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Blowing-up solutions of multi-order fractional differential equations with the periodic boundary condition

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Abstract

In this paper, we analyze the boundary value problem of a class of multi-order fractional differential equations involving the standard Caputo fractional derivative with the general periodic boundary conditions:

$$\begin{cases} L(D)u(t) = f(t,u(t)), & t \in [0,T], T > 0, \\ u(0) = u(T) > 0, & u'(0) = u'(T) > 0, \end{cases}$$

where $L(D) = \sum_{i=0}^{n} a_i D^{S_i}$, $1 \leq S_0 < \cdots < S_{n-1} < S_n < 2$, $a_i \in \mathbb{R}$, $a_n \neq 0$, and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous operation. We get the Green's function in terms of the Laplace transform. We obtain the existence and uniqueness of solution for the class of multi-order fractional differential equations. We investigate the blowing-up solutions to the special case $f(t,u(t)) = |u(t)|^p$, $a_i \geq 0$, and give an upper bound on the blow-up time $T_{\max}$.

MSC: Primary 34A08; secondary 34B15

Keywords: fractional differential equations; periodic boundary problem; multi-order Mittag-Leffler functions; blowing-up solutions

1 Introduction

The idea of derivatives of noninteger order initially appeared in the letter from Leibnizs to L'Hospital in 1695. For many years, studies of the theory of fractional order were mainly constrained to the field of pure theoretical mathematics. One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives. Another difficulty is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character. However, during the last 30 years fractional calculus has started to attract much more attention of physicists and mathematicians. Many researchers found that derivatives of noninteger order are very suitable for the description of various physical phenomena such as rheology, damping laws and diffusion processes. These findings invoked the growing interest in studies of the fractal calculus in various fields such as physics, chemistry and engineering. Existence results for nonlinear fractional differential equations with integral boundary conditions [1] and anti-periodic fractional boundary conditions [2] have been investigated. Bazhlekova [3] studied a linear initial value problem and derived fundamental solution and impulse response solution.

Ahmad and Nieto [4] investigated the existence and uniqueness of solutions for an anti-periodic fractional boundary value problem given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), {}^cD^r x(t)), & t \in [0,T], T > 0, 1 < q \leq 2, 0 < r \leq 1, \\ x(0) = -x(T), \quad {}^cD^p x(0) = -{}^cD^p x(T), & 0 < p < 1, \end{cases}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function.

In [5], the authors investigated the existence and uniqueness of solutions to a class of Caputo-type multi-order fractional differential equations with the initial value problem

$$\begin{cases} ({}^cD^\mu y)(x) - \sum_{i=1}^n \lambda_i ({}^cD^{\mu_i} y)(x) = g(x), \\ y^{(k)}(0) = c_k, \end{cases}$$

where $\lambda_i, c_k \in \mathbb{R}$, $k = 0, \dots, m-1$, $m-1 < \mu \leq m$, $\mu > \mu_1 > \cdots > \mu_n \geq 0$, $m_i - 1 < \mu_i \leq m_i$, $m_i \in \mathbb{N}$, $i = 1, \dots, n$.

Stojanović [6] analyzed the existence and uniqueness of solutions for the nonlinear multi-order fractional differential equation

$$\begin{cases} L(D)u(t) = f(t, u(t)), & t \in [0,T], T > 0, \\ u(0) = u(T), \end{cases}$$

where $L(D) = \sum_{i=1}^n \lambda_i {}^cD^{\alpha_i}$, $0 \leq S_0 < \cdots < S_{n-1} < S_n < 1$, $\lambda_i \in \mathbb{R}$, $\lambda_n \neq 0$. Kirane and Malik in [7] studied the profile of blowing-up solutions of the system

$$\begin{cases} u'(t) + D^\alpha(u - u(0))(t) = v^q(t), & t > 0, \\ v'(t) + D^\beta(v - v(0))(t) = u^r(t), & t > 0, \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \end{cases}$$

where $u > 0$, $v > 0$, $0 < \alpha, \beta < 1$. Then Alsaedi et al. in [8] were concerned with blowing-up solutions of the nonlinear fractional system

$$\begin{cases} u'(t) - D^\alpha(u - u(0))(t) = u^p(t) v^q(t), & t > 0, \\ v'(t) - D^\beta(v - v(0))(t) = u^r(t) v^s(t), & t > 0, \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \end{cases}$$

where $u > 0$, $v > 0$, $p, q, r, s \in \mathbb{R}^+$.

In this paper, we analyze nonlinear boundary value problems of the multi-order fractional differential equations

$$L(D)u(t) = f\bigl(t, u(t)\bigr), \quad t \in [0,T], T > 0, \tag{1}$$

with the boundary condition

$$u(0) = u(T) > 0, \qquad u'(0) = u'(T) > 0, \tag{2}$$

where $L(D) = \sum_{i=1}^{n} a_i {}^{c}D^{S_i}$, $1 \leq S_0 < \cdots < S_{n-1} < S_n < 2$, $a_i \in \mathbb{R}$, $a_n \neq 0$, ${}^{c}D^{S_i}$ $(i = 1, 2, \ldots, n)$ are the standard Caputo fractional derivatives, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous operation.

