Zhang *Advances in Difference Equations* (2017) 2017:101
DOI 10.1186/s13662-017-1161-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A resonant boundary value problem for the fractional p-Laplacian equation

Bo Zhang*

*Correspondence:
zhbahhb@chnu.edu.cn
School of Mathematical Sciences, Huaibei Normal University, Huaibei, 235000, PR China
Information College, Huaibei Normal University, Huaibei, 235000, PR China

Abstract
The purpose of this paper is to study the solvability of a resonant boundary value problem for the fractional p-Laplacian equation. By using the continuation theorem of coincidence degree theory, we obtain a new result on the existence of solutions for the considered problem.

MSC: 34A08; 34B15

Keywords: resonant boundary value problem; fractional p-Laplacian equation; continuation theorem

1 Introduction

In this paper, we establish an existence theorem of solutions for the following resonant boundary value problem with p-Laplacian operator:

$$\begin{cases} {}_0^cD_t^\beta \phi_p({}_0^cD_t^\alpha x) = f(t, x, {}_0^cD_t^\alpha x), & t \in [0,1], \\ x(0) = 0, \quad {}_0^cD_t^\alpha x(0) = {}_0^cD_t^\alpha x(1), \end{cases} \tag{1.1}$$

where $0 < \alpha, \beta \le 1$ are constants, ${}_0^cD_t^\alpha$ is a Caputo fractional derivative, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $\phi_p : \mathbb{R} \to \mathbb{R}$ is a p-Laplacian operator defined by

$$\phi_p(s) = |s|^{p-2}s \quad (s \neq 0), \qquad \phi_p(0) = 0, \quad p > 1.$$

Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $1/p + 1/q = 1$.

Fractional calculus is a generalization of ordinary differentiation and integration, and fractional differential equations appear in various fields (see [1–4]). Recently, because of the intensive development of fractional calculus theory and its applications, the initial and boundary value problems (BVPs for short) of fractional differential equations have gained popularity (see [5–15] and the references therein).

In [11], by using the coincidence degree theory for Fredholm operators, the authors considered the existence of solutions for BVP (1.1). Notice that ${}_0^cD_t^\beta \phi_p({}_0^cD_t^\alpha)$ is nonlinear, and so it is not a Fredholm operator. Thus there is a gap in the proof of the main result, and we fix this gap in the present paper.

2 Preliminaries

For convenience of the reader, we will introduce some necessary basic knowledge about fractional calculus theory (see [2, 4]).

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$ {}_0I_t^\alpha u = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, $$

provided that the right-hand side integral is pointwise defined in $(0, +\infty)$.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$ \begin{aligned} {}_0^cD_t^\alpha u &= {}_0I_t^{n-\alpha} \frac{\mathrm{d}^n u}{\mathrm{d}t^n} \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds, \end{aligned} $$

where n is the smallest integer greater than or equal to α, provided that the right-hand side integral is pointwise defined in $(0, +\infty)$.

Lemma 2.1 (See [1]) *Let $\alpha > 0$. Assume that $u, {}_0^cD_t^\alpha u \in L([0,T], \mathbb{R})$. Then the following equality holds*:

$$ {}_0I_t^\alpha {}_0^cD_t^\alpha u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, $$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, here n is the smallest integer greater than or equal to α.

Next we present some notations and an abstract existence result (see [16]).

Let X, Y be real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, and $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$ \begin{aligned} &\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L, \\ &X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q. \end{aligned} $$

It follows that

$$ L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L $$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \overline{\Omega} \neq \varnothing$, then the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.2 (See [16]) *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(1) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$,
(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$,
(3) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \operatorname{Ker} Q$.

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

In this paper, we let $Z = C([0,1],\mathbb{R})$ with the norm $\|z\|_\infty = \max_{t\in[0,1]} |z(t)|$ and take

$$X = \{x = (x_1,x_2)^\top | x_1,x_2 \in Z\}$$

with the norm

$$\|x\|_X = \max\{\|x_1\|_\infty, \|x_2\|_\infty\}.$$

By means of the linear functional analysis theory, we can prove that X is a Banach space.

