), x(t-h(t)), \int_{t-k(t)}^{t} x(s)\,ds)$ at $\int_{t-k(t)}^{t} s(\theta)\,d\theta$ with the derivative of $f(x(t), x(t-h(t)), \int_{t-k(t)}^{t} x(s)\,ds)$ with respect to $\int_{t-k(t)}^{t} x(s)\,ds$,

and

$$
\begin{aligned}
\delta &= \frac{1}{2\lambda_{\min}(I_N \otimes G_2)} \bigl(\varepsilon_1 + c_2\varepsilon_3 + c_2 d_2^* \varepsilon_5\bigr), \\
\tau &= \frac{1}{2\lambda_{\min}(I_N \otimes G_3)} \bigl(\varepsilon_2 + c_3\varepsilon_4 + c_3 d_3^* \varepsilon_6\bigr), \\
\eta &= \frac{1}{\lambda_{\min}(I_N \otimes G_1)} \biggl(\lambda_{\max}\bigl(I_N \otimes J(t)\bigr) + c_1 \lambda_{\max}(A)\lambda_{\max}(G_1) + \frac{c_2}{2(1-\beta)}\lambda_{\max}(I_N \otimes G_2) \\
&\quad + \frac{c_3 k^2}{2}\lambda_{\max}(I_N \otimes G_3) + \frac{1}{2\varepsilon_1}\lambda_{\max}\bigl(I_N \otimes J_h(t) J_h^T(t)\bigr) + \frac{1}{2\varepsilon_2}\lambda_{\max}\bigl(I_N \otimes J_k(t) J_k^T(t)\bigr) \\
&\quad + \frac{c_2}{2\varepsilon_3}\lambda_{\max}\bigl(BB^T\bigr)\lambda_{\max}\bigl(G_2 G_2^T\bigr) + \frac{c_3}{2\varepsilon_4}\lambda_{\max}\bigl(CC^T\bigr)\lambda_{\max}\bigl(G_3 G_3^T\bigr) \\
&\quad + \frac{c_2 d_2^*}{2\varepsilon_5}\lambda_{\max}\bigl(I_N \otimes G_2 G_2^T\bigr) + \frac{c_3 d_3^*}{2\varepsilon_6}\lambda_{\max}\bigl(I_N \otimes G_3 G_3^T\bigr)\biggr), \\
\xi(t) &= \biggl(e^T(t), e^T\bigl(t-h(t)\bigr), \biggl(\int_{t-k}^{t} e(s)\,ds\biggr)^T\biggr)^T.
\end{aligned}
$$

Theorem 3.1 *For some given synchronization scaling function matrix* $\alpha(t)$, *the complex dynamical networks* (1) *with time-varying delay satisfying Assumption* 1 *and the target system can realize modified function projective synchronization by the hybrid adaptive control law as shown in* (5) *if there exist positive constants* ε_i, $i = 1,2,\dots,6$, *and by taking appropriate* d_1^*, d_2^* *and* d_3^* *such that*

$$
d_1^* - \frac{\eta}{c_1} > 0, \qquad (8)
$$

$$
d_2^* - \frac{1}{\varepsilon_5}\biggl(\frac{\varepsilon_1}{c_2} + \varepsilon_3 - \lambda_{\min}(I_N \otimes G_2)\biggr) > 0, \qquad (9)
$$

$$d_3^* - \frac{1}{\varepsilon_6}\left(\frac{\varepsilon_2}{c_3} + \varepsilon_4 - \lambda_{\min}(I_N \otimes G_3)\right) > 0. \tag{10}$$

Then the controlled complex dynamical network (1) *is modified function projective synchronization.*

Proof Since $f(\cdot)$ is continuous differentiable, it is easy to know that the origin of the nonlinear system (7) is an asymptotically stable equilibrium point if it is an asymptotically stable equilibrium point of the following linear time-varying delays systems:

$$\begin{aligned}
\dot{e}_i(t) &= J(t)e_i(t) + J_h(t)e_i\big(t-h(t)\big) + J_k(t)\int_{t-k(t)}^{t} e_i(s)\,ds + c_1\sum_{j=1}^{N} a_{ij}G_1e_j(t) \\
&\quad + c_2\sum_{j=1}^{N} b_{ij}G_2e_j\big(t-h(t)\big) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k(t)}^{t} e_j(s)\,ds - c_1d_{i1}(t)G_1e_i(t) \\
&\quad - c_2d_{i2}(t)G_2e_i\big(t-h(t)\big) - c_3d_{i3}(t)G_3\int_{t-k(t)}^{t} e_i(s)\,ds, \quad i=1,\ldots,N, \qquad (11)\\
\dot{d}_{i1}(t) &= q_{i1}e_i^T(t)G_1e_i(t), \quad i=1,\ldots,N, \\
\dot{d}_{i2}(t) &= q_{i2}e_i^T(t)G_2e_i\big(t-h(t)\big), \quad i=1,\ldots,N, \\
\dot{d}_{i3}(t) &= q_{i3}e_i^T(t)G_3\left[\int_{t-k(t)}^{t} e_i(s)\,ds\right], \quad i=1,\ldots,N.
\end{aligned}$$

Construct the following Lyapunov-Krasovskii functional candidate:

$$\begin{aligned}
V(t) &= \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{N}\frac{c_1}{q_{i1}}\big(d_{i1}(t)-d_1^*\big)^2 \\
&\quad + \frac{c_2}{2(1-\beta)}\sum_{i=1}^{N}\int_{t-h(t)}^{t} e_i^T(s)G_2e_i(s)\,ds + \frac{1}{2}\sum_{i=1}^{N}\frac{c_2}{q_{i2}}\big(d_{i2}(t)-d_2^*\big)^2 \\
&\quad + \frac{c_3k}{2}\sum_{i=1}^{N}\int_{-k}^{0}\int_{t+s}^{t} e_i^T(\theta)G_3e_i(\theta)\,d\theta\,ds + \frac{1}{2}\sum_{i=1}^{N}\frac{c_3}{q_{i3}}\big(d_{i3}(t)-d_3^*\big)^2. \qquad (12)
\end{aligned}$$

Taking the derivative of $V(t)$ along the trajectories of system (7), we have the following:

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N} e_i^T(t)J(t)e_i(t) + \sum_{i=1}^{N} e_i^T(t)J_h(t)e_i\big(t-h(t)\big) + \sum_{i=1}^{N} e_i^T(t)J_k(t)\int_{t-k(t)}^{t} e_i(s)\,ds \\
&\quad + c_1\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)a_{ij}G_1e_j(t) + c_2\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)b_{ij}G_2e_j\big(t-h(t)\big) \\
&\quad - c_2\sum_{i=1}^{N} e_i^T(t)d_{i2}(t)G_2e_i\big(t-h(t)\big) - c_1\sum_{i=1}^{N} e_i^T(t)d_{i1}(t)G_1e_i(t) \\
&\quad + c_3\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)c_{ij}G_3\int_{t-k(t)}^{t} e_j(s)\,ds - c_3\sum_{i=1}^{N} e_i^T(t)d_{i3}(t)G_3\int_{t-k(t)}^{t} e_i(s)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\frac{c_1}{q_{i1}}\big(d_{i1}(t)-d_1^*\big)q_{i1}e_i^T(t)G_1e_i(t)+\frac{c_2}{2(1-\beta)}\sum_{i=1}^{N}e_i^T(t)G_2e_i(t)\\
&-\frac{c_2(1-\dot{h}(t))}{2(1-\beta)}\sum_{i=1}^{N}e_i^T\big(t-h(t)\big)G_2e_i\big(t-h(t)\big)+\frac{c_3k^2}{2}\sum_{i=1}^{N}e_i^T(t)G_3e_i(t)\\
&+\sum_{i=1}^{N}\frac{c_2}{q_{i2}}\big(d_{i2}(t)-d_2^*\big)q_{i2}e_i^T(t)G_1e_i\big(t-h(t)\big)-\frac{c_3k}{2}\int_{t-k}^{t}e_i^T(s)G_3e_i(s)\,ds\\
&+\sum_{i=1}^{N}\frac{c_3}{q_{i3}}\big(d_{i3}(t)-d_3^*\big)q_{i3}e_i^T(t)G_3\int_{t-k(t)}^{t}e_i(s)\,ds\\
\le{}&\sum_{i=1}^{N}e_i^T(t)J(t)e_i(t)+\sum_{i=1}^{N}e_i^T(t)J_h(t)e_i\big(t-h(t)\big)+\sum_{i=1}^{N}e_i^T(t)J_k(t)\int_{t-k(t)}^{t}e_i(s)\,ds\\
&+c_1\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)a_{ij}G_1e_j(t)+c_2\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)b_{ij}G_2e_j\big(t-h(t)\big)\\
&+c_3\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)c_{ij}G_3\int_{t-k(t)}^{t}e_j(s)\,ds+\frac{c_2}{2(1-\beta)}\sum_{i=1}^{N}e_i^T(t)G_2e_i(t)\\
&-\frac{c_2}{2}\sum_{i=1}^{N}e_i^T\big(t-h(t)\big)G_2e_i\big(t-h(t)\big)-c_2d_2^*\sum_{i=1}^{N}e_i^T(t)G_2e_i\big(t-h(t)\big)\\
&-c_1d_1^*\sum_{i=1}^{N}e_i^T(t)G_1e_i(t)+\frac{c_3k^2}{2}\sum_{i=1}^{N}e_i^T(t)G_3e_i(t)-\frac{c_3k}{2}\sum_{i=1}^{N}\int_{t-k}^{t}e_i^T(s)G_3e_i(s)\,ds\\
&-c_3d_3^*\sum_{i=1}^{N}e_i^T(t)G_3\int_{t-k(t)}^{t}e_i(s)\,ds. \qquad (13)
\end{aligned}
$$

Let $e(t)=(e_1(t),\dots,e_N(t))\in R^{n\times N}$, $e(t-h(t))=(e_1(t-h(t)),\dots,e_N(t-h(t)))\in R^{n\times N}$, $\int_{t-k(t)}^{t}e(s)\,ds=\int_{t-k(t)}^{t}(e_1(s),e_2(s),\dots,e_N(s))\,ds\in R^{n\times N}$, we have

$$
\begin{aligned}
\dot{V}(t)\le{}&e^T(t)\big(I_N\otimes J(t)\big)e(t)+e^T(t)\big(I_N\otimes J_h(t)\big)e\big(t-h(t)\big)+c_1e^T(t)(A\otimes G_1)e(t)\\
&+c_2e^T(t)(B\otimes G_2)e\big(t-h(t)\big)-c_1d_1^*e^T(t)(I_N\otimes G_1)e(t)\\
&+c_3e^T(t)(C\otimes G_3)\int_{t-k(t)}^{t}e(s)\,ds-c_2d_2^*e^T(t)(I_N\otimes G_2)e\big(t-h(t)\big)\\
&-c_3d_3^*e^T(t)(I_N\otimes G_3)\int_{t-k(t)}^{t}e(s)\,ds+\frac{c_2}{2(1-\beta)}e^T(t)(I_N\otimes G_2)e(t)\\
&-\frac{c_2}{2}e^T\big(t-h(t)\big)(I_N\otimes G_2)e\big(t-h(t)\big)+\frac{c_3k^2}{2}e^T(t)(I_N\otimes G_3)e(t)\\
&+e^T(t)\big(I_N\otimes J_k(t)\big)\int_{t-k(t)}^{t}e(s)\,ds-\frac{c_3k}{2}\int_{t-k}^{t}e^T(s)(I_N\otimes G_3)e(s)\,ds. \qquad (14)
\end{aligned}
$$