This equation is a generalization of the classical relaxation equation, and it governs some fractional relaxation processes.

We investigate the blowing-up solutions to the special case

$$\begin{cases} L(D)u(t) = |u(t)|^p, & t > 0, \\ u(0) = u(T) > 0, & u'(0) = u'(T) > 0, \end{cases}$$

where $L(D) = \sum_{i=0}^{n} a_i {}^{c}D^{S_i}$, $1 \leq S_0 < \cdots < S_{n-1} < S_n < 2$, $a_i \geq 0$, $a_n \neq 0$, T is a positive constant, and we give an upper bound on the blow-up time $T_{\max}$.

The rest of this paper is organized as follows. In Section 2, we introduce some basic definitions and notations. In Section 3, we find the Green's function for a multi-order fractional differential equation, we prove the existence and uniqueness theorems for the equations. We investigate the blowing-up solutions to the special case $f(t, u(t)) = |u(t)|^p$, $a_i \geq 0$, $u(0) > 0$, and give an upper bound on the blow-up time $T_{\max}$.

2 Preliminaries

In this section, we introduce preliminary facts and some basic results, which are used throughout this paper (refer to [9–15]).

Definition 2.1 Let $C_\mu = \{f(x) | f(x) = x^p f_1(x), f_1 \in C[0, +\infty), p > \mu\}$. If $f \in C_\mu$, we define the Riemann-Liouville fractional integral operator of order α of a function f as follows:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t)\, dt, \quad \alpha > 0, x > 0,$$

where $J^0 f(x) = f(x)$.

Definition 2.2 The Caputo fractional derivative ${}^{c}D_{0+}^{\alpha}$ of $f(x)$ is defined as

$${}^{c}D_{0+}^{\alpha} f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - t)^{m-\alpha-1} f^m(t)\, dt,$$

where $m - 1 < \alpha \leq m$, $m \in N$, $x > 0$, $f \in C_{-1}^{m}$.

For brevity of notation, let us take ${}^{c}D_{0+}^{\alpha}$ as D^α.

The two-parametric Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \beta > 0, \alpha > 0, z \in \mathbb{C}.$$

The Laplace transform of the Caputo derivative is

$$L\{D^\alpha f(t)\}(s) = s^\alpha \tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0^+), \quad n-1 < \alpha \le n.$$

The Laplace transform of the two-parametric Mittag-Leffler function is

$$L\{t^{\beta-1}E_{\alpha,\beta}(\pm at^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{(s^\alpha \mp a)}, \quad \operatorname{Re}(s) > |a|^{\frac{1}{\alpha}}, \operatorname{Re}(\beta) > 0,$$
$$L\{t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm at^\alpha)\}(s) = \frac{k!s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}}, \quad \operatorname{Re}(s) > |a|^{\frac{1}{\alpha}}, \operatorname{Re}(\beta) > 0,$$

where $E^{(k)}_{\alpha,\beta}(y) = \frac{d^k}{dy^k}E_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{(j+k)!y^j}{j!\Gamma(\alpha j+\alpha k+\beta)}$, $k = 0,1,2,\ldots$.

Let us denote by $C[0,T]$ the Banach space of all continuous real-valued functions defined on $[0,T]$, $T > 0$ with the norm

$$\|u\|_\infty = \max\{|u(t)| : t \in [0,T]\}, \quad T > 0.$$

Let us denote by $C^n[0,T]$ the class of all real functions on $[0,T]$ which have a continuous nth order derivative. S denotes the class of functions $\alpha : \mathbb{R}^+ \to [0,1)$ satisfying the condition $\alpha(t_n) \to 1$, which implies $t_n \to 0$. B denotes the class of increasing functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(x) < x$ for all $x > 0$ and $\frac{\phi(x)}{x} \in S$. $(C[0,T],d)$ denotes a metric space where $d(u,v) = \max_{t\in[0,T]} |u(t) - v(t)|$. Obviously, $(C[0,T],d)$ is a complete metric space.

Lemma 2.1 (see [13]) *Let (M,d) be a complete metric space and let $T : M \to M$. Suppose that there exists $\alpha \in S$ such that for each $u,v \in M$,*

$$d(T(x),T(y)) \le \alpha(d(u,v))d(u,v),$$

then T has a unique fixed point $z \in M$ and $\{T^n(x)\}$ converges to z for each $x \in M$.

