Zhou et al. *Advances in Difference Equations* (2017) 2017:52
DOI 10.1186/s13662-017-1107-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Research on the Poincaré center-focus problem of some cubic differential systems by using a new method

Zhengxin Zhou*, Fangfang Mao and Yuexin Yan

*Correspondence:
zxzhou@yzu.edu.cn
School of Mathematical Sciences,
Yangzhou University, Yangzhou,
China

Abstract

In this article, we discuss the Poincaré center-focus problem of some cubic differential systems by using a new method (Mironenko's method) and obtain some new sufficient conditions for a critical point to be a center. By using this method we not only solve a center-focus problem, but also at the same time, we open a class of differential systems, which do not have to be polynomial differential systems, with the same qualitative behavior at the critical point.

MSC: 34A12; 34A34; 34C14

Keywords: cubic system; Poincaré center-focus problem; Mironenko's method; reflecting function

1 Introduction and preliminaries

Consider the cubic system

$$\begin{cases} x' = p_1(x,y) + p_2(x,y) + p_3(x,y), \\ y' = q_1(x,y) + q_2(x,y) + q_3(x,y), \end{cases} \tag{1.1}$$

where $p_i(x,y)$ and $q_i(x,y)$ are homogeneous polynomials in x and y of degree i ($i = 1,2,3$). Taking $x = r\cos\theta$, $y = r\sin\theta$, then (1.1) becomes

$$\frac{dr}{d\theta} = \frac{\sum_{i=0}^{2} b_i(\theta)r^i}{\sum_{i=0}^{2} a_i(\theta)r^i} r, \tag{1.2}$$

where $a_i(\theta)$ and $b_i(\theta)$ ($i = 0,1,2,\ldots,n$) are polynomials in $\cos\theta$ and $\sin\theta$. By [1–5], we know that the origin $(0,0)$ of (1.1) is a center, if and only if equation (1.2) has a center at $r = 0$, *i.e.*, all the solutions nearby $r = 0$ are 2π-periodic.

As usually, we often apply the method of Lyapunov and Poincaré to study center-focus problem. A new technique recently developed by Christopher and Sadowskii and others. Polynomial ideals and invariant algebraic curves are sought and appropriate Dulac functions constructed [1–6]. But for high-order polynomial systems and general planar systems, to give the center conditions is very difficult. Necessary and sufficient conditions are known for very few classes of systems. There are well-known conditions for quadratic systems and the problem has been resolved for systems in which P and Q are cubic polynomials without quadratic terms and other some particular forms. In this paper, we will apply the Mironenko's (reflecting function method) method to study the Poincaré center-problem of the certain cubic differential systems which are in the general form.

Now, we simply introduce the concept of the reflecting function, which will be used throughout the rest of this article.

Consider the differential system

$$x' = X(t,x) \quad \bigl(t \in I \subset R, x \in D \subset R^n, 0 \in I\bigr), \tag{1.3}$$

which has a continuously differentiable right-hand side and general solution $\phi(t;t_0,x_0)$.

Definition 1.1 For system (1.3), $F(t,x) := \phi(-t,t,x)$ is called its *reflecting function* [7].

By this, for any solution $x(t)$ of (1.3), we have $F(t,x(t)) = x(-t)$, $F(0,x) = x$ and $F(t,x)$ is a reflecting function of system (1.3) if and only if it is a solution of the Cauchy problem

$$F_t + F_x X(t,x) + X(-t,F) = 0, \qquad F(0,x) = x. \tag{1.4}$$

Theorem 1.2 ([7]) *If system* (1.3) *is* 2ω*-periodic with respect to* t, *and* $F(t,x)$ *is its reflecting function*, *then* $T(x) := F(-\omega,x)$ *is the Poincaré mapping of* (1.3) *over the period* $[-\omega,\omega]$, *and the solution* $x = \phi(t;-\omega,x_0)$ *of* (1.3) *defined on* $[-\omega,\omega]$ *is* 2ω*-periodic if and only if* x_0 *is a fixed point of* $T(x)$.

From this theorem, we know that if system (1.3) and its reflecting function $F(t,x)$ are 2ω-periodic with respect to t, then all the solutions of (1.3) defined on $[-\omega,\omega]$ are 2ω-periodic.

Definition 1.3 If the reflecting functions of two differential systems coincide in their common domain, then these systems are said to be *equivalent* [7].

Theorem 1.4 ([7]) *If* $F(t,x)$ *is the reflecting function of* (1.3), *then one system is equivalent to* (1.3) *if and only if this system can be expressed as follows*:

$$x' = X(t,x) + F_x^{-1} G(t,x) - G\bigl(-t,F(t,x)\bigr), \tag{1.5}$$

where $G(t,x)$ *is an arbitrary vector function such that the every solution of the system* (1.5) *is uniquely determined by its initial conditions*.

All the equivalent 2ω-periodic systems have a common Poincaré mapping over the period $[-\omega,\omega]$, and the qualitative behavior of the periodic solutions of these systems are the same. By this one can study the qualitative behavior of the solutions of a complicated system by using a simple differential system [7].

There are many papers which are also devoted to investigations of qualitative behavior of solutions of differential systems by help of reflecting functions [7–16].