Applying Lemmas 2.2, 2.3 and 2.4 gives

$$
\begin{aligned}
&e^T(t)\big(I_N\otimes J_h(t)\big)e\big(t-h(t)\big)\\
&\quad\le\frac{1}{2\varepsilon_1}e^T(t)\big(I_N\otimes J_h(t)J_h^T(t)\big)e(t)+\frac{\varepsilon_1}{2}e^T\big(t-h(t)\big)e\big(t-h(t)\big), \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&e^T(t)\big(I_N \otimes J_k(t)\big)\int_{t-k(t)}^{t} e(s)\,ds \\
&\quad\le \frac{1}{2\varepsilon_2} e^T(t)\big(I_N \otimes J_k(t)J_k^T(t)\big)e(t) + \frac{\varepsilon_2}{2}\left(\int_{t-k(t)}^{t} e^T(s)\,ds\right)^T\left(\int_{t-k(t)}^{t} e^T(s)\,ds\right), \qquad (16)\\
&c_2 e^T(t)(B \otimes G_2)e\big(t-h(t)\big) \\
&\quad\le \frac{c_2}{2\varepsilon_3} e^T(t)\big(BB^T \otimes G_2G_2^T\big)e(t) + \frac{c_2\varepsilon_3}{2} e^T\big(t-h(t)\big)e\big(t-h(t)\big), \qquad (17)\\
&c_3 e^T(t)(C \otimes G_3)\int_{t-k(t)}^{t} e(s)\,ds \\
&\quad\le \frac{c_3}{2\varepsilon_4} e^T(t)\big(CC^T \otimes G_3G_3^T\big)e(t) + \frac{c_3\varepsilon_4}{2}\left(\int_{t-k(t)}^{t} e^T(s)\,ds\right)^T\left(\int_{t-k(t)}^{t} e^T(s)\,ds\right), \qquad (18)\\
&-c_2 d_2^* e^T(t)(I_N \otimes G_2)e\big(t-h(t)\big) \\
&\quad\le \frac{c_2 d_2^*}{2\varepsilon_5} e^T(t)\big(I_N \otimes G_2G_2^T\big)e(t) + \frac{c_2 d_2^*\varepsilon_5}{2} e^T\big(t-h(t)\big)e\big(t-h(t)\big), \qquad (19)\\
&-c_3 d_3^* e^T(t)(I_N \otimes G_3)\int_{t-k(t)}^{t} e(s)\,ds \\
&\quad\le \frac{c_3 d_3^*}{2\varepsilon_6} e^T(t)\big(I_N \otimes G_3G_3^T\big)e(t) + \frac{c_3 d_3^*\varepsilon_6}{2}\left(\int_{t-k(t)}^{t} e^T(s)\,ds\right)^T\left(\int_{t-k(t)}^{t} e^T(s)\,ds\right). \qquad (20)
\end{aligned}
$$

Hence, according to (14)-(15), we have

$$
\begin{aligned}
\dot{V}(t) \le{}& e^T(t)\bigg(I_N \otimes J(t) + c_1(A \otimes G_1) - c_1 d_1^*(I_N \otimes G_1) + \frac{c_2}{2(1-\beta)}(I_N \otimes G_2) \\
&+ \frac{c_3 k^2}{2}(I_N \otimes G_3) + \frac{1}{2\varepsilon_1}\big(I_N \otimes J_h(t)J_h^T(t)\big) + \frac{1}{2\varepsilon_2}\big(I_N \otimes J_k(t)J_k^T(t)\big) \\
&+ \frac{c_2}{2\varepsilon_3}\big(BB^T \otimes G_2G_2^T\big) + \frac{c_3}{2\varepsilon_4}\big(CC^T \otimes G_3G_3^T\big) + \frac{c_2 d_2^*}{2\varepsilon_5}\big(I_N \otimes G_2G_2^T\big) \\
&+ \frac{c_2 d_3^*}{2\varepsilon_6}\big(I_N \otimes G_3G_3^T\big)\bigg)e(t) - e^T\big(t-h(t)\big)\bigg(\frac{c_2}{2}(I_N \otimes G_2) \\
&+ \delta(I_N \otimes G_2)\bigg)e\big(t-h(t)\big) - \left(\int_{t-k}^{t} e(s)\,ds\right)^T\bigg(\frac{c_3}{2}(I_N \otimes G_3) \\
&- \tau(I_N \otimes G_3)\bigg)\left(\int_{t-k}^{t} e(s)\,ds\right) \\
\le{}& \big(\eta - c_1 d_1^*\big)e^T(t)(I_N \otimes G_1)e(t) - e^T\big(t-h(t)\big)\left(\frac{c_2}{2} - \delta\right)(I_N \otimes G_2)e\big(t-h(t)\big) \\
&- \left(\int_{t-k}^{t} e(s)\,ds\right)^T\left(\frac{c_3}{2} - \tau\right)(I_N \otimes G_3)\left(\int_{t-k}^{t} e(s)\,ds\right). \qquad (21)
\end{aligned}
$$

It is obvious that there exist sufficiently large positive constants d_1^*, d_2^* and d_3^* such that

$$
d_1^* - \frac{\eta}{c_1} > 0, \qquad (22)
$$

$$
d_2^* - \frac{1}{\varepsilon_5}\left(\frac{\varepsilon_1}{c_2} + \varepsilon_3 - \lambda_{\min}(I_N \otimes G_2)\right) > 0, \qquad (23)
$$

$$d_3^* - \frac{1}{\varepsilon_6}\left(\frac{\varepsilon_2}{c_3} + \varepsilon_4 - \lambda_{\min}(I_N \otimes G_3)\right) > 0. \tag{24}$$

We can choose d_1^*, d_2^* and d_3^* satisfying (22), (23) and (24), respectively. Since G_1, G_2 and G_3 are positive definite matrices, we know that $\dot{V}(t) \leq 0$ and $\dot{V}(t) = 0$ if and only if $\xi(t) = 0$. Hence, the set $\mathcal{W} = \{\xi(t) = 0, d_{1i} = d_1^*, d_{2i} = d_2^*, d_{3i} = d_3^*\}$ is the invariant set contained in $\mathcal{W}_1 = \{\xi(t) = 0 : \dot{V}(t) = 0\}$ for system (7). According to LaSalle's invariance principle [49] and Lyapunov stability theory, for any initial condition, every solution of system (7) approaches $\mathcal{W}$ as $t \longrightarrow \infty$, which indicates that $\|e_i(t)\| \longrightarrow 0$, $i = 1, 2, \ldots, N$. This means that the function projective synchronization between the delayed complex dynamical network (1) and the reference node $s(t)$ is achieved under hybrid adaptive control (5). The proof is completed. □

Remark 4 If $f(x_i(t), x_i(t-h(t)), \int_{t-k(t)}^{t} x_i(s)\,ds) = f(x(t))$, $h(t) = h$, $k(t) = 0$ and c_3=0, then system (1) reduces to the following network (4) presented in [7]. According to Theorem 3.1, we obtain the following corollary for the synchronization of network (4).

Corollary 3.2 *For some given synchronization scaling function matrix $\alpha(t)$, the complex dynamical network* (1) *and the target system can realize modified function projective synchronization by the hybrid adaptive control law as shown in* (5) *if there exist positive constants ε_i, $i = 3, 5$, and by taking appropriate d_1^* and d_2^* such that*

$$d_1^* - \frac{\omega}{c_1} > 0, \tag{25}$$

$$d_2^* + \frac{\lambda_{\min}(I_N \otimes G_2)}{\varepsilon_5} > 0, \tag{26}$$

where

$$\begin{aligned}\omega = {}& \frac{1}{\lambda_{\min}(I_N \otimes G_1)}\Big(\lambda_{\max}\big(I_N \otimes J(t)\big) + c_1\lambda_{\max}(A)\lambda_{\max}(G_1) \\ & + \frac{c_2}{2}\lambda_{\max}(I_N \otimes G_2)\frac{c_2}{2\varepsilon_3}\lambda_{\max}\big(BB^T\big)\lambda_{\max}\big(G_2G_2^T\big) \\ & + \frac{c_2 d_2^*}{2\varepsilon_5}\lambda_{\max}\big(I_N \otimes G_2G_2^T\big)\Big).\end{aligned}$$

Then the controlled complex dynamical networks (4) *is modified function projective synchronization.*

Proof The proof is similar to that of Theorem 3.1. Indeed, by setting $J_h(t) = 0$, $J_k(t) = 0$, $k = 0$, $\beta = 0$ and $c_3 = 0$, one may easily derive the result, and hence the proof is omitted. □

Remark 5 The authors in [13, 21] presented the synchronization of complex dynamical networks via hybrid control, which is dependent on a nonlinear function $f(\cdot)$. But in this paper, the controller (5) is independent of the nonlinear function $f(\cdot)$. Therefore, for removing the nonlinear function $f(\cdot)$, we employ some new techniques that make the implementation of controller easier with practice. This theorem can be applied to a great many complex dynamical networks in the real world.

3.2 MFPS under hybrid adaptive pinning control

Without loss of generality, assume that the first l nodes $1 \leq i \leq l$ are selected and pinned with the adaptive pinning control, which is described by

$$\mathcal{U}_i(t) = u_{i1}(t) + \bar{u}_{i2}(t), \quad i = 1, 2, \dots, N, \tag{27}$$

where

$$\begin{aligned} u_{i1}(t) &= \dot{\alpha}(t)s(t), \quad i = 1, 2, \dots, N, \\ \bar{u}_{i2}(t) &= -c_1 d_{i1}(t) G_1 e_i(t) - c_2 d_{i2}(t) G_2 e_i\bigl(t - h(t)\bigr) \\ &\quad - c_3 d_{i3}(t) G_3 \int_{t-k(t)}^{t} e_i(s)\,ds, \quad i = 1, 2, \dots, l, \\ \bar{u}_{i2}(t) &= 0, \quad i = l+1, l+2, \dots, N, \end{aligned}$$

and the updating laws are defined in (6). The controllers $u_{i1}(t)$ and $\bar{u}_{i2}(t)$ are different types of controllers, i.e., $u_{i1}(t)$ is the nonlinear control and $u_{i2}(t)$ is the adaptive pinning control. Let us set

$$\begin{aligned} \Sigma_1 &= \frac{1}{\lambda_{\min}(I_N \otimes G_1)} \biggl(\lambda_{\max}\bigl(I_N \otimes J(t)\bigr) + \frac{c_2}{2(1-\beta)} \lambda_{\max}(I_N \otimes G_2) + \frac{c_3 k^2}{2} \lambda_{\max}(I_N \otimes G_3) \\ &\quad + \frac{1}{2\varepsilon_1} \lambda_{\max}\bigl(I_N \otimes J_h(t) J_h^T(t)\bigr) + \frac{1}{2\varepsilon_2} \lambda_{\max}\bigl(I_N \otimes J_k(t) J_k^T(t)\bigr) + \frac{c_3 \bar{d}_3^*}{2\varepsilon_6} \lambda_{\max}\bigl(I_N \otimes G_3 G_3^T\bigr) \\ &\quad + \frac{c_3}{2\varepsilon_4} \lambda_{\max}\bigl(CC^T\bigr) \lambda_{\max}\bigl(G_3 G_3^T\bigr) + \frac{c_2 \bar{d}_2^*}{2\varepsilon_5} \lambda_{\max}\bigl(I_N \otimes G_2 G_2^T\bigr) \\ &\quad + \frac{c_2}{2\varepsilon_3} \lambda_{\max}\bigl(BB^T\bigr) \lambda_{\max}\bigl(G_2 G_2^T\bigr) \biggr), \\ \Sigma_2 &= \frac{1}{2\lambda_{\min}(I_N \otimes G_2)} \bigl(\varepsilon_1 + c_2 \varepsilon_3 + c_2 \bar{d}_2^* \varepsilon_5\bigr), \\ \Sigma_3 &= \frac{1}{2\lambda_{\min}(I_N \otimes G_3)} \bigl(\varepsilon_2 + c_3 \varepsilon_4 + c_3 \bar{d}_3^* \varepsilon_6\bigr). \end{aligned}$$

Theorem 3.3 *For some given synchronization scaling function* $\alpha(t)$, *the complex dynamical network* (1) *with time-varying delay satisfying Assumption* 1 *and the target system can realize function projective synchronization by the adaptive pinning control law as shown in* (27) *if there exist positive constants* ε_i, $i = 1, 2, \dots, 6$, *and by taking appropriate* $\bar{d}_{1i}^*$, $i = 1, 2, \dots, l$, $\bar{d}_2^*$ *and* $\bar{d}_3^*$ *such that*

$$\lambda_{\max}\left(\frac{A_l + A_l^T}{2}\right) < -\frac{\Sigma_1}{c_1}, \tag{28}$$

$$\bar{d}_2^* - \frac{1}{\varepsilon_5}\left(\frac{\varepsilon_1}{c_2} + \varepsilon_3 - \lambda_{\min}(I_N \otimes G_2)\right) > 0, \tag{29}$$

$$\bar{d}_3^* - \frac{1}{\varepsilon_6}\left(\frac{\varepsilon_2}{c_3} + \varepsilon_4 - \lambda_{\min}(I_N \otimes G_3)\right) > 0. \tag{30}$$

Then the controlled complex dynamical network is modified function projective synchronization.