3 Main results

Lemma 3.1 *The fractional differential equation*

$$L(D)u(t) = f(t,u(t)), \quad t \in [0,T], T > 0,$$

with the boundary condition $u(0) = u(T)$, $u'(0) = u'(T)$ is equivalent to the fractional integral equation

$$u(t) = \int_0^T G(t,s)f(s,u(s))\,ds,$$

where $G(t,s)$ is the following Green's function:

For $0 \le s < t$,

$$G(t,s) = \widetilde{C}(t) + \frac{\widetilde{A}(t)\widetilde{C}(T)(1-\widetilde{E}(T)) + \widetilde{A}(t)\widetilde{B}(T)\widetilde{F}(T) + \widetilde{B}(t)\widetilde{C}(T)\widetilde{D}(T) + \widetilde{B}(t)\widetilde{F}(T)(1-\widetilde{A}(T))}{(1-\widetilde{E}(T))(1-\widetilde{A}(T)) - \widetilde{B}(T)\widetilde{D}(T)};$$

For $t \le s < T$,

$$G(t,s) = \frac{\widetilde{A}(t)\widetilde{C}(T)(1-\widetilde{E}(T)) + \widetilde{A}(t)\widetilde{B}(T)\widetilde{F}(T) + \widetilde{B}(t)\widetilde{C}(T)\widetilde{D}(T) + \widetilde{B}(t)\widetilde{F}(T)(1-\widetilde{A}(T))}{(1-\widetilde{E}(T))(1-\widetilde{A}(T)) - \widetilde{B}(T)\widetilde{D}(T)},$$

where

$$\begin{aligned}
\widetilde{A}(t) &= \sum_{r=0}^{n} \frac{\alpha_r}{\alpha_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0+\cdots+k_{n-2}=m} (m; k_0, \ldots, k_{n-2}) \\
&\quad \times \prod_{i=0}^{n-2} \left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} t^{\alpha m+\beta-2} E_{\alpha,\beta-1}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right), \\
\widetilde{B}(t) &= \sum_{r=0}^{n} \frac{\alpha_r}{\alpha_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0+\cdots+k_{n-2}=m} (m; k_0, \ldots, k_{n-2}) \\
&\quad \times \prod_{i=0}^{n-2} \left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right), \\
\widetilde{C}(t) &= \frac{1}{\alpha_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0+\cdots+k_{n-2}=m} (m; k_0, \ldots, k_{n-2}) \\
&\quad \times \prod_{i=0}^{n-2} \left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} (t-s)^{\alpha m+\gamma-1} E_{\alpha,\gamma}^{(m)}\left(-\frac{\alpha_{n-1}(t-s)^{\alpha}}{\alpha_n}\right), \\
\widetilde{D}(t) &= \sum_{r=0}^{n} \frac{\alpha_r}{\alpha_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0+\cdots+k_{n-2}=m} (m; k_0, \ldots, k_{n-2}) \prod_{i=0}^{n-2} \left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} t^{\alpha m+\beta-3} \\
&\quad \times \left[(\alpha m+\beta-2) E_{\alpha,\beta-1}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right) - \alpha t^{\alpha}\frac{\alpha_{n-1}}{\alpha_n} E_{\alpha,\beta-1}^{(m+1)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)\right], \\
\widetilde{E}(t) &= \sum_{r=0}^{n} \frac{\alpha_r}{\alpha_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0+\cdots+k_{n-2}=m} (m; k_0, \ldots, k_{n-2}) \prod_{i=0}^{n-2} \left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} t^{\alpha m+\beta-2} \\
&\quad \times \left[(\alpha m+\beta-1) E_{\alpha,\beta}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right) - \alpha t^{\alpha}\frac{\alpha_{n-1}}{\alpha_n} E_{\alpha,\beta}^{(m+1)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)\right], \\
\widetilde{F}(t) &= \frac{1}{\alpha_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0+\cdots+k_{n-2}=m} (m; k_0, \ldots, k_{n-2}) \prod_{i=0}^{n-2} \left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} (t-s)^{\alpha m+\gamma-2} \\
&\quad \times \left[(\alpha m+\gamma-1) E_{\alpha,\gamma}^{(m)}\left(-\frac{\alpha_{n-1}(t-s)^{\alpha}}{\alpha_n}\right) - \alpha\frac{\alpha_{n-1}}{\alpha_n}(t-s)^{\alpha} E_{\alpha,\gamma}^{(m+1)}\left(-\frac{\alpha_{n-1}(t-s)^{\alpha}}{\alpha_n}\right)\right],
\end{aligned}$$

and $(m; k_0, \ldots, k_{n-2})$, $k_0, \ldots, k_{n-2} \ge 0$, $m = k_0 + \cdots + k_{n-2}$ are the multinomial coefficients,

$$\alpha = S_n - S_{n-1}, \qquad \beta = S_n + \sum_{j=0}^{n-2}(S_{n-1}-S_j)k_j - S_r + 2, \qquad \gamma = S_n + \sum_{j=0}^{n-2}(S_{n-1}-S_j)k_j.$$