3 Main result

We will establish the existence theorem of solutions for BVP (1.1).

Theorem 3.1 *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous. Assume that*

(H_1) *there exist nonnegative functions* $a,b,c \in Z$ *such that*

$$|f(t,u,v)| \le a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1}, \quad \forall (t,u,v) \in [0,1] \times \mathbb{R}^2,$$

(H_2) *there exists a constant* $B > 0$ *such that*

$$vf(t,u,v) > 0 \ (or\ < 0), \quad \forall t \in [0,1], u \in \mathbb{R}, |v| > B.$$

Then BVP (1.1) *has at least one solution provided that*

$$\gamma := \frac{2}{\Gamma(\beta+1)}\left(\frac{\|b\|_\infty}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_\infty\right) < 1.$$

Consider BVP of the linear differential system as follows:

$$\begin{cases} {}_0^cD_t^\alpha x_1 = \phi_q(x_2), & t \in [0,1], \\ {}_0^cD_t^\beta x_2 = f(t,x_1,\phi_q(x_2)), & t \in [0,1], \\ x_1(0) = 0, & x_2(0) = x_2(1). \end{cases} \tag{3.1}$$

Obviously, if $x = (x_1,x_2)^\top$ is a solution of BVP (3.1), then x_1 must be a solution of BVP (1.1). Therefore, to prove BVP (1.1) has solutions, it suffices to show that BVP (3.1) has solutions.

Define the operator $L : \operatorname{dom} L \subset X \to X$ by

$$Lx = \begin{pmatrix} {}_0^cD_t^\alpha x_1 \\ {}_0^cD_t^\beta x_2 \end{pmatrix}, \tag{3.2}$$

where

$$\operatorname{dom} L = \{x \in X|{}_0^cD_t^\alpha x_1, {}_0^cD_t^\beta x_2 \in Z, x_1(0) = 0, x_2(0) = x_2(1)\}.$$

Let $N : X \to X$ be the Nemytskii operator defined by

$$Nx(t) = \begin{pmatrix} \phi_q(x_2(t)) \\ f(t, x_1(t), \phi_q(x_2(t))) \end{pmatrix}, \quad \forall t \in [0,1]. \tag{3.3}$$

Then BVP (3.1) is equivalent to the following operator equation:

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

Now, in order to prove Theorem 3.1, we give some lemmas.

Lemma 3.1 *Let L be defined by* (3.2), *then*

$$\operatorname{Ker} L = \{x \in X|x_1(t) = 0, x_2(t) = c, \forall t \in [0,1], c \in \mathbb{R}\}, \tag{3.4}$$

$$\operatorname{Im} L = \{y \in X|{}_0I_t^\beta y_2(1) = 0\}. \tag{3.5}$$

Proof By Lemma 2.1, the equation $Lx = 0$ has solutions

$$x_1(t) = c_1, \qquad x_2(t) = c_2, \quad c_1, c_2 \in \mathbb{R}.$$

Thus, from the boundary value condition $x_1(0) = 0$, one has that (3.4) holds.

Let $y \in \operatorname{Im} L$, then there exists a function $x \in \operatorname{dom} L$ such that $y_2 = {}_0^cD_t^\beta x_2$. So, by Lemma 2.1, we have

$$x_2(t) = c + {}_0I_t^\beta y_2(t), \quad c \in \mathbb{R}.$$

Hence, from the boundary value condition $x_2(0) = x_2(1)$, we get (3.5).