Proof Similarly to the proof of Theorem 3.1, we can get

$$\begin{aligned}
\dot{e}_i(t) &= J(t)e_i(t) + J_h(t)e_i\bigl(t-h(t)\bigr) + J_k(t)\int_{t-k(t)}^{t} e_i(s)\,ds + c_1\sum_{j=1}^{N} a_{ij}G_1e_j(t) \\
&\quad + c_2\sum_{j=1}^{N} b_{ij}G_2e_j\bigl(t-h(t)\bigr) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k(t)}^{t} e_j(s)\,ds - d_{i1}(t)G_1e_i(t) \\
&\quad - d_{i2}(t)G_2e_i\bigl(t-h(t)\bigr) - d_{i3}(t)G_3\int_{t-k(t)}^{t} e_i(s)\,ds, \quad i=1,\dots,l, \\
\dot{d}_{i1}(t) &= q_{i1}e_i^T(t)G_1e_i(t), \quad i=1,\dots,l, \\
\dot{d}_{i2}(t) &= q_{i2}e_i^T(t)G_2e_i\bigl(t-h(t)\bigr), \quad i=1,\dots,l, \\
\dot{d}_{i3}(t) &= q_{i3}e_i^T(t)G_3\biggl[\int_{t-k(t)}^{t} e_i(s)\,ds\biggr], \quad i=1,\dots,l, \\
\dot{e}_i(t) &= J(t)e_i(t) + J_h(t)e_i\bigl(t-h(t)\bigr) + J_{k_1}(t)\int_{t-k(t)}^{t} e_i(s)\,ds \\
&\quad + c_1\sum_{j=1}^{N} a_{ij}G_1e_j(t) + c_2\sum_{j=1}^{N} b_{ij}G_2e_j\bigl(t-h(t)\bigr) \\
&\quad + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k(t)}^{t} e_j(s)\,ds, \quad i=l+1,l+2,\dots,N.
\end{aligned} \tag{31}$$

Choose the Lyapunov-Krasovskii functional candidate as follows:

$$\begin{aligned}
V(t) &= \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{1}{2}\sum_{i=1}^{l}\frac{c_1}{q_{i1}}\bigl(d_{i1}(t)-d_{1i}^*\bigr)^2 \\
&\quad + \frac{c_2}{2(1-\beta)}\sum_{i=1}^{N}\int_{t-h(t)}^{t} e_i^T(s)G_2e_i(s)\,ds + \frac{1}{2}\sum_{i=1}^{l}\frac{c_2}{q_{i2}}\bigl(d_{i2}(t)-d_{2i}^*\bigr)^2 \\
&\quad + \frac{c_3k}{2}\sum_{i=1}^{N}\int_{-k}^{0}\int_{t+s}^{t} e_i^T(\theta)G_3e_i(\theta)\,d\theta\,ds + \frac{1}{2}\sum_{i=1}^{l}\frac{c_3}{q_{i3}}\bigl(d_{i3}(t)-d_{3i}^*\bigr)^2.
\end{aligned} \tag{32}$$

Taking the derivative of $V(t)$ along the trajectories of system (31), we have the following:

$$\begin{aligned}
\dot{V}(t) &\le \sum_{i=1}^{N} e_i^T(t)J(t)e_i(t) + \sum_{i=1}^{N} e_i^T(t)J_h(t)e_i\bigl(t-h(t)\bigr) + \sum_{i=1}^{N} e_i^T(t)J_k(t)\int_{t-k(t)}^{t} e_i(s)\,ds \\
&\quad + c_1\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)a_{ij}G_1e_j(t) + c_2\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)b_{ij}G_2e_j\bigl(t-h(t)\bigr) \\
&\quad - \frac{c_2}{2}\sum_{i=1}^{N} e_i^T\bigl(t-h(t)\bigr)G_2e_i\bigl(t-h(t)\bigr) + \frac{c_2}{2(1-\beta)}\sum_{i=1}^{N} e_i^T(t)G_2e_i(t) \\
&\quad + c_3\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)c_{ij}G_3\int_{t-k(t)}^{t} e_j(s)\,ds - c_2\sum_{i=1}^{l} e_i^T(t)d_{2i}^*G_2e_i\bigl(t-h(t)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&- c_1 \sum_{i=1}^{l} e_i^T(t) d_{1i}^* G_1 e_i(t) + \frac{c_3 k^2}{2} \sum_{i=1}^{N} e_i^T(t) G_3 e_i(t) \\
&- \frac{c_3 k}{2} \sum_{i=1}^{N} \int_{t-k}^{t} e_i^T(s) G_3 e_i(s)\, ds \\
&- c_3 \sum_{i=1}^{l} e_i^T(t) d_{3i}^* G_3 \int_{t-k(t)}^{t} e_i(s)\, ds. \qquad (33)
\end{aligned}
$$

Let $\bar{I}_N = \operatorname{diag}(\underbrace{1,\ldots,1}_{l}, \underbrace{0,\ldots,0}_{N-l})$, $D_1^* = \operatorname{diag}(d_{11}^*, d_{12}^*, \ldots, d_{1l}^*, 0, \ldots, 0) \in R^{N\times N}$, $D_2^* = \operatorname{diag}(d_{21}^*, d_{22}^*, \ldots, d_{2l}^*, 0, \ldots, 0) \in R^{N\times N}$ and $D_3^* = \operatorname{diag}(d_{31}^*, d_{32}^*, \ldots, d_{3l}^*, 0, \ldots, 0) \in R^{N\times N}$. Using a method similar to that of Theorem 3.1, we can get

$$
\begin{aligned}
\dot{V}(t) \le{}& e^T(t)\bigg(I_N \otimes J(t) + c_1(A \otimes G_1) - c_1\big(D_1^* \otimes G_1\big) + \frac{c_2}{2(1-\beta)}(I_N \otimes G_2) \\
&+ \frac{c_3 k^2}{2}(I_N \otimes G_3) + \frac{1}{2\varepsilon_1}\big(I_N \otimes J_h(t) J_h^T(t)\big) + \frac{1}{2\varepsilon_2}\big(I_N \otimes J_k(t) J_k^T(t)\big) \\
&+ \frac{c_2}{2\varepsilon_3}\big(BB^T \otimes G_2 G_2^T\big) + \frac{c_3}{2\varepsilon_4}\big(CC^T \otimes G_3 G_3^T\big) \\
&+ \frac{c_2 \bar{d}_2^*}{2\varepsilon_5}\big(I_N \otimes G_2 G_2^T\big) + \frac{c_2 \bar{d}_3^*}{2\varepsilon_6}\big(I_N \otimes G_3 G_3^T\big)\bigg) e(t) \\
&- e^T\big(t-h(t)\big)\bigg(\frac{c_2}{2}(I_N \otimes G_2) - \Sigma_2(I_N \otimes G_2)\bigg) e\big(t-h(t)\big) \\
&- \bigg(\int_{t-k}^{t} e(s)\, ds\bigg)^T \bigg(\frac{c_3}{2}(I_N \otimes G_3) - \Sigma_3(I_N \otimes G_3)\bigg)\bigg(\int_{t-k}^{t} e(s)\, ds\bigg) \\
\le{}& e^T(t)\big[\big(\Sigma_1 I_N + c_1 A - c_1 D_1^*\big) \otimes G_1\big] e(t) \\
&- e^T\big(t-h(t)\big)\bigg(\frac{c_2}{2} - \Sigma_2\bigg)(I_N \otimes G_2) e\big(t-h(t)\big) \\
&- \bigg(\int_{t-k}^{t} e(s)\, ds\bigg)^T \bigg(\frac{c_3}{2} - \Sigma_3\bigg)(I_N \otimes G_3)\bigg(\int_{t-k}^{t} e(s)\, ds\bigg) \\
={}& e^T(t)\big[\big(\Pi - c_1 D_1^*\big) \otimes G_1\big] e(t) \\
&- e^T\big(t-h(t)\big)\bigg(\frac{c_2}{2} - \Sigma_2\bigg)(I_N \otimes G_2) e\big(t-h(t)\big) \\
&- \bigg(\int_{t-k}^{t} e(s)\, ds\bigg)^T \bigg(\frac{c_3}{2} - \Sigma_3\bigg)(I_N \otimes G_3)\bigg(\int_{t-k}^{t} e(s)\, ds\bigg), \qquad (34)
\end{aligned}
$$

where $\bar{d}_2^* = \max_{1\le i\le l}\{d_{2i}^*\}$, $\bar{d}_3^* = \max_{1\le i\le l}\{d_{3i}^*\}$ and $\Pi = \Sigma_1 I_N + c_1 A$. Note that the matrix Π is symmetric. Let

$$
\Pi - c_1 D_1^* = \begin{bmatrix} \tilde{\Pi}_1 - c_1 \tilde{D}_{1l}^* & \tilde{\Pi}_3 \\ \tilde{\Pi}_3^T & \tilde{\Pi}_2 \end{bmatrix},
$$

where $\tilde{\Pi}_2$ is the minor matrix of Π by removing its first l ($1 \le l < N$) row column pairs, $\tilde{\Pi}_1$ and $\tilde{\Pi}_3$ are matrices with appropriate dimensions, $\tilde{D}_{1l}^* = \operatorname{diag}(d_{11}^*, d_{12}^*, \ldots, d_{1l}^*)$.

If $\lambda_{\max}(\frac{A_l+A_l^T}{2}) < -\frac{\Sigma_1}{c_1}$ and Lemma 2.5, we have $\tilde{\Pi}_2 < 0$. Therefore, one can choose suitable positive constants $d_{1i}^* > 0$, $i = 1,2,\ldots,l$, such that $d_{1i}^* > \lambda_{\max}(\tilde{\Pi}_1 - \tilde{\Pi}_3\tilde{\Pi}_2^{-1}\tilde{\Pi}_3^T)$. It follows from Lemma 2.7 and $\tilde{\Pi}_2 < 0$ that $\Pi - c_1 D_1^* < 0$. Then, by $G_1 > 0$ and (34), we can conclude that

$$\begin{aligned} \dot{V}(t) \leq{}& -e^T\bigl(t-h(t)\bigr)\left(\frac{c_2}{2} - \Sigma_2\right)(I_N \otimes G_2)e\bigl(t-h(t)\bigr) \\ & - \left(\int_{t-k}^{t} e(s)\,ds\right)^T \left(\frac{c_3}{2} - \Sigma_3\right)(I_N \otimes G_3)\left(\int_{t-k}^{t} e(s)\,ds\right). \end{aligned} \tag{35}$$

We only need to choose the suitable positive constants $\bar{d}_2^*$ and $\bar{d}_3^*$ such that

$$\bar{d}_2^* - \frac{1}{\varepsilon_5}\left(\frac{\varepsilon_1}{c_2} + \varepsilon_3 - \lambda_{\min}(I_N \otimes G_2)\right) > 0, \tag{36}$$

$$\bar{d}_3^* - \frac{1}{\varepsilon_6}\left(\frac{\varepsilon_2}{kc_3} + \varepsilon_4 - \lambda_{\min}(I_N \otimes G_3)\right) > 0. \tag{37}$$

The remaining proof is similar to that of Theorem 3.1 and is omitted. □

Remark 6 The nodes pinned for directed networks are chosen as follows.

Step I: Choose some appropriate parameters ε_i, $i = 1,2,\ldots,6$, and by taking appropriate $\bar{d}_{1i}^*$, $i = 1,2,\ldots,l$, $\bar{d}_2^*$ and $\bar{d}_3^*$ such that the conditions in Theorem 3.3 are feasible.

Step II: The l pinned nodes are sorted according to the pinned-node selection scheme studied in [10] for the pinning controlled error dynamical network (31); so, the nodes to be pinned are chosen in the particular order. Let $l = 1$, if the first inequalities of Theorem 3.3 are satisfied, then the least number is 1; otherwise, go to next step.

Step III: If condition (28) is not satisfied, increase l ($l = l + 1$) gradually and add more network nodes to the pined node based on the order in step II particularly until condition (28) holds.

For undirected networks, e.g., the small-world network [3], the scale-free network [5] and the Watts-Strogatz network [4], we can randomly choose a set of pinned nodes to satisfy condition (28) by increasing the number of pinned nodes l.