Proof By the Laplace transform of Eq. (1), we get

$$\sum_{k=0}^{n} \alpha_k s^{S_k}\tilde{u}(s) - \sum_{k=0}^{n} \alpha_k s^{S_k-1}u(0) - \sum_{k=0}^{n} \alpha_k s^{S_k-2}u'(0) = \tilde{f}\bigl(s,u(s)\bigr).$$

Now taking the inverse Laplace transform, we obtain

$$
\begin{aligned}
u(t) &= u(0)\sum_{r=0}^{n} L^{-1}\left\{\frac{\alpha_r s^{S_r-1}}{\sum_{k=0}^{n}\alpha_k s^{S_k}}\right\} + u'(0)\sum_{r=0}^{n} L^{-1}\left\{\frac{\alpha_r s^{S_r-2}}{\sum_{k=0}^{n}\alpha_k s^{S_k}}\right\} + L^{-1}\left\{\frac{\tilde{f}(s,u(s))}{\sum_{k=0}^{n}\alpha_k s^{S_k}}\right\} \\
&= u(0)\Biggl\{\sum_{r=0}^{n}\frac{\alpha_r}{\alpha_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k_0+\cdots+k_{n-2}=m}(m;k_0,\ldots,k_{n-2}) \\
&\quad\times\prod_{i=0}^{n-2}\left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} t^{\alpha m+\beta-2}E_{\alpha,\beta-1}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)\Biggr\} \\
&\quad+u'(0)\Biggl\{\sum_{r=0}^{n}\frac{\alpha_r}{\alpha_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k_0+\cdots+k_{n-2}=m}(m;k_0,\ldots,k_{n-2}) \\
&\quad\times\prod_{i=0}^{n-2}\left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} t^{\alpha m+\beta-1}E_{\alpha,\beta}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)\Biggr\} \\
&\quad+\int_0^t\frac{1}{\alpha_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k_0+\cdots+k_{n-2}=m}(m;k_0,\ldots,k_{n-2})\prod_{i=0}^{n-2}\left(\frac{\alpha_i}{\alpha_n}\right)^{k_i} \\
&\quad\times(t-s)^{\alpha m+\gamma-1}E_{\alpha,\gamma}^{(m)}\left(-\frac{\alpha_{n-1}(t-s)^{\alpha}}{\alpha_n}\right)f\bigl(s,u(s)\bigr)\,ds,
\end{aligned}
$$

where $\alpha = S_n - S_{n-1}$, $\beta = S_n + \sum_{j=0}^{n-2}(S_{n-1}-S_j)k_j - S_r + 2$, $\gamma = S_n + \sum_{j=0}^{n-2}(S_{n-1}-S_j)k_j$.

Let $t = T$, we have

$$
u(T) = u(0)\widetilde{A}(T) + u'(0)\widetilde{B}(T) + \int_0^T \widetilde{C}(T)f\bigl(s,u(s)\bigr)\,ds.
$$

In view of the boundary condition $u(0) = u(T) > 0$, we get

$$
\begin{aligned}
u(0) &= \frac{u'(0)\widetilde{B}(T) + \int_0^T \widetilde{C}(T)f(s,u(s))\,ds}{1-\widetilde{A}(T)}, \\
u'(t) &= \int_0^t\frac{1}{\alpha_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k_0+\cdots+k_{n-2}=m}(m;k_0,\ldots,k_{n-2})\prod_{i=0}^{n-2}\left(\frac{\alpha_i}{\alpha_n}\right)^{k_i}(t-s)^{\alpha m+\gamma-2} \\
&\quad\times\left[(\alpha m+\gamma-1)E_{\alpha,\gamma}^{(m)}\left(-\frac{\alpha_{n-1}(t-s)^{\alpha}}{\alpha_n}\right)-\alpha\frac{\alpha_{n-1}}{\alpha_n}(t-s)^{\alpha}E_{\alpha,\gamma}^{(m+1)}\left(-\frac{\alpha_{n-1}(t-s)^{\alpha}}{\alpha_n}\right)\right] \\
&\quad\times f\bigl(s,u(s)\bigr)\,ds \\
&\quad+u'(0)\Biggl\{\sum_{r=0}^{n}\frac{\alpha_r}{\alpha_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k_0+\cdots+k_{n-2}=m}(m;k_0,\ldots,k_{n-2})\prod_{i=0}^{n-2}\left(\frac{\alpha_i}{\alpha_n}\right)^{k_i}t^{\alpha m+\beta-2} \\
&\quad\times\left[(\alpha m+\beta-1)E_{\alpha,\beta}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)-\alpha t^{\alpha}\frac{\alpha_{n-1}}{\alpha_n}E_{\alpha,\beta}^{(m+1)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)\right]\Biggr\} \\
&\quad+u(0)\Biggl\{\sum_{r=0}^{n}\frac{\alpha_r}{\alpha_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{k_0+\cdots+k_{n-2}=m}(m;k_0,\ldots,k_{n-2})\prod_{i=0}^{n-2}\left(\frac{\alpha_i}{\alpha_n}\right)^{k_i}t^{\alpha m+\beta-3} \\
&\quad\times\left[(\alpha m+\beta-2)E_{\alpha,\beta-1}^{(m)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)-\alpha t^{\alpha}\frac{\alpha_{n-1}}{\alpha_n}E_{\alpha,\beta-1}^{(m+1)}\left(-\frac{\alpha_{n-1}t^{\alpha}}{\alpha_n}\right)\right]\Biggr\}.
\end{aligned}
$$