On the other hand, suppose that $y \in X$ satisfies ${}_0I_t^\beta y_2(1) = 0$. Let $x_1 = {}_0I_t^\alpha y_1$, $x_2 = {}_0I_t^\beta y_2(t)$, then $x = (x_1, x_2)^\top \in \operatorname{dom} L$ and $Lx = y$. That is, $y \in \operatorname{Im} L$. The proof is complete. □

Lemma 3.2 *Let L be defined by* (3.2), *then L is a Fredholm operator of index zero. And the projectors* $P : X \to X$, $Q : X \to X$ *can be defined as*

$$Px(t) = \begin{pmatrix} 0 \\ x_2(0) \end{pmatrix}, \quad \forall t \in [0,1],$$

$$Qy(t) = \begin{pmatrix} 0 \\ \Gamma(\beta+1){}_0I_t^\beta y_2(1) \end{pmatrix}, \quad \forall t \in [0,1].$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be written as*

$$K_P y = \begin{pmatrix} {}_0I_t^\alpha y_1 \\ {}_0I_t^\beta y_2 \end{pmatrix}.$$

Proof For any $y \in X$, one has

$$\begin{aligned} Q^2 y &= Q\begin{pmatrix} 0 \\ \Gamma(\beta+1){}_0I_t^\beta y_2(1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \Gamma(\beta+1){}_0I_t^\beta y_2(1)\cdot\Gamma(\beta+1){}_0I_t^\beta 1(1) \end{pmatrix} \\ &= Qy. \end{aligned} \tag{3.6}$$

Let $y^* = y - Qy$, then we get from (3.6) that

$$\begin{aligned} {}_0I_t^\beta y_2^*(1) &= {}_0I_t^\beta y_2(1) - {}_0I_t^\beta (Qy_2)(1) \\ &= \frac{1}{\Gamma(\beta+1)}\big((Qy_2)(t) - \big(Q^2 y_2\big)(t)\big) \\ &= 0, \end{aligned}$$

which yields $y^* \in \operatorname{Im} L$. So $X = \operatorname{Im} L + \operatorname{Im} Q$. Since $\operatorname{Im} L \cap \operatorname{Im} Q = \{(0,0)^\top\}$, we have $X = \operatorname{Im} L \oplus \operatorname{Im} Q$. Hence

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L = 1.$$

Thus L is a Fredholm operator of index zero.

For $y \in \operatorname{Im} L$, by the definition of operator K_P, we have

$$\begin{aligned} LK_P y &= \begin{pmatrix} {}_0^cD_t^\alpha\, {}_0I_t^\alpha y_1 \\ {}_0^cD_t^\beta\, {}_0I_t^\beta y_2 \end{pmatrix} \\ &= y. \end{aligned} \tag{3.7}$$

On the other hand, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, one has

$$x_1(0) = x_2(0) = x_2(1) = 0.$$

Thus, from Lemma 2.1, we get

$$\begin{aligned} K_P Lx(t) &= \begin{pmatrix} {}_0I_t^\alpha\, {}_0^cD_t^\alpha x_1(t) \\ {}_0I_t^\beta\, {}_0^cD_t^\beta x_2(t) \end{pmatrix} \\ &= \begin{pmatrix} x_1(t) - x_1(0) \\ x_2(t) - x_2(0) \end{pmatrix} \\ &= x(t). \end{aligned} \tag{3.8}$$

Hence, combining (3.7) with (3.8), we know $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. The proof is complete. □

Lemma 3.3 *Let N be defined by* (3.3). *Assume $\Omega \subset X$ is an open bounded subset such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, *then N is L-compact on $\overline{\Omega}$.*

Proof From the continuity of ϕ_q and f, we obtain $K_P(I-Q)N$ is continuous in X and $QN(\overline{\Omega})$, $K_P(I-Q)N(\overline{\Omega})$ are bounded. Moreover, there exists a constant $T>0$ such that

$$\left\|(I-Q)Nx\right\|_X \leq T, \quad \forall x \in \overline{\Omega}. \tag{3.9}$$