Remark 7 In Theorem 3.3, we investigated the MFPS of complex dynamical networks via hybrid control, where the control $u_{i1}(t)$ is a nonlinear control (not pinning control) to apply for every node. By using the principle of function projective synchronization, this control needs to be applied for every node. And $\bar{u}_{i2}(t)$ is an adaptive pinning control to apply for the first l nodes $1 \leq i \leq l$ by using the principle of pinning control approach. This technique for applying both of controls has been considered in [21].

Remark 8 If we investigate the dynamical nodes without delays and ignore the adaptive linear feedback control, which contains time-varying delay error linear term and distributed time-varying delay error linear term, we can see the general model of the complex dynamical networks in [13, 21]. By comparison, this paper contains discrete and distributed time-varying delays in dynamical nodes and adaptive linear feedback control simultaneously. Furthermore, it also develops the pre-existing research.

Remark 9 However, there is room for improvement. First, the time-varying delays are still necessarily differentiable. So, we should remove them, which means that fast time-varying delays are allowed. Second, even though the hybrid pinning adaptive control can reduce the number of controllers, it cannot reduce the control cost. Hence, combining the intermittent control technique and the pinning control strategy should be considered together.

4 Numerical examples

In this section, we present three examples to illustrate the effectiveness and the reduced conservatism of our result.

Example 4.1 We consider the perturbed Chua's circuit system with mixed time-varying delays used as an uncoupled node in network (1) to show the effectiveness of the proposed control scheme. The perturbed Chua's circuit system with mixed time-varying delays (drive system) is given by [16]

$$\begin{aligned}
\dot{x}_1(t) &= p\left(x_2\big(t-h(t)\big) - \frac{1}{7}\big(2x_1^3(t) - x_1(t)\big)\right), \\
\dot{x}_2(t) &= x_1(t) - sx_2(t) + x_3\big(t-h(t)\big), \\
\dot{x}_3(t) &= qx_2(t) + r\int_{t-k(t)}^{t} x_1^2(s)\,ds,
\end{aligned} \tag{38}$$

and we take system (38) as identical nodes of the network (response networks), which is given by

$$\begin{aligned}
\begin{pmatrix} \dot{x}_{i1}(t) \\ \dot{x}_{i2}(t) \\ \dot{x}_{i3}(t) \end{pmatrix} &= \begin{pmatrix} p(x_{i2}(t-h(t)) - \frac{1}{7}(2x_{i1}^3(t) - x_{i1}(t))) \\ x_{i1}(t) - sx_{i2}(t) + x_{i3}(t-h(t)) \\ qx_{i2}(t) + r\int_{t-k(t)}^{t} x_{i1}^2(s)\,ds \end{pmatrix} + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t) \\
&\quad + c_2\sum_{j=1}^{N} b_{ij}G_2x_j\big(t-h(t)\big) + c_3\sum_{j=1}^{N} c_{ij}G_3\int_{t-k(t)}^{t} x_j(s)\,ds \\
&\quad + \mathcal{U}_i(t), \quad i = 1,2,\ldots,N,
\end{aligned} \tag{39}$$

where p, q, r and s are real positive constants. It is well known that system (38) exhibits chaotic behavior with the parameters p, q, r and s being chosen as $p = 7$, $q = -\frac{100}{7}$, $r = 0.07$ and $s = 1.5$. The initial condition function $\phi(t) = [0.65\cos t, 0.3\cos t, -0.2\cos t]^T$, the time-varying delay functions $h(t) = 0.1 + 0.1\sin^2 t$ and $k(t) = 0.1\cos^2 t$ are shown in Figure 1. It is stable at the equilibrium point $s(t) = 0$, $s(t-h(t)) = 0$, $\int_{t-k(t)}^{t} s(\theta)\,d\theta = 0$ and Jacobian matrices are

$$J(t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1.5 & 0 \\ 0 & -\frac{100}{7} & 0 \end{bmatrix}, \qquad J_h(t) = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad J_k(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

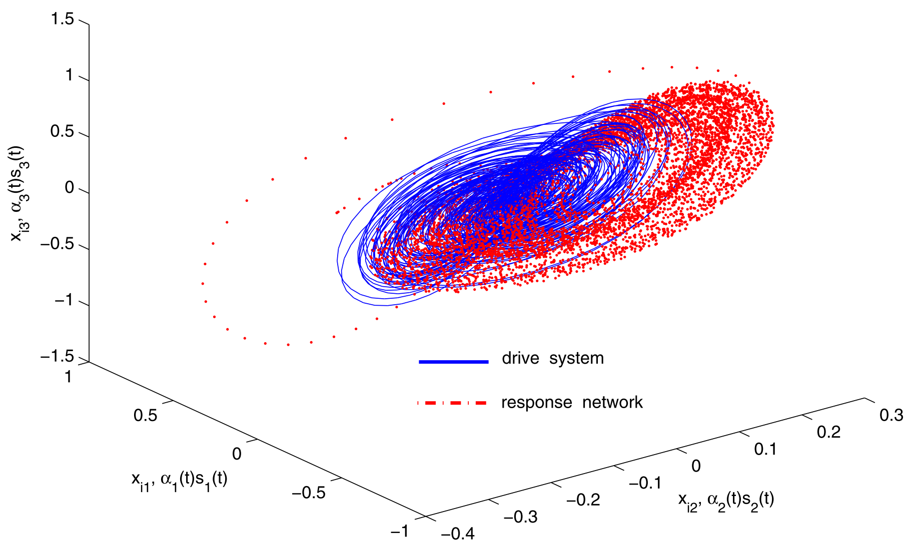

Figure 1 Chaotic behavior of drive system (38) and response network (39) with the time-varying scaling function matrix $\alpha(t) = \mathrm{diag}(0.6\sin(\frac{2\pi}{15}), 0.7\sin(\frac{2\pi}{15}), 0.75\sin(\frac{2\pi}{15}))$.

The parameters are selected as follows: the time-varying scaling function matrix $\alpha(t) = \mathrm{diag}(0.6\sin(\frac{2\pi}{15}), 0.7\sin(\frac{2\pi}{15}), 0.75\sin(\frac{2\pi}{15}))$, the coupling strength $c_1 = 0.4$, $c_2 = 0.3$, $c_3 = 0.5$, the inner-coupling matrices are

$$G_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad G_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the coupling configuration matrices are given respectively as follows:

$$A = \begin{bmatrix} -5 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & -3 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -4 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -3 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & -4 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} -4 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & -3 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -4 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & -3 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -3 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix},$$

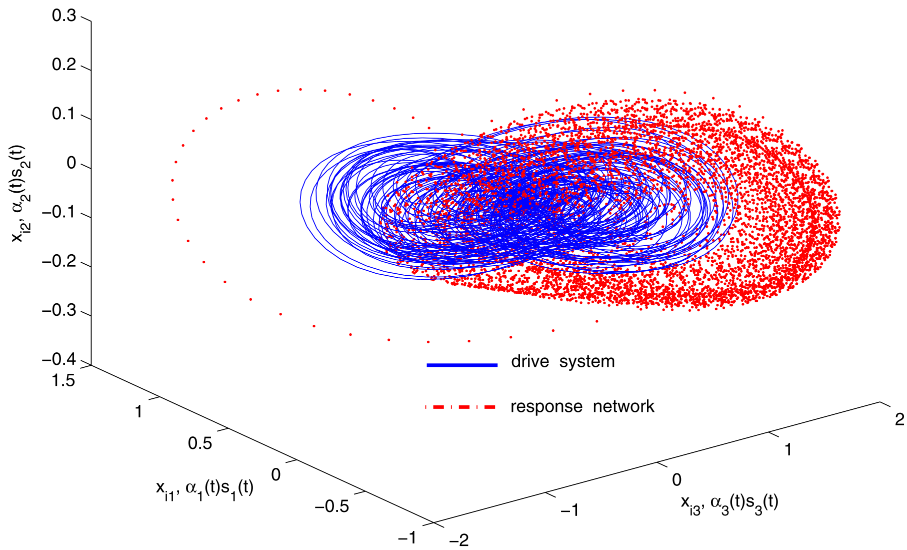

Figure 2 Chaotic behavior of drive system (38) and response network (39) with the time-varying scaling function matrix $\alpha(t) = \mathrm{diag}(0.6\sin(\frac{2\pi}{15}), 0.7\sin(\frac{2\pi}{15}), 0.75\sin(\frac{2\pi}{15}))$.

$$C = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Solution: From conditions (8)-(10) of Theorem 3.1 and with positive constants $\varepsilon_1 = 9.86$, $\varepsilon_2 = 8.75$, $\varepsilon_3 = 12.9$, $\varepsilon_4 = 10.2$, $\varepsilon_5 = 13.10$, $\varepsilon_6 = 10.70$, one can check that the last three conditions in Theorem 3.1 are satisfied. From the conditions of Theorem 3.1, we obtain $d_1^* > 5.7340$, $d_2^* > 4.2786$ and $d_3^* > 2.9042$.

The numerical simulations are carried out using the explicit Runge-Kutta-like method (dde45), interpolation and extrapolation by spline of the third order. Figures 1 and 2 show the chaotic behavior of drive system (38) and response network (39). Figure 3 shows the modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39), where $e_{ij}(t) = x_{ij}(t) - \alpha_j(t)s_j(t)$ for $i = 1, \dots, 8$, $j = 1, 2, 3$, without hybrid adaptive control (5). Figure 4 shows the modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39) with hybrid adaptive control (5). Figure 5 gives the evolution of adaptive feedback gain $d_{i1}(t)$, $d_{i2}(t)$ and $d_{i3}(t)$ ($i = 1, 2, \dots, 8$). We assume $l = 5$, i.e., the number of nodes to be controlled is five. Figure 6 shows the modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39) with hybrid adaptive pinning control (27). Figure 7 gives the evolution of adaptive pinning feedback gain $d_{i1}(t)$, $d_{i2}(t)$ and $d_{i3}(t)$ ($i = 1, 2, \dots, 5$).

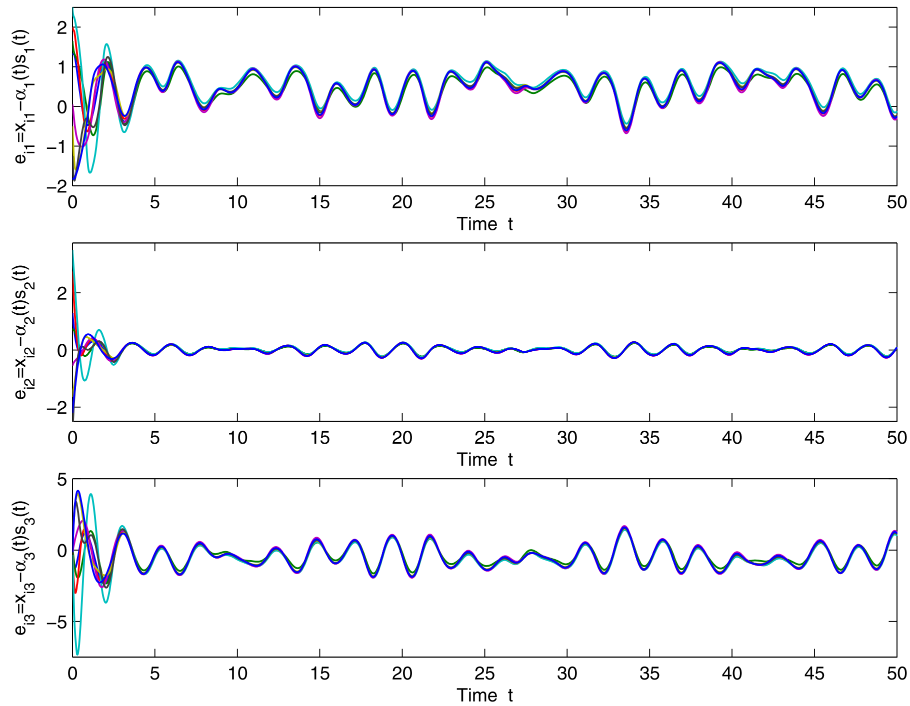

Figure 3 The modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39), $i = 1, 2, \ldots, 8$, without hybrid adaptive control (5).