Applying the boundary condition $u'(0) = u'(T)$ to the above equation, we get

$$u'(0) = \frac{u(0)\widetilde{D}(T) + \int_0^T \widetilde{F}(T)f(s,u(s))\,ds}{1 - \widetilde{E}(T)}.$$

Substituting the above value of $u'(0)$, $u(0)$ in $u(t)$, we obtain

$$\begin{aligned} u(t) = {} & \int_0^t \widetilde{C}(t)f\bigl(s,u(s)\bigr)\,ds + \int_0^T \frac{\widetilde{A}(t)\widetilde{C}(T)(1-\widetilde{E}(T)) + \widetilde{A}(t)\widetilde{B}(T)\widetilde{F}(T)}{(1-\widetilde{E}(T))(1-\widetilde{A}(T)) - \widetilde{B}(T)\widetilde{D}(T)} f\bigl(s,u(s)\bigr)\,ds \\ & + \int_0^T \frac{\widetilde{B}(t)\widetilde{C}(T)\widetilde{D}(T) + \widetilde{B}(t)\widetilde{F}(T)(1-\widetilde{A}(T))}{(1-\widetilde{E}(T))(1-\widetilde{A}(T)) - \widetilde{B}(T)\widetilde{D}(T)} f\bigl(s,u(s)\bigr)\,ds. \end{aligned}$$

Hence the proof is over. □

Theorem 3.1 *Boundary value problem* (1)-(2) *has the unique solution if the following conditions hold*:

(C_1) *The function* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $T > 0$ *is continuous*;
(C_2) *There exists* $\phi \in B$ *such that*

$$\bigl|f(t,y) - f(t,x)\bigr| \le \frac{1}{\hat{G}}\phi\bigl(|y-x|\bigr), \quad \forall x,y \in \mathbb{R}.$$

Proof Let $M = C([0,T],\mathbb{R})$. Then (M,d) is a complete metric space, where

$$d(u,v) = \sup_{t\in[0,T]} \bigl|u(t) - v(t)\bigr|.$$

Let the operator

$$F : M \to M, \qquad F(u) = \int_0^T G(t,s)f\bigl(s,u(s)\bigr)\,ds,$$

where $G(t,s)$ is the Green's function corresponding to boundary conditions (2).
For $u \neq v$,

$$\begin{aligned} d\bigl(F(u),F(v)\bigr) &= \sup_{t\in[0,T]} \bigl|Fu(t) - Fv(t)\bigr| \\ &\le \sup_{t\in[0,T]} \int_0^T \bigl|G(t,s)\bigr| \cdot \bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|\,ds \\ &\le \sup_{t\in[0,T]} \int_0^T \bigl|G(t,s)\bigr| \frac{1}{\hat{G}}\phi\bigl(\bigl|u(s)-v(s)\bigr|\bigr)\,ds \\ &\le \phi\bigl(d(u,v)\bigr)\frac{1}{\hat{G}} \sup_{t\in[0,T]} \int_0^T \bigl|G(t,s)\bigr|\,ds \\ &= \phi\bigl(d(u,v)\bigr) = \alpha\bigl(d(u,v)\bigr)d(u,v). \end{aligned}$$

Therefore, there exists $\alpha \in S$ such that $d(Fu,Fv) \le \alpha(d(u,v))d(u,v)$, $\forall u,v \in M$. Thus by Lemma 2.1, F has a unique fixed point. Hence boundary value problem (1)-(2) has the unique solution. □

We can prove the following existence and uniqueness theorems for boundary value problem (1)-(2) (refer to [1]).

Theorem 3.2 *Boundary value problem* (1)-(2) *has at least one solution if the following conditions hold*:

(D_1) *The function* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $T > 0$ *is continuous*;
(D_2) *There exist* $p \in C([0,T], \mathbb{R}^+)$ *and* $\psi : (0,\infty) \to (0,\infty)$ *continuous and nondecreasing such that* $|f(t,v)| \leq p(t)\psi(|v|)$ *for* $t \in [0,T]$ *and* $v \in \mathbb{R}$;
(D_3) *There exists a constant* $M > 0$ *such that* $M > \hat{p}\psi(M)\hat{G}$, *where* $\hat{p} = \sup_{t\in[0,T]}\{p(t)\}$.