Thus, in view of the Arzelà-Ascoli theorem, we need only to prove $K_P(I-Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1$, $x \in \overline{\Omega}$, one has

$$\begin{aligned}
&\left|K_P(I-Q)Nx(t_2) - K_P(I-Q)Nx(t_1)\right| \\
&\quad = \begin{pmatrix} {}_0I_t^\alpha((I-Q)Nx)_1(t_2) - {}_0I_t^\alpha((I-Q)Nx)_1(t_1) \\ {}_0I_t^\beta((I-Q)Nx)_2(t_2) - {}_0I_t^\beta((I-Q)Nx)_2(t_1) \end{pmatrix}.
\end{aligned}$$

From (3.9), we have

$$\begin{aligned}
&\left|{}_0I_t^\alpha\big((I-Q)Nx\big)_1(t_2) - {}_0I_t^\alpha\big((I-Q)Nx\big)_1(t_1)\right| \\
&\quad = \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}\big((I-Q)Nx\big)_1(s)\,ds \right. \\
&\qquad \left. - \int_0^{t_1}(t_1-s)^{\alpha-1}\big((I-Q)Nx\big)_1(s)\,ds\right| \\
&\quad \leq \frac{T}{\Gamma(\alpha)}\left\{\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\right\} \\
&\quad = \frac{T}{\Gamma(\alpha+1)}\left[t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha\right].
\end{aligned}$$

Since t^α is uniformly continuous on $[0,1]$, we get $(K_P(I-Q)N(\overline{\Omega}))_1 \subset Z$ is equicontinuous. A similar proof can show that $(K_P(I-Q)N(\overline{\Omega}))_2 \subset Z$ is also equicontinuous. Hence, we obtain $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. The proof is complete. □

Finally, we give the proof of Theorem 3.1.

Proof of Theorem 3.1 Let

$$\Omega_1 = \left\{x \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid Lx = \lambda Nx, \lambda \in (0,1)\right\}.$$

For $x \in \Omega_1$, we have $x_1(0) = 0$ and $Nx \in \operatorname{Im} L$. So, by Lemma 2.1, we get

$$x_1 = {}_0I_t^\alpha {}_0^cD_t^\alpha x_1.$$

Thus one has

$$\left|x_1(t)\right| \leq \frac{1}{\Gamma(\alpha+1)}\left\|{}_0^cD_t^\alpha x_1\right\|_\infty, \quad \forall t \in [0,1].$$

That is,

$$\|x_1\|_\infty \leq \frac{1}{\Gamma(\alpha+1)}\left\|{}_0^cD_t^\alpha x_1\right\|_\infty. \tag{3.10}$$

From $Nx \in \operatorname{Im} L$ and (3.5), we obtain

$$\begin{aligned} 0 &= {}_0I_t^\beta (Nx)_2(1) \\ &= \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\bigl(s, x_1(s), \phi_q\bigl(x_2(s)\bigr)\bigr)\, ds. \end{aligned}$$

Then, by the integral mean value theorem, there exists a constant $\xi \in (0,1)$ such that

$$f\bigl(\xi, x_1(\xi), \phi_q\bigl(x_2(\xi)\bigr)\bigr) = 0.$$

So, by (H_2), we have $|x_2(\xi)| \le B^{p-1}$. From Lemma 2.1, we get

$$x_2(t) = x_2(\xi) - {}_0I_t^{\beta}{}_0^cD_t^\beta x_2(\xi) + {}_0I_t^{\beta}{}_0^cD_t^\beta x_2(t),$$

which together with

$$\bigl|{}_0I_t^{\beta}{}_0^cD_t^\beta x_2(t)\bigr| \le \frac{1}{\Gamma(\beta+1)} \bigl\|{}_0^cD_t^\beta x_2\bigr\|_\infty, \quad \forall t \in [0,1]$$

yields

$$\|x_2\|_\infty \le B^{p-1} + \frac{2}{\Gamma(\beta+1)} \bigl\|{}_0^cD_t^\beta x_2\bigr\|_\infty. \tag{3.11}$$