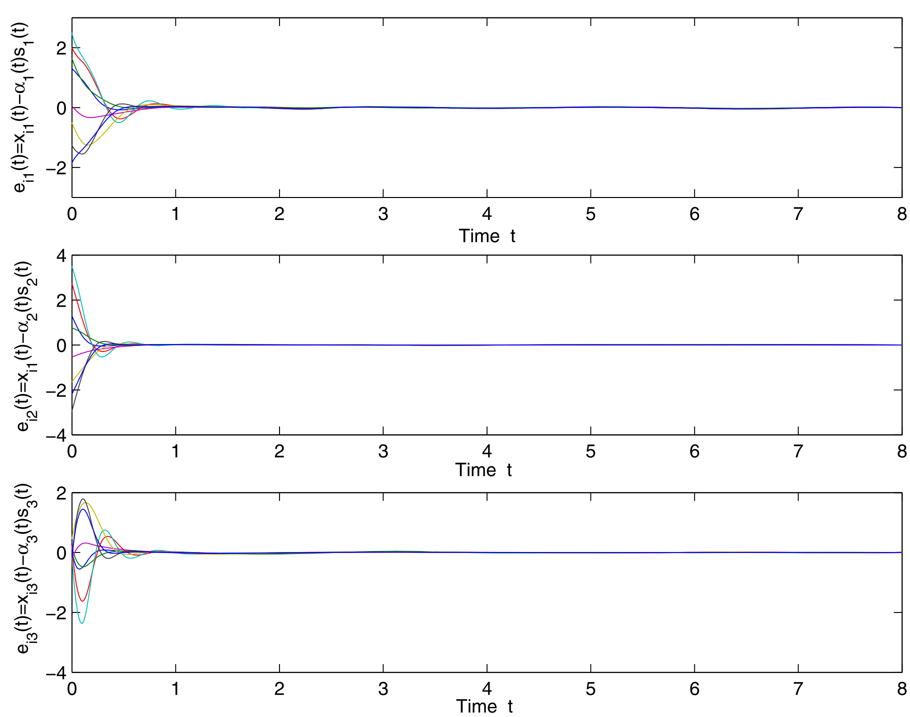

Figure 4 The modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39), $i = 1, 2, \ldots, 8$, with hybrid adaptive control (5).

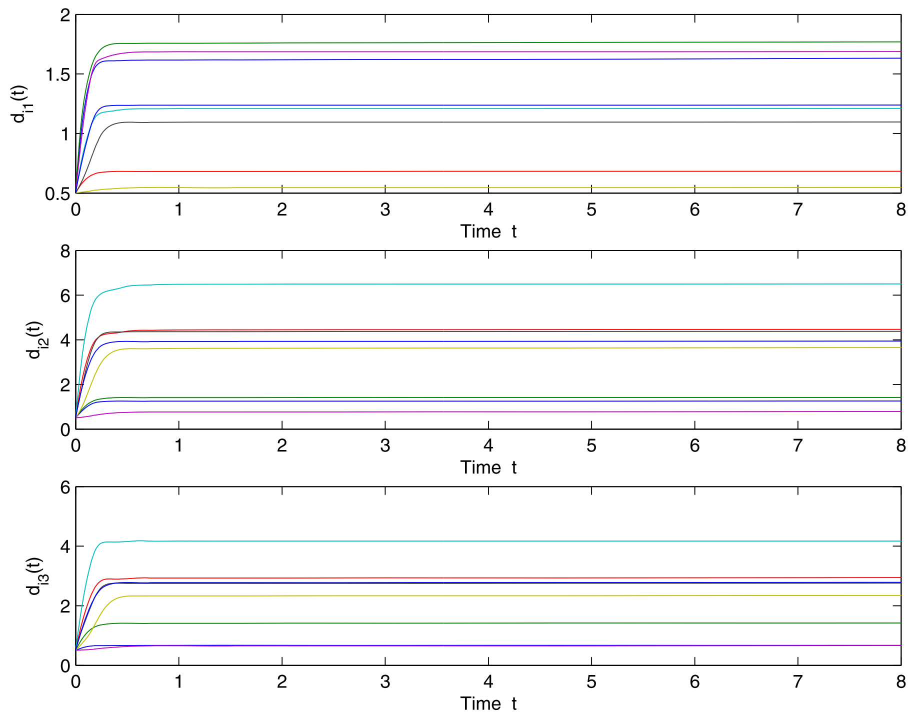

Figure 5 The evolution of adaptive feedback gain $d_{i1}(t)$, $d_{i2}(t)$ and $d_{i3}(t)$ ($i = 1, 2, \ldots, 8$).

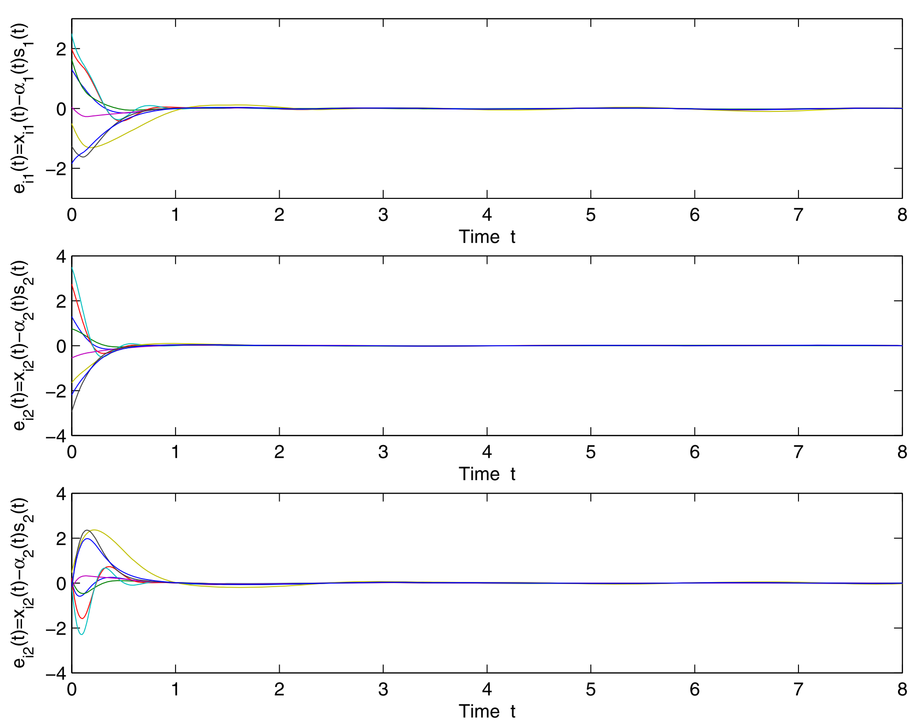

Figure 6 The modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39), $i = 1, 2, \ldots, 5$, with hybrid adaptive pinning control (27).

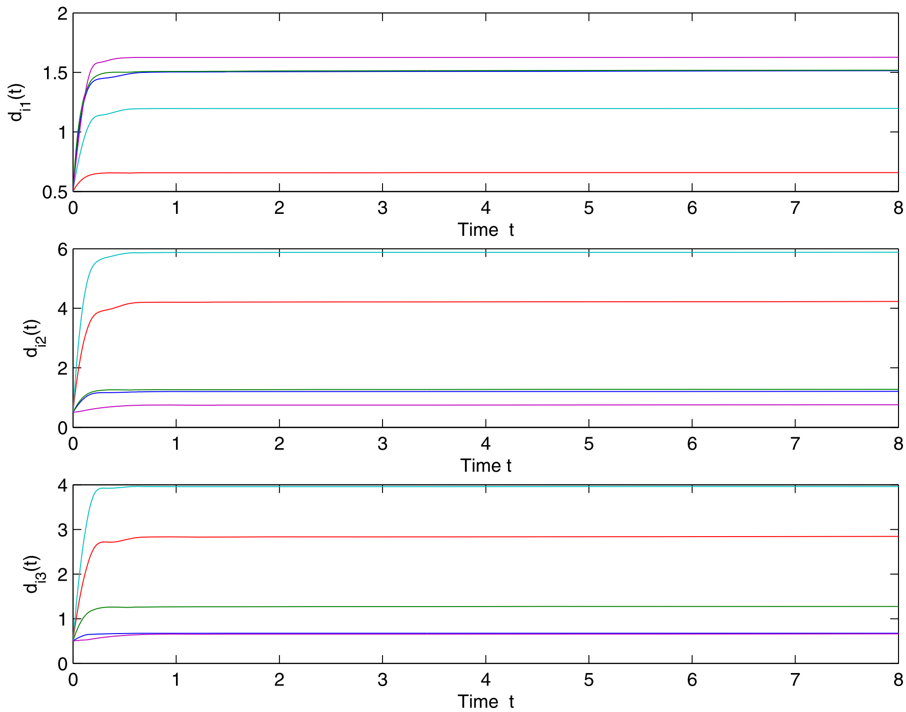

Figure 7 The evolution of adaptive feedback gain $d_{i1}(t)$, $d_{i2}(t)$ and $d_{i3}(t)$ ($i = 1, 2, \ldots, 5$).

Remark 10 The advantage of Example 4.1 is that the discrete and distributed time-varying delays are different values, i.e., $h(t) = 0.1 + 0.1\sin^2 t$, $k(t) = 0.1\cos^2 t$. Moreover, in these examples we still consider discrete and distributed time-varying delays in the dynamical nodes and the hybrid coupling term simultaneously. Hence the synchronization conditions in [11] cannot be applied to these examples.

Example 4.2 In this example, the drive dynamical system and response dynamical networks with coupling time delay, respectively, in which each node is a unified chaotic system with coupling time delay, were proposed by [20], which can be described by

$$\begin{aligned}
\dot{x}_1(t) &= (25\theta + 10)\bigl(x_2(t) - x_1(t)\bigr),\\
\dot{x}_2(t) &= (28 - 35\theta)x_1(t) + (29\theta - 1)x_2(t) - x_{i1}(t)x_3(t),\\
\dot{x}_3(t) &= x_1(t)x_2(t) - \frac{8+\theta}{3}x_3(t)
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
\begin{pmatrix}\dot{x}_{i1}(t)\\ \dot{x}_{i2}(t)\\ \dot{x}_{i3}(t)\end{pmatrix} &= \begin{pmatrix}(25\theta + 10)(x_{i2}(t) - x_{i1}(t))\\ (28 - 35\theta)x_{i1}(t) + (29\theta - 1)x_{i2}(t) - x_{i1}(t)x_{i3}(t)\\ x_{i1}(t)x_{i2}(t) - \frac{8+\theta}{3}x_{i3}(t)\end{pmatrix} + c_1\sum_{j=1}^{N} a_{ij}G_1x_j(t)\\
&\quad + c_2\sum_{j=1}^{N} b_{ij}G_2x_j(t-h) + \mathcal{U}_i(t), \quad i = 1, 2, \ldots, N,
\end{aligned} \tag{41}$$

where $\theta \in [0,1]$ is a system parameter. It is stable at the equilibrium point $s(t) = 0$, and the Jacobian matrix is

$$J(t) = \begin{bmatrix} -25\theta - 10 & 25\theta + 10 & 0 \\ 28 - 35\theta & 29\theta - 1 & 0 \\ 0 & 0 & -\frac{8+\theta}{3} \end{bmatrix}.$$

The parameters are selected as follows: the coupling strength $c_1 = 1.7$ and $c_2 = 0.4$, the inner-coupling matrices are

$$G_1 = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 4 & 2 \\ -1 & 2 & 5 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

and the coupling configuration matrices A and B are given in Example 4.1, respectively, the time-varying scaling function matrix $\alpha(t) = \mathrm{diag}(0.45e^{\sin(t)}, 0.5e^{\sin(t)}, 0.4e^{\sin(t)})$. Since time-varying scaling function matrices $\alpha_i(t)$, $i = 1, 2, 3$, are different values, the synchronization conditions derived in [12, 13, 21, 36] cannot be applied to these examples.