Theorem 3.3 *Assume that there exists* $k > 0$ *such that*

$$\bigl|f(t,y) - f(t,x)\bigr| \leq K|y-x|, \quad \forall x,y \in \mathbb{R}, t \in [0,T].$$

If $K\hat{G} < 1$, *then there exists the unique solution for boundary value problem* (1)-(2).

The above analysis can be performed for the fractional differential equations

$$L(D)u(t) = f\bigl(t, u(t)\bigr), \quad t \in [0,T], T > 0, \tag{3}$$

with the general periodic and antiperiodic boundary conditions

$$au(0) + bu(T) = 0, \qquad cu'(0) + du'(T) = 0, \quad a,b,c,d \in \mathbb{R}, \tag{4}$$

where $L(D) = a_n D^{S_n} + a_{n-1}D^{S_{n-1}} + \cdots + a_0 D^{S_0}$, $1 \leq S_0 < \cdots < S_{n-1} < S_n < 2$, $a_i \in \mathbb{R}$, $a_n \neq 0$, D^{S_i} $(i = 1,2,\ldots,n)$ are the standard Caputo fractional derivatives, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R} \to \mathbb{R}$) is a continuous operation.

From Theorems 3.1 and 3.2, the solution of boundary value problem (1)-(2) can be extended to the interval $[0, 2T]$. Let $\tilde{u}$ be the solution of (1)-(2) on $[0,T]$, then by means of $\int_0^T G(t,s)f(s,\tilde{u}(s))\,ds$ is continuous and Lemma 3.1, boundary value problem (1)-(2) has a solution

$$\tilde{\tilde{u}} = \int_0^T G(t,s)f\bigl(s,\tilde{u}(s)\bigr)\,ds + \int_T^{2T} G(t,s)f\bigl(s,u(s)\bigr)\,ds,$$

on $[T, 2T]$.

The pair of functions

$$u(t) = \begin{cases} \tilde{u}(t), & t \in [0,T], \\ \tilde{\tilde{u}}(t), & t \in [T,2T], \end{cases}$$

is the solution of boundary value problem (1)-(2) on $[0, 2T]$. We can continue in the same way until $T \to \infty$.

We focus on the blowing-up solution of the following boundary value problem of a class of multi-order fractional differential equations involving the Caputo derivative:

$$L(D)u(t) = \bigl|u(t)\bigr|^p, \quad t > 0, \tag{5}$$

where $L(D) = a_n {}^{c}D^{S_n} + a_{n-1} {}^{c}D^{S_{n-1}} + \cdots + a_0 {}^{c}D^{S_0}$, $1 \le S_0 < \cdots < S_{n-1} < S_n < 2$, $a_i \ge 0$, with the boundary condition

$$u(0) = u(T) = u_0 > 0, \qquad u'(0) = u'(T) = u'_0. \tag{6}$$

By means of the above analysis and Theorem 3.2, boundary value problem (5)-(6) has a continuous solution.

The relation between the Riemann-Liouville and the Caputo fractional derivatives is

$${}^{c}D^{\alpha} u(t) = {}^{\mathrm{RL}}D^{\alpha}\bigl[u(t) - u(0) - u'(0)t\bigr], \quad 1 \le \alpha < 2.$$

Therefore, boundary problem (5)-(6) is equivalent to the following boundary problem:

$$L(D)\bigl[u(t) - u(0) - u'(0)t\bigr] = \bigl|u(t)\bigr|^p, \quad t > 0, \tag{7}$$

where $L(D) = \sum_{i=0}^{n} a_i {}^{\mathrm{RL}}D^{S_i}$, $1 \le S_0 < \cdots < S_{n-1} < S_n < 2$, $a_i \ge 0$, with the boundary condition

$$u(0) = u(T) = u_0 > 0, \qquad u'(0) = u'(T) = u'_0. \tag{8}$$

Let the test function considered in [16]

$$\varphi(t) = \begin{cases} T^{-\lambda}(T-t)^{\lambda}, & t \in [0,T], \\ 0, & t > T. \end{cases}$$