From $Lx = \lambda Nx$, one has

$${}_0^cD_t^\alpha x_1 = \lambda \phi_q(x_2), \tag{3.12}$$

$${}_0^cD_t^\beta x_2 = \lambda f\bigl(t, x_1, \phi_q(x_2)\bigr). \tag{3.13}$$

By (3.12), we have

$$\bigl\|{}_0^cD_t^\alpha x_1\bigr\|_\infty \le \|x_2\|_\infty^{q-1},$$

which together with (3.10) yields

$$\|x_1\|_\infty \le \frac{1}{\Gamma(\alpha+1)} \|x_2\|_\infty^{q-1}. \tag{3.14}$$

By (3.13) and (H_1), we obtain

$$\bigl\|{}_0^cD_t^\beta x_2\bigr\|_\infty \le \|a\|_\infty + \|b\|_\infty \|x_1\|_\infty^{p-1} + \|c\|_\infty \|x_2\|_\infty,$$

which together with (3.11) and (3.14) yields

$$\begin{aligned} \bigl\|{}_0^cD_t^\beta x_2\bigr\|_\infty &\le \|a\|_\infty + \frac{\Gamma(\beta+1)\gamma}{2} \|x_2\|_\infty \\ &\le \|a\|_\infty + \frac{\Gamma(\beta+1)\gamma B^{p-1}}{2} + \gamma \bigl\|{}_0^cD_t^\beta x_2\bigr\|_\infty. \end{aligned} \tag{3.15}$$

Since $\gamma < 1$, we get from (3.15) that there exists a constant $M_0 > 0$ such that

$$\left\| {}_0^c D_t^\beta x_2 \right\|_\infty \leq M_0.$$

Thus, combining (3.11) with (3.14), we have

$$\|x_2\|_\infty \leq B^{p-1} + \frac{2M_0}{\Gamma(\beta+1)} := M_1,$$
$$\|x_1\|_\infty \leq \frac{M_1^{q-1}}{\Gamma(\alpha+1)} := M_2.$$

Hence

$$\|x\|_X \leq \max\{M_1, M_2\} := M,$$

which means Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \operatorname{Ker} L | Nx \in \operatorname{Im} L\}.$$

For $x \in \Omega_2$, we have ${}_0I_t^\beta (Nx)_2(1) = 0$ and $x_1(t) = 0$, $x_2(t) = c$, $c \in \mathbb{R}$. Thus one has

$$\int_0^1 (1-s)^{\beta-1} f\big(s, 0, \phi_q(c)\big)\, ds = 0,$$

which together with (H_2) yields $|c| \leq B^{p-1}$. Hence

$$\|x\|_X \leq \max\{0, B^{p-1}\} = B^{p-1},$$

which means Ω_2 is bounded.

By (H_2), one has

$$\phi_p(v) f(t,u,v) > 0, \quad \forall t \in [0,1], u \in \mathbb{R}, |v| > B \tag{3.16}$$

or

$$\phi_p(v) f(t,u,v) < 0, \quad \forall t \in [0,1], u \in \mathbb{R}, |v| > B. \tag{3.17}$$

When (3.16) is true, let

$$\Omega_3 = \big\{x \in \operatorname{Ker} L | \lambda x + (1-\lambda) QNx = 0, \lambda \in [0,1]\big\}.$$

For $x \in \Omega_3$, we have $x_1(t) = 0$, $x_2(t) = c$, $c \in \mathbb{R}$ and

$$\lambda c + (1-\lambda)\beta \int_0^1 (1-s)^{\beta-1} f\big(s, 0, \phi_q(c)\big)\, ds = 0. \tag{3.18}$$

If $\lambda = 0$, we get from (3.16) that $|c| \leq B^{p-1}$. If $\lambda \in (0,1]$, we assume $|c| > B^{p-1}$. Thus, by (3.16), we obtain

$$\lambda c^2 + (1-\lambda)\beta \int_0^1 (1-s)^{\beta-1} \phi_p\bigl(\phi_q(c)\bigr) f\bigl(s, 0, \phi_q(c)\bigr)\, ds > 0,$$

which contradicts (3.18). Hence, Ω_3 is bounded.