Mironenko's method [7] was introduced and applied to the study of Poincaré center-focus problem in [13], in which the author has stated in detail how to apply this method to calculate the focus quantities and derive the center conditions and use the reflecting function of a particular differential system to study the qualitative behavior of the periodic solutions of its equivalent systems. In this paper we observe that any differential equation that takes the form of (1.2) has an analytic reflecting function and the calculation can be carried out easily. Thus by Mironenko's method, the center-focus problem of a wide range of systems that are equivalent to (1.2) can be solved automatically. To be more specific, we will study the center-focus problem of some cubic differential systems (1.1) by looking for the rational reflecting function of (1.2). We will give a set of new sufficient conditions under which the critical point of (1.1) is a center. Meanwhile, we discover a class of differential systems, which is equivalent to (1.1) but not necessarily polynomial, with the same character at its critical point.

Lemma 1.5

(1) *If* $F(t,x) = \sum_{i=0}^{n} f_i(t)x^i$ *is a reflecting function of one differential system, then* $n = 1$*, i.e.,* $F = f_0(t) + f_1(t)x$*.*

(2) *If* $F(t,x) = \frac{Q(t,x)}{P(t,x)}$ *is a reflecting function of one differential system,* P, Q *are relatively prime polynomials* ($(P,Q) = 1$), $P = \sum_{i=0}^{n} p_i(t)x^i$ ($n \geq 1$, $p_n p_0 \neq 0$), $Q = \sum_{j=0}^{m} q_j(t)x^j$, $p_i(t)$, $q_j(t)$ *are continuously differentiable functions.*

Then $m = n = 1$*, i.e.,* $F = \frac{q_0(t)+q_1(t)x}{p_0(t)+p_1(t)x}$.

Proof (1) If $F = \sum_{i=0}^{n} f_i(t)x^i$ is a reflecting function of one differential system, by [7], we have $F(-t, F(t,x)) \equiv x$, *i.e.*,

$$x = \sum_{i=0}^{n} f_i(-t)F^i(t,x),$$

it implies

$$x - f_0(-t) = F(t,x)\sum_{i=1}^{n} f_i(-t)F^{i-1}(t,x),$$

from this we get $F(t,x) | x - f_0(-t)$ ($x - f_0(t)$ is divisible by $F(t,x)$), thus $n = 1$, *i.e.*, $F(t,x) = f_0(t) + f_1(t)x$.

(2) If $F(t,x) = \frac{Q(t,x)}{P(t,x)}$ is a reflecting function of one differential system, by [7], we get $F(-t, F(t,x)) \equiv x$, *i.e.*,

$$x\left(\sum_{i=0}^{n} \bar{p}_i P^{n-i} Q^i\right) = P^{n-m}\left(\sum_{i=0}^{m} \bar{q}_i P^{m-i} Q^i\right), \tag{1.6}$$

where $\bar{p}_i = p_i(-t)$, $\bar{q}_i = q_i(-t)$.

Case 1. If $n > m$, from (1.6) we get

$$P\left[P^{n-m-1}\left(\sum_{i=0}^{m} \bar{q}_i P^{m-i} Q^i\right) - x\left(\sum_{i=0}^{n-1} \bar{p}_i P^{n-1-i} Q^i\right)\right] = x\bar{p}_n Q^n,$$

it implies $P | \bar{p}_n x Q^n$. Since $(P,Q) = 1$ and $p_0 \neq 0$, so $P | \bar{p}_n$, *i.e.*, $n = 0$, this is contradiction with $n \geq 1$. Therefore, $n \leq m$.

Case 2. If $n < m$, from (1.6) we have

$$P\left[P^{m-n-1}x\left(\sum_{i=0}^{n}\bar{p}_iP^{n-i}Q^i\right)-\sum_{i=0}^{m-1}\bar{q}_iP^{m-1-i}Q^i\right]=\bar{q}_mQ^m,$$

which yields $P|\bar{q}_mQ^m$, as $(P,Q)=1$, so $P|\bar{q}_m$, *i.e.*, $n=0$, this is contradiction with $n\geq 1$. Thus $m=n$.

Case 3. If $m=n$, from (1.6), we get

$$x\big(\bar{p}_0P^n+\bar{p}_1P^{n-1}Q+\cdots+\bar{p}_nQ^n\big)=\bar{q}_0P^n+\bar{q}_1P^{n-1}Q+\cdots+\bar{q}_nQ^n. \tag{1.7}$$

For the coefficients of the degree n^2+1 of x on both sides of (1.7) we have

$$\bar{p}_0p_n^n+\bar{p}_1p_n^{n-1}q_n+\cdots+\bar{p}_nq_n^n=0,$$

i.e.,

$$\bar{p}_0+\bar{p}_1\frac{q_n}{p_n}+\cdots+\bar{p}_n\left(\frac{q_n}{p_n}\right)^n=0,$$

it implies $P=(x-\frac{\bar{q}_n}{\bar{p}_n})\tilde{P}$, $\tilde{P}$ is a polynomial of degree $n-1$ with respect to x.

From (1.7) follows

$$\sum_{i=0}^{n}(\bar{p}_ix-\bar{q}_i)P^{n-i}Q^i=0,$$

by this we get

$$P\sum_{i=0}^{n-1}(\bar{p}_ix-\bar{q}_i)P^{n-1-i}Q^i=-(\bar{p}_nx-\bar{q}_n)Q^n,$$

which implies $P|(\bar{p}_nx-\bar{q}_n)Q^n$, and so $\tilde{P}|Q^n$. As, $(P,Q)=1$, thus $\partial\tilde{P}=0$, and $\partial P=1$, *i.e.*, $n=1$.

The proof is completed. □

By Lemma 1.5, to seek the analytic rational reflecting function of (1.2), only need to search the reflecting function in the form of $F=f_0(t)+f_1(t)x$ and $F=\frac{q_0(t)+q_1(t)x}{p_0(t)+p_1(t)x}$. Thus, in this paper, we