For $1 \le \alpha < 2$, $\lambda > p\alpha - 1$, it satisfies

$$\begin{aligned} &\int_0^T {}^{\mathrm{RL}}D_{T-}^{\alpha}\varphi(t)\,dt = C_{\alpha,\lambda}T^{1-\alpha}, \qquad C_{\alpha,\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(2-\alpha+\lambda)}, \\ &\int_0^T t \cdot {}^{\mathrm{RL}}D_{T-}^{\alpha}\varphi(t)\,dt = C_{\alpha-1,\lambda}T^{2-\alpha}, \qquad C_{\alpha-1,\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(3-\alpha+\lambda)}, \\ &\int_0^T \varphi^{1-p}(t)\bigl|{}^{\mathrm{RL}}D_{T-}^{\alpha}\varphi(t)\bigr|^p\,dt = C_{p,\alpha}T^{1-p\alpha}, \qquad C_{p,\alpha} = \frac{1}{\lambda - p\alpha + 1}\left[\frac{\Gamma(\lambda+1)}{\Gamma(2-\alpha+\lambda)}\right]^p, \end{aligned}$$

where ${}^{\mathrm{RL}}D_{T-}^{\alpha}$ is the right-sided (RL) fractional derivative defined by

$${}^{\mathrm{RL}}D_{T-}^{\alpha}f(t) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_t^T (s-t)^{1-\alpha}f(s)\,ds, \quad 1 \le \alpha < 2.$$

Theorem 3.4 *Let* $1 < p < \frac{S_n}{S_n - S_0}$ *and* $u_0 > 0$, *then any solution to boundary problem* (7)-(8) *blows up in a finite time* $T_{\max}$. *Furthermore, an upper bound on the blow-up time* $T_{\max}$ *is given by* $(\frac{K}{u_0})^r$, *where* $r = \frac{p-1}{pS_0 - pS_n + S_n}$, $K = n^{q-1} \cdot a_{\max}^{q} \cdot a_{\min}^{-1} C_{q,S_0} C_{S_n,\lambda}^{-1}$, *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof The proof is by contradiction. Suppose $u(t)$ is a global solution of boundary problem (7)-(8).

Multiplying Eq. (7) by the function $\varphi(t)$ and integrating over $[0,T]$, we obtain

$$\sum_{i=0}^{n} a_i \int_0^T \varphi(t) \cdot {}^{\mathrm{RL}}D^{S_i}\bigl[u(t) - u(0) - u'(0)t\bigr]\,dt = \int_0^T \varphi(t) \cdot \bigl|u(t)\bigr|^p\,dt.$$

The formula for the integration by parts in $[0, T]$ is given by (see [9])

$$\int_0^T f(t)^{\mathrm{RL}}D^{\alpha} g(t)\,dt = \int_0^T g(t)^{\mathrm{RL}}D^{\alpha}_{T-} f(t)\,dt. \tag{9}$$

By virtue of (9), we obtain

$$\begin{aligned}
&\sum_{i=0}^{n} a_i \int_0^T u(t) \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt \\
&\quad= \sum_{i=0}^{n} a_i \int_0^T u_0 \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt \\
&\qquad+ \sum_{i=0}^{n} a_i u_0' \int_0^T t \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt + \int_0^T \varphi(t) \cdot \bigl|u(t)\bigr|^p\,dt.
\end{aligned}$$

Using Hölder's inequality, for $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned}
\int_0^T u(t) \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt &\le \left[\int_0^T \bigl|u(t)\bigr|^p \cdot \varphi(t)\,dt\right]^{\frac{1}{p}} \\
&\quad\times \left[\int_0^T \bigl|\varphi(t)\bigr|^{-\frac{q}{p}} \cdot \bigl|{}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\bigr|^q\,dt\right]^{\frac{1}{q}}, && (10)\\
\int_0^T u(t) \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt &\le C^{\frac{1}{q}}_{q,S_i} T^{\frac{1-qS_i}{q}} \left[\int_0^T \bigl|u(t)\bigr|^p \cdot \varphi(t)\,dt\right]^{\frac{1}{p}}. && (11)
\end{aligned}$$

Let $N = \int_0^T |u(t)|^p \cdot \varphi(t)\,dt$, we get

$$\begin{aligned}
&\sum_{i=0}^{n} a_i \int_0^T u(t) \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt \le N^{\frac{1}{p}} \sum_{i=0}^{n} a_i C^{\frac{1}{q}}_{q,S_i} T^{\frac{1-qS_i}{q}}, \\
&\sum_{i=0}^{n} a_i \int_0^T u_0 \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt \le N^{\frac{1}{p}} \sum_{i=0}^{n} a_i C^{\frac{1}{q}}_{q,S_i} T^{\frac{1-qS_i}{q}}, \qquad (12)\\
&\int_0^T \bigl|u(t)\bigr|^p \cdot \varphi(t)\,dt = N \le N^{\frac{1}{p}} \sum_{i=0}^{n} a_i C^{\frac{1}{q}}_{q,S_i} T^{\frac{1-qS_i}{q}},
\end{aligned}$$

then

$$N^{\frac{1}{q}} \le \sum_{i=0}^{n} a_i C^{\frac{1}{q}}_{q,S_i} T^{\frac{1-qS_i}{q}}. \tag{13}$$