When (3.17) is true, let

$$\Omega_3' = \bigl\{x \in \operatorname{Ker} L | -\lambda x + (1-\lambda) QNx = 0, \lambda \in [0,1]\bigr\}.$$

A similar proof can show Ω_3' is also bounded.

Set

$$\Omega = \bigl\{x \in X | \|x\|_X < \max\bigl\{M, B^{p-1}\bigr\} + 1\bigr\}.$$

Clearly, $\Omega_1 \cup \Omega_2 \cup \Omega_3 \subset \Omega$ (or $\Omega_1 \cup \Omega_2 \cup \Omega_3' \subset \Omega$). It follows from Lemma 3.2 and 3.3 that L (defined by (3.2)) is a Fredholm operator of index zero and N (defined by (3.3)) is L-compact on $\overline{\Omega}$. Moreover, based on the above proof, the conditions (1) and (2) of Lemma 2.2 are satisfied. Define the operator $H : \overline{\Omega} \times [0,1] \to X$ by

$$H(x, \lambda) = \pm\lambda x + (1-\lambda) QNx.$$

Then, from the above proof, we have

$$H(x, \lambda) \neq 0, \quad \forall x \in \partial\Omega \cap \operatorname{Ker} L.$$

Thus, by the homotopy property of degree, we get

$$\begin{aligned} \deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg\bigl(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg\bigl(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\bigr) \\ &= \deg(\pm I, \Omega \cap \operatorname{Ker} L, 0) \\ &\neq 0. \end{aligned}$$

Hence, condition (3) of Lemma 2.2 is also satisfied.

Therefore, by using Lemma 2.2, the operator equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. Namely, BVP (1.1) has at least one solution in X. The proof is complete. □

Competing interests
The author declares that he has no competing interests.

Acknowledgements
This work was supported by the Natural Science Research Foundation of Colleges and Universities in Anhui Province (KJ2016A648).

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 17 February 2017 Accepted: 23 March 2017 Published online: 04 April 2017

References

1. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Podlubny, I: Fractional Differential Equation. Academic Press, San Diego (1999)
3. Sabatier, J, Agrawal, OP, Machado, JAT (eds.): Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
4. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
5. Agarwal, RP, O'Regan, D, Stanek, S: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57-68 (2010)
6. Bai, Z, Lü, H: Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Anal. Appl. **311**, 495-505 (2005)
7. Benchohra, M, Hamani, S, Ntouyas, SK: Boundary value problems for differential equations with fractional order and nonlocal conditions. Nonlinear Anal. TMA **71**, 2391-2396 (2009)
8. Chen, T, Liu, W, Yang, C: Antiperiodic solutions for Lienard-type differential equation with *p*-Laplacian operator. Bound. Value Probl. **2010**, 194824 (2010)
9. Chen, T, Liu, W, Hu, Z: A boundary value problem for fractional differential equation with *p*-Laplacian operator at resonance. Nonlinear Anal. TMA **75**, 3210-3217 (2012)
10. Chen, T, Liu, W: An anti-periodic boundary value problem for the fractional differential equation with a *p*-Laplacian operator. Appl. Math. Lett. **25**, 1671-1675 (2012)
11. Chen, T, Liu, W, Liu, J: Existence of solutions for some boundary value problems of fractional *p*-Laplacian equation at resonance. Bull. Belg. Math. Soc. Simon Stevin **20**, 503-517 (2013)
12. Darwish, MA, Ntouyas, SK: On initial and boundary value problems for fractional order mixed type functional differential inclusions. Comput. Math. Appl. **59**, 1253-1265 (2010)
13. El-Shahed, M, Nieto, JJ: Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order. Comput. Math. Appl. **59**, 3438-3443 (2010)
14. Jiang, W: The existence of solutions to boundary value problems of fractional differential equations at resonance. Nonlinear Anal. TMA **74**, 1987-1994 (2011)
15. Su, X: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. **22**, 64-69 (2009)
16. Mawhin, J: Topological Degree Methods in Nonlinear Boundary Value Problems. CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence (1979)