Solution: From conditions (25)-(26) of Corollary 3.2 and with positive constants $\varepsilon_3 = 13.45$ and $\varepsilon_5 = 11.15$, one can check that the last two conditions in Corollary 3.2 are satisfied. From the conditions of Corollary 3.2, we can obtain $d_1^* > 6.0773$, $d_2^* > -0.5683$. Figures 8 and 9 show the chaotic behavior of drive system (40) and response network (41). Figure 10 shows the function projective synchronization errors between the states of

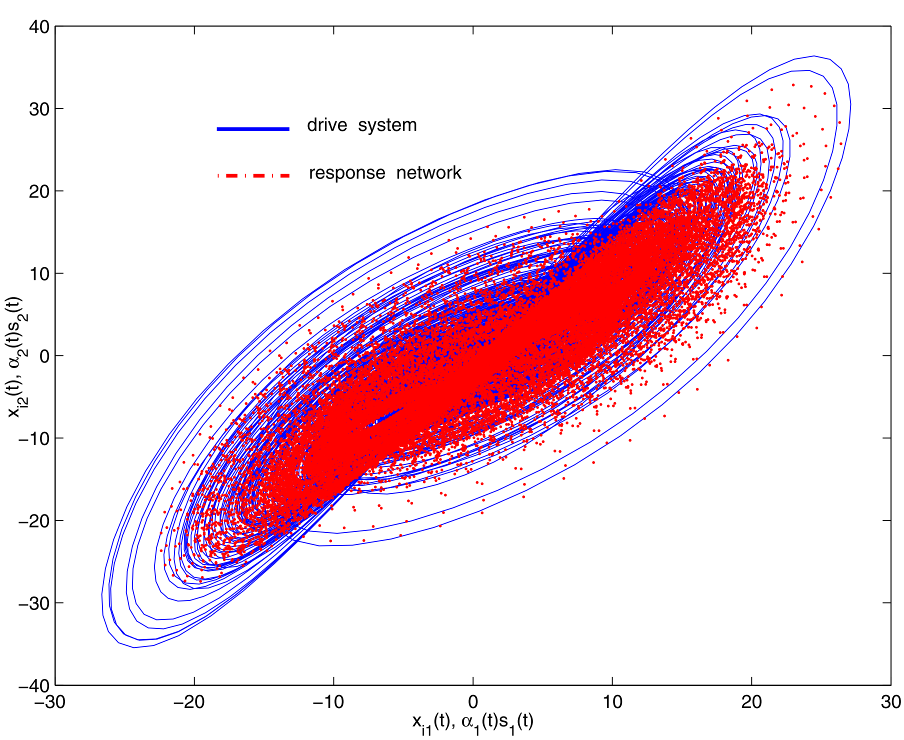

Figure 8 Chaotic behavior of drive system (40) and response network (41) with the time-varying scaling function matrix $\alpha(t) = \mathrm{diag}(0.45e^{\sin(t)}, 0.5e^{\sin(t)}, 0.4e^{\sin(t)})$.

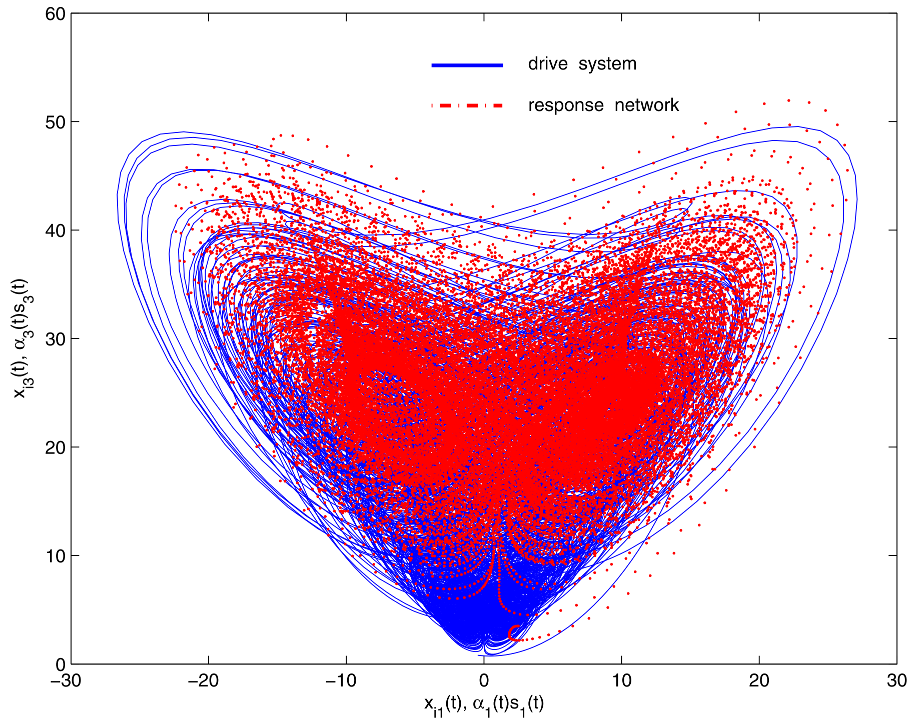

Figure 9 Chaotic behavior of drive system (40) and response network (41) with the time-varying scaling function matrix $\alpha(t) = \mathrm{diag}(0.45e^{\sin(t)}, 0.5e^{\sin(t)}, 0.4e^{\sin(t)})$.

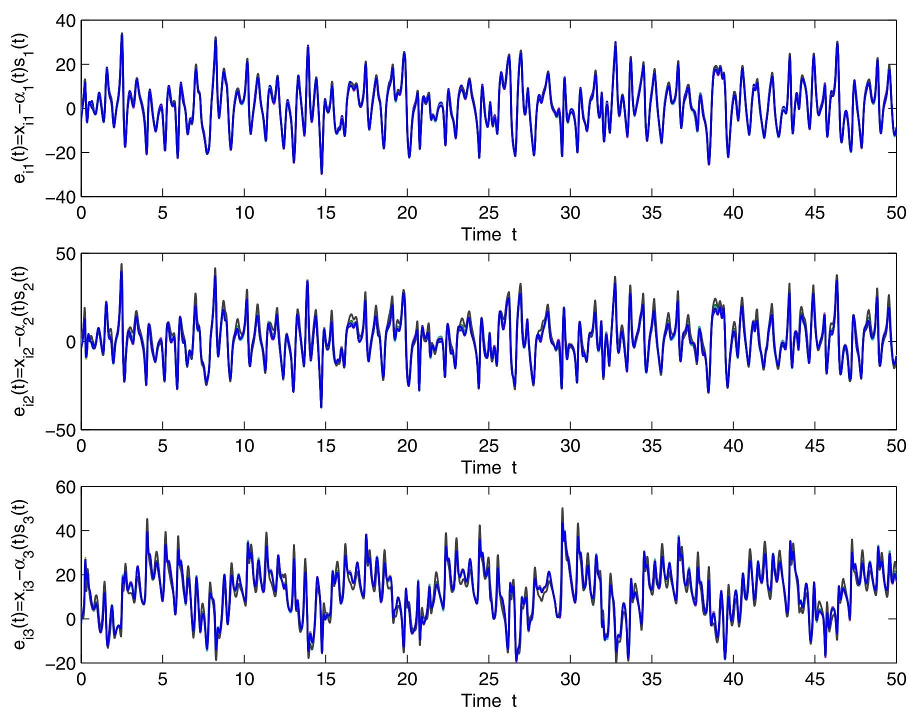

Figure 10 The modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (40) and node $x_i(t)$ (41), $i = 1, \ldots, 8$, without hybrid adaptive control (5).

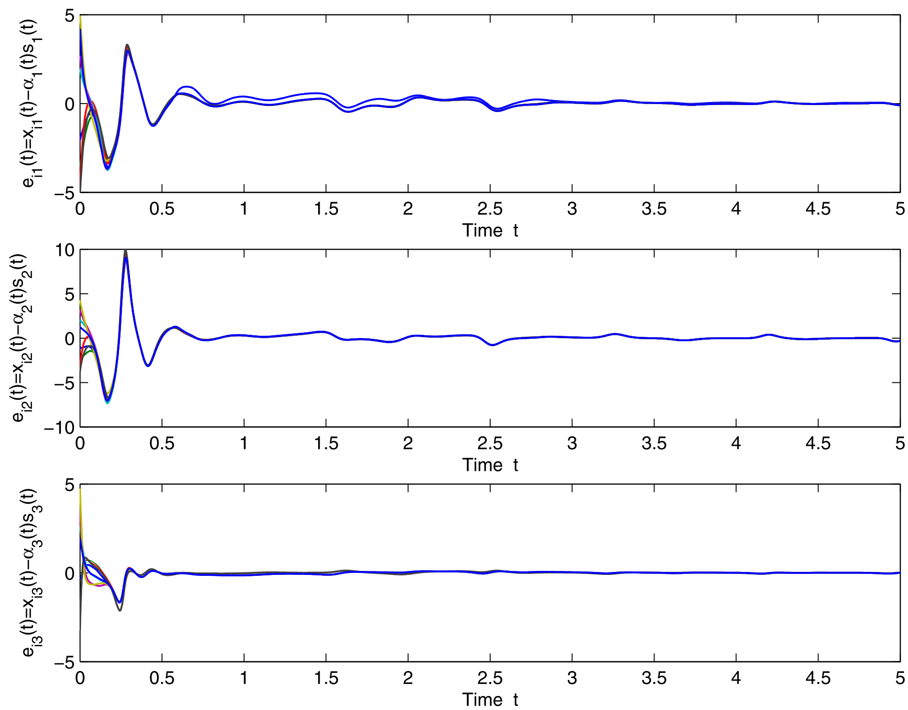

Figure 11 The modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (40) and node $x_i(t)$ (41), $i = 1,\dots,8$, with hybrid adaptive control (5).

isolate node $\alpha(t)s(t)$ (40) and node $x_i(t)$ (41), $i = 1,\dots,8$, without hybrid adaptive control (5). Figure 11 shows the function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (40) and node $x_i(t)$ (41), $i = 1,\dots,8$, with hybrid adaptive control (5). Figure 12 gives the evolution of adaptive feedback gain $d_{i1}(t)$ and $d_{i2}(t)$ $(i = 1,2,\dots,8)$.

Remark 11 In Examples 4.1 and 4.2, we see that every state variable of the error networks of (39) and (41) is unstable without control. After applying controllers (5) and (27), all the state variables of the error networks of (39) and (41) quickly converge to 0. That shows the effectiveness of the controllers.

Example 4.3 Now we investigate the pining MFPS of a large-scale undirected Watts-Strogatz network with 20 identical nodes of the perturbed Chua's circuit system with mixed time-varying delays given in Example 4.1. The parameters are selected as follows: the time-varying scaling function matrix $\alpha(t) = \operatorname{diag}(0.6\sin(\frac{2\pi}{15}), 0.7\sin(\frac{2\pi}{15}), 0.75\sin(\frac{2\pi}{15}))$, the coupling strength $c_1 = 2$, $c_2 = 0.5$, $c_3 = 0.4$, the inner-coupling matrices G_1, G_2 and G_3 are given in Example 4.1, respectively. For a Watts-Strogatz network here, we set the parameters $[N = 20, K = 2, \beta = 0.75]$, $[N = 20, K = 1, \beta = 0.75]$ and $[N = 20, K = 1, \beta = 0.5]$. Then the coupling matrices A, B and C can be randomly generated by the Watts-Strogatz model as shown in Figures 13-15, respectively.

Now we study how to select pinned nodes of a network. Since A is an undirected Watts-Strogatz network, the pinned nodes can be randomly chosen for the convenience of practical applications. We randomly choose seven network nodes, i.e., $l = 7$, and the feedback control gains are chosen as $d_{1i} = 20$, $i = 1,2,\dots,7$, $\bar{d}_2^* = 1.6535$, $\bar{d}_3^* = 2.7566$ and with

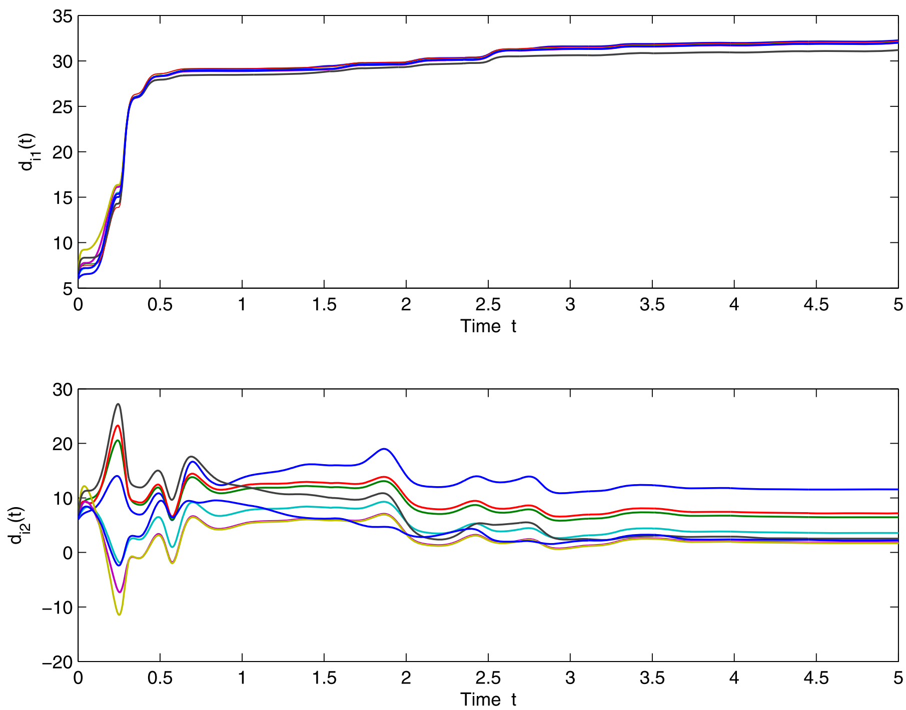

Figure 12 The evolution of adaptive feedback gain $d_{i1}(t)$ and $d_{i2}(t)$ ($i = 1, 2, \ldots, 8$).