By inequalities (10)-(13), we obtain

$$\begin{aligned}
n \cdot a_{\min} \cdot u_0 C_{S_n,\lambda} T^{1-S_n} &\le \sum_{i=0}^{n} a_i \int_0^T u_0 \cdot {}^{\mathrm{RL}}D^{S_i}_{T-}\varphi(t)\,dt \\
&\le \sum_{i=0}^{n} a_i C^{\frac{1}{q}}_{q,S_i} T^{\frac{1-qS_i}{q}} \times \left[\sum_{i=0}^{n} a_i C^{\frac{1}{q}}_{q,S_i} T^{\frac{1-qS_i}{q}}\right]^{\frac{q}{p}}
\end{aligned}$$

$$\begin{aligned} &= \left[\sum_{i=0}^{n} a_i C_{q,S_i}^{\frac{1}{q}} T^{\frac{1-qS_i}{q}}\right]^q \leq \left[n a_{\max} C_{q,S_0}^{\frac{1}{q}} T^{\frac{1-qS_0}{q}}\right]^q \\ &= n^q \cdot a_{\max}^q \cdot C_{q,S_0} T^{1-qS_0}, \end{aligned}$$

where $a_{\min} = \min_{0\leq i\leq n}\{a_i\}$, $a_{\max} = \max_{0\leq i\leq n}\{a_i\}$.

We get

$$u_0 \leq n^{q-1} \cdot a_{\max}^q \cdot a_{\min}^{-1} C_{q,S_0} C_{S_n,\lambda}^{-1} T^{S_n-qS_0}. \tag{14}$$

Letting $T \to \infty$, by (14) we obtain the contradiction $0 < u_0 \leq 0$. To obtain an estimation on the blow-up time,

$$u_0 \leq K T^{S_n-qS_0},$$

where $K = n^{q-1} \cdot a_{\max}^q \cdot a_{\min}^{-1} C_{q,S_0} C_{S_n,\lambda}^{-1}$, and $S_n - qS_0 < 0$.

Therefore, a bound on the blowing-up time is given by

$$T_{\max} \leq \left(\frac{K}{u_0}\right)^{\frac{1}{qS_0-S_n}}.$$

This completed the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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References

1. Benchohra, M, Ouaar, F: Existence results for nonlinear fractional differential equations with integral boundary conditions. Bull. Math. Anal. Appl. **2**(4), 7-15 (2012)
2. Wang, F, Liu, Z: Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order. Adv. Differ. Equ. **2012**, 116 (2012)
3. Bazhlekova, E: Properties of the fundamental and the impulse response solutions of multi-term fractional differential equations. In: Complex Analysis and Applications '13 (Proc. of International Conference), 31 Oct.-2 Nov. 2013, pp. 55-64. Inst. Math. Inform. - Bulg. Acad. Sci., Sofia (2013)
4. Ahmad, B, Nieto, JJ: Anti-periodic fractional boundary value problems with nonlinear term depending on lower order derivative. Fract. Calc. Appl. Anal. **15**(3), 451-462 (2012)
5. Luchko, Y, Gorenflo, R: An operational method for solving fractional differential equations with the Caputo derivatives. Acta Math. Vietnam. **24**, 207-233 (1999)
6. Stojanović, M: Existence-uniqueness result for a nonlinear *n*-term fractional equation. J. Math. Anal. Appl. **353**, 244-255 (2009)
7. Kirane, M, Malik, S: Profile of blowing-up solutions to a nonlinear system of fractional differential equations. Nonlinear Anal., Theory Methods Appl. **73**(12), 3723-3736 (2010)

8. Alsaedi, A, Ahmad, B, Kirane, MBM, Al Musalhi, FSK, Alzahrani, F: Blowing-up solutions for a nonlinear time-fractional system. Bull. Math. Sci. (2016). doi:10.1007/s13373-016-0087-0
9. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
10. Daftardar-Gejji, V: Fractional Calculus: Theory and Applications. Narosa, New Delhi (2013)
11. Jumarie, G: Laplace's transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative. Appl. Math. Lett. **22**, 1659-1664 (2009)
12. Hu, L: Existence of solutions to a coupled system of fractional differential equations with infinite-point boundary value conditions at resonance. Adv. Differ. Equ. **2016**, 200 (2016)
13. He, JH: Some applications of nonlinear fractional differential equations and their approximations. Bull. Sci. Technol. Soc. **15**(2), 86-90 (1999)
14. Liu, SL, Li, HL, Dai, Q, Liu, JP: Existence and uniqueness results for nonlocal integral boundary value problems for fractional differential equations. Adv. Differ. Equ. **2016**, 122 (2016)
15. He, Y: Existence and multiplicity of positive solutions for singular fractional differential equations with integral boundary value conditions. Adv. Differ. Equ. **2016**, 31 (2016)
16. Furati, KM, Kirane, M: Necessary conditions for the existence of global solutions to systems of fractional differential equations. Fract. Calc. Appl. Anal. **11**, 281-298 (2008)