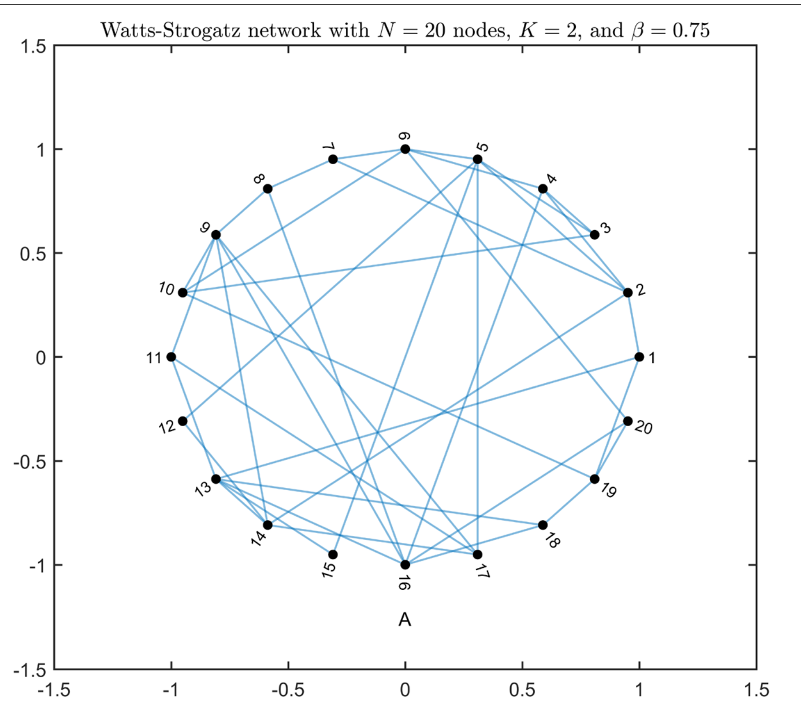

Figure 13 The topology structure of Watts-Strogatz complex network with $N = 20$, $K = 2$ and $\beta = 0.75$.

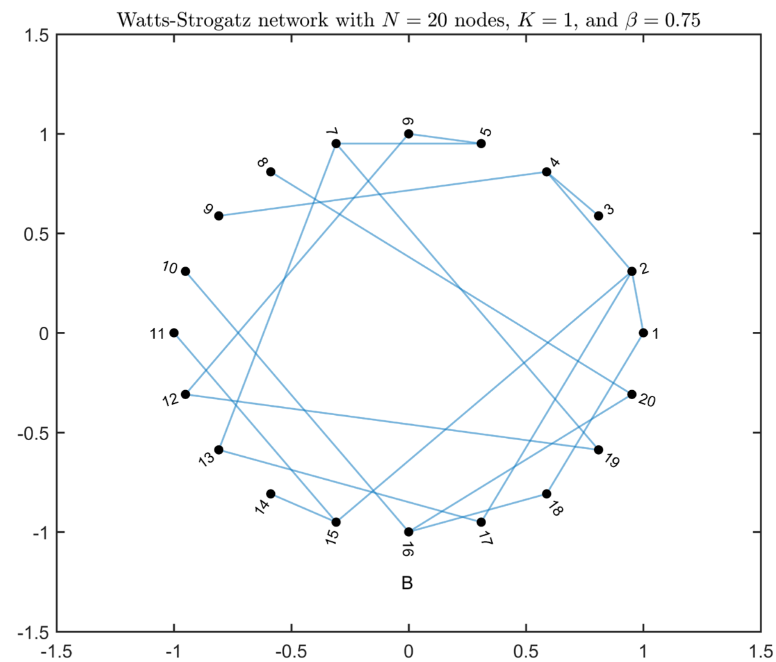

Figure 14 The topology structure of Watts-Strogatz complex network with $N = 20$, $K = 1$ and $\beta = 0.75$.

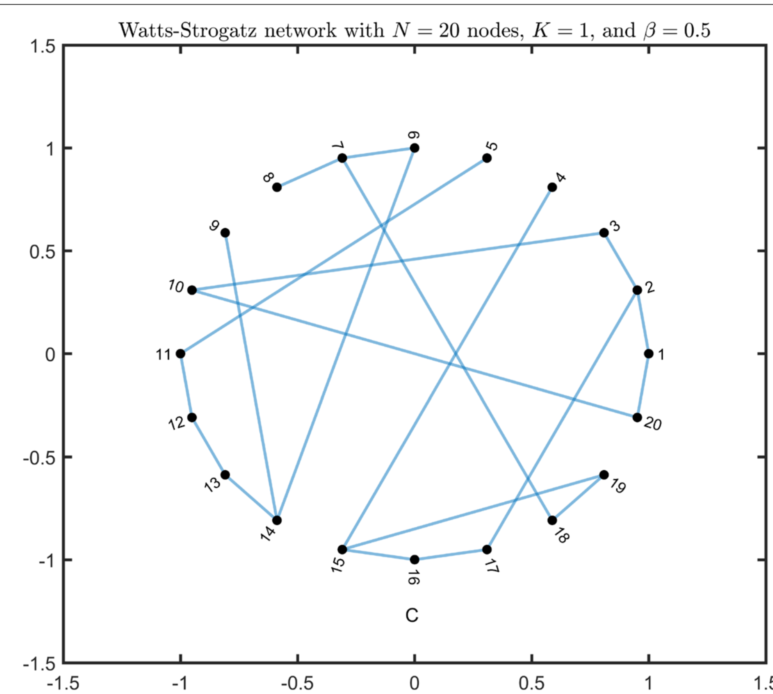

Figure 15 The topology structure of Watts-Strogatz complex network with $N = 20$, $K = 1$ and $\beta = 0.5$.

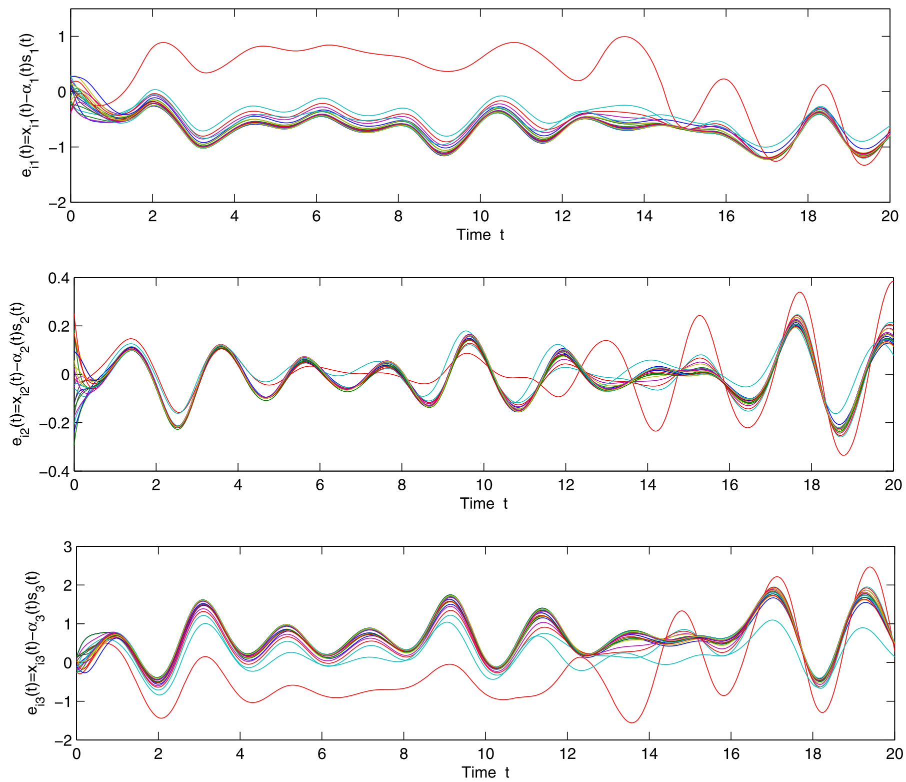

Figure 16 The modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39) with Watts-Strogatz network, where $e_{ij}(t) = x_{ij}(t) - \alpha_j(t)s_j(t)$ for $i = 1, \dots, 20$, $j = 1, 2, 3$, without hybrid adaptive pinning control.

positive constants $\varepsilon_1 = 7.67$, $\varepsilon_2 = 8.75$, $\varepsilon_3 = 12.65$, $\varepsilon_4 = 15.21$, $\varepsilon_5 = 14.13$, $\varepsilon_6 = 13.72$. By a simple numerical calculation, we see that pinning condition (28) holds:

$$\lambda_{\max}\left(\frac{A_7 + A_7^T}{2}\right) = -0.7431 < -\frac{\Sigma_1}{c_1} = -\frac{1.2980}{2} = -0.6490.$$

Figure 16 shows the modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39) with Watts-Strogatz network, where $e_{ij}(t) = x_{ij}(t) - \alpha_j(t)s_j(t)$ for $i = 1, \dots, 20$, $j = 1, 2, 3$, without hybrid adaptive pinning control (5). Figure 17 shows the modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39) with Watts-Strogatz network and hybrid adaptive pinning control (27). Figure 18 gives the evolution of adaptive pinning feedback gain $d_{i1}(t)$, $d_{i2}(t)$ and $d_{i3}(t)$ $(i = 1, 2, \dots, 7)$.

5 Conclusions

In this paper, modified function projective synchronization (MFPS) for complex dynamical networks with mixed time-varying and hybrid coupling delays was investigated. It is assumed that the coupling configuration matrix need not be symmetric or irreducible and it contains state coupling, time-varying delay coupling and distributed time-varying delay coupling. Firstly, we considered MFPS via either hybrid control or hybrid pinning control with nonlinear and adaptive linear feedback control, which contains error linear term,

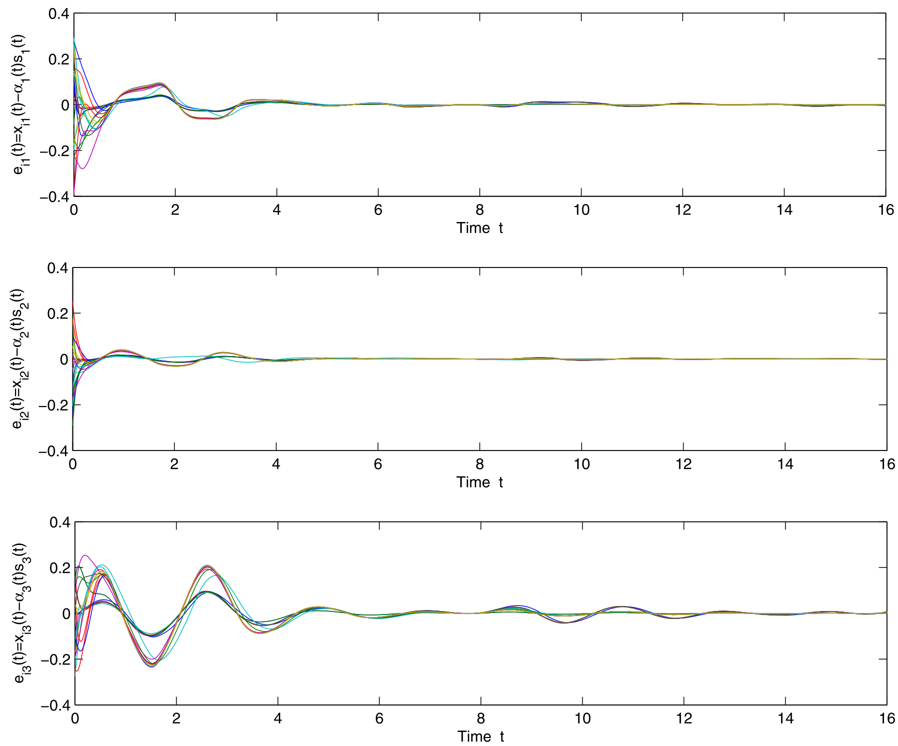

Figure 17 **The modified function projective synchronization errors between the states of isolate node $\alpha(t)s(t)$ (38) and node $x_i(t)$ (39) with Watts-Strogatz network and hybrid adaptive pinning control.**

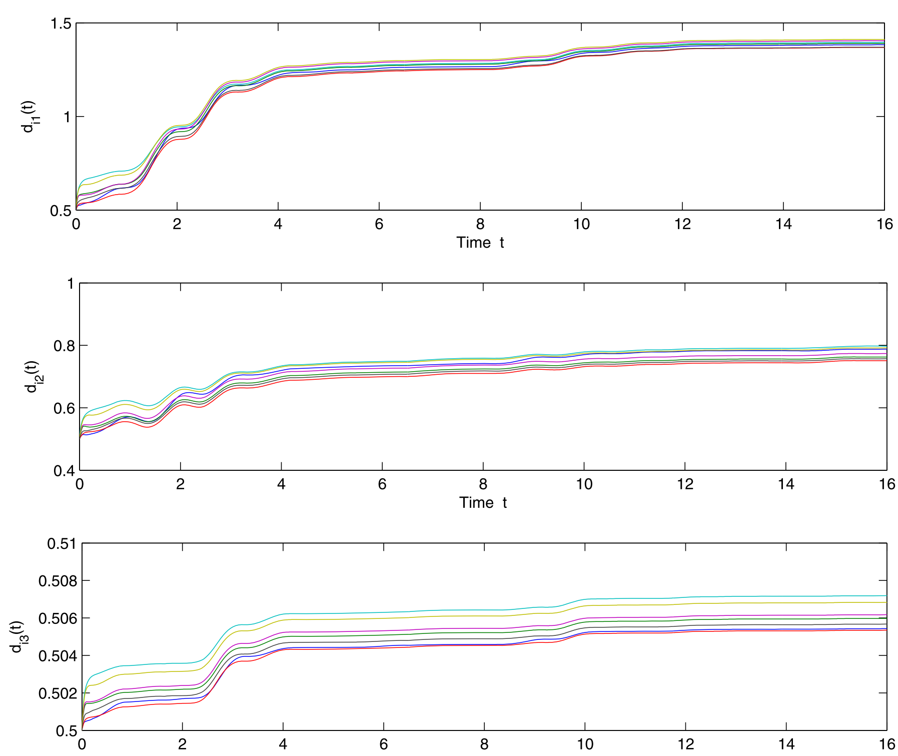

Figure 18 **The evolution of adaptive pinning feedback gain $d_{i1}(t)$, $d_{i2}(t)$ and $d_{i3}(t)$ ($i = 1, 2, \ldots, 7$).**

time-varying delay error linear term and distributed time-varying delay error linear term. Secondly, by using a novel Lyapunov-Krasovskii functional, a new adaptive control technique, the parameter update law and the technique of dealing with some integral terms, improved MFPS criteria of delayed CDNs with asymmetric coupling delays are obtained. In addition, the pinning nodes can be randomly selected. Finally, numerical examples are included to show the effectiveness of the proposed hybrid adaptive control and hybrid adaptive pinning control scheme. The results in this paper generalize and improve the corresponding results of the recent works.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Albert, R, Jeong, H, Barabasi, AL: Diameter of the world wide web. Nature **401**, 130-131 (1999)
2. Williams, RJ, Martinez, ND: Simple rules yield complex food webs. Nature **404**, 180-183 (2000)
3. Watts, DJ, Strogatz, SH: Collective dynamics of 'small world' networks. Nature **393**, 440-442 (1998)
4. Strogatz, SH: Exploring complex networks. Nature **410**, 268-276 (2001)
5. Wang, XF, Chen, G: Synchronization in scale-free dynamical networks: robustness and fragility. IEEE Trans. Circuits Syst. I **49**, 54-62 (2002)
6. Wu, X, Lu, H: Generalized projective synchronization between two different general complex dynamical networks with delayed coupling. Phys. Lett. A **374**, 3932-3941 (2010)
7. Wu, X, Lu, H: Hybrid synchronization of the general delayed and non-delayed complex dynamical networks via pinning control. Neurocomputing **89**, 168-177 (2012)
8. Li, K, Guan, S, Gong, X, Lai, CH: Synchronization stability of general complex dynamical networks with time-varying delays. Phys. Lett. A **372**, 7133-7139 (2008)
9. Wang, Y, Zhang, H, Wang, XY, Yang, D: Networked synchronization control of coupled dynamic networks with time-varying delay. IEEE Trans. Syst. Man Cybern., Part B, Cybern. **40**, 1468-1479 (2010)
10. Song, Q, Cao, J, Liu, F: Pinning-controlled synchronization of hybrid-coupled complex dynamical networks with mixed time-delays. Int. J. Robust Nonlinear Control **22**, 690-706 (2012)
11. Li, B: Pinning adaptive hybrid synchronization of two general complex dynamical networks with mixed coupling. Appl. Math. Model. **40**, 2983-2998 (2016)
12. Du, H: Function projective synchronization in complex dynamical networks with or without external disturbances via error feedback control. Neurocomputing **173**, 1443-1449 (2016)
13. Du, H, Shi, P, Lu, N: Function projective synchronization in complex dynamical networks with time delay via hybrid feedback control. Nonlinear Anal., Real World Appl. **14**, 1182-1190 (2013)
14. Nian, F, Wang, XY, Niu, Y, Lin, D: Module-phase synchronization in complex dynamic system. Appl. Math. Comput. **217**, 2481-2489 (2010)
15. Cai, S, Lei, X, Liu, Z: Outer synchronization between two hybrid-coupled delayed dynamical networks via aperiodically adaptive intermittent pinning control. Complexity **21**, 593-605 (2016)
16. Botmart, T, Niamsup, P: Exponential synchronization of complex dynamical network with mixed time-varying and hybrid coupling delays via intermittent control. Adv. Differ. Equ. **2014**, 116 (2014)
17. Sivaranjani, K, Rakkiyappan, R: Pinning sampled-data synchronization of complex dynamical networks with Markovian jumping and mixed delays using multiple integral approach. Complexity **21**, 622-632 (2016)
18. Cui, B, Lou, X: Synchronization of chaotic recurrent neural networks with time-varying delays using nonlinear feedback control. Chaos Solitons Fractals **39**, 288-294 (2009)

19. He, P, Jing, CG, Fan, T, Chen, CZ: Robust decentralized adaptive synchronization of general complex networks with coupling delayed and uncertainties. Complexity **19**, 10-26 (2014)
20. Zhao, YP, He, P, Nik, HS, Ren, JC: Robust adaptive synchronization of uncertain complex networks with multiple time-varying coupled delays. Complexity **20**, 62-73 (2014)
21. Shi, L, Zhu, H, Zhong, S, Shi, K, Cheng, J: Function projective synchronization of complex networks with asymmetric coupling via adaptive and pinning feedback control. ISA Trans. **65**, 81-87 (2016)
22. Wu, X, Lu, H: Generalized projective synchronization between two different general complex dynamical networks with delayed coupling. Phys. Lett. A **374**, 3932-3941 (2010)
23. Zhang, H, Wang, XY, Lin, XH: Topology identification and module-phase synchronization of neural network with time delay. IEEE Trans. Syst. Man Cybern. Syst. (2016). doi:10.1109/TSMC.2016.2523935
24. Xie, C, Xua, Y, Tong, D: Synchronization of time varying delayed complex networks via impulsive control. Optik **125**, 3781-3787 (2014)
25. Lin, D, Wang, XY: Observer-based decentralized fuzzy neural sliding mode control for interconnected unknown chaotic systems via network structure adaptation. Fuzzy Sets Syst. **161**, 2066-2080 (2010)
26. Ma, XH, Wang, JA: Pinning outer synchronization between two delayed complex networks with nonlinear coupling via adaptive periodically intermittent control. Neurocomputing **199**, 197-203 (2016)
27. Yu, R, Zhang, H, Wang, Z, Wang, J: Synchronization of complex dynamical networks via pinning scheme design under hybrid topologies. Neurocomputing **214**, 210-217 (2016)
28. Yu, WW, Chen, GR, Lu, JH: On pinning synchronization of complex dynamical networks. Automatica **45**, 429-435 (2009)
29. Rakkiyappan, R, Sakthivel, N: Cluster synchronization for T-S fuzzy complex networks using pinning control with probabilistic time-varying delays. Complexity **21**, 59-77 (2014)
30. Wu, Y, Li, C, Yang, A, Song, L, Wu, Y: Pinning adaptive anti-synchronization between two general complex dynamical networks with non-delayed and delayed coupling. Appl. Math. Comput. **218**, 7445-7452 (2012)
31. Chen, T, Liu, X, Lu, W: Pinning complex networks by a single controller. IEEE Trans. Circuits Syst. I, Regul. Pap. **54**, 1317-1326 (2007)
32. Guo, X, Li, L: A new synchronization algorithm for delayed complex dynamical networks via adaptive control approach. Commun. Nonlinear Sci. Numer. Simul. **17**, 4395-4403 (2012)
33. Zhou, J, Lu, JA, Lu, J: Pinning adaptive synchronization of a general complex dynamical network. Automatica **44**, 996-1003 (2008)
34. Botmart, T, Niamsup, P: Adaptive control and synchronization of the perturbed Chua system. Math. Comput. Simul. **75**, 37-55 (2007)
35. Wu, YQ, Li, CP, Wu, YJ, Kurths, J: Generalized synchronization between two different complex networks. Commun. Nonlinear Sci. Numer. Simul. **17**, 349-355 (2012)
36. Wu, X, Lu, H: Generalized function projective (lag, anticipated and complete) synchronization between two different complex networks with nonidentical nodes. Commun. Nonlinear Sci. Numer. Simul. **17**, 3005-3021 (2012)
37. Fan, YQ, Ying, KY, Wang, YH, Wang, LY: Projective synchronization adaptive control for different chaotic neural networks with mixed time delays. Optik **127**, 2551-2557 (2016)
38. Wang, XY, Yijie, H: Projective synchronization of fractional order chaotic system based on linear separation. Phys. Lett. A **372**, 435-441 (2008)
39. Li, B: Hybrid synchronization of two complex delayed dynamical networks with nonidentical topologies and mixed coupling. Complexity **20**, 62-73 (2016)
40. Zhang, R, Yang, Y, Xu, Z, Hu, M: Function projective synchronization in drive-response dynamical network. Phys. Lett. A **374**, 3025-3038 (2010)
41. Du, H, Zeng, Q, Wang, C: Modified function projective synchronization of chaotic system. Chaos Solitons Fractals **42**, 2399-2404 (2009)
42. Zheng, S: Partial switched modified function projective synchronization of unknown complex nonlinear systems. Optik **126**, 3854-3858 (2015)
43. Wang, S, Zheng, S, Zhang, B, Guo, H: Modified function projective lag synchronization of uncertain complex networks with time-varying coupling strength. Optik **127**, 4716-4725 (2016)
44. Gu, K, Kharitonov, VL, Chen, J: Stability of Time-Delay System. Birkhäuser, Boston (2003)
45. MacDuffee, M: The Theory of Matrices. Dover, New York (2004)
46. Wilkinson, JH: The Algebraic Eigenvalue Problem. Oxford University Press, London (1965)
47. Chen, T, Liu, X, Lu, W: Pinning complex networks by a single controller. IEEE Trans. Circuits Syst. I, Regul. Pap. **54**, 1317-1326 (2007)
48. Song, Q, Cao, J: On pinning synchronization of directed and undirected complex dynamical networks. IEEE Trans. Circuits Syst. I **57**, 672-680 (2010)
49. Hale, JH: Theory of Functional Differential Equations. Springer, New York (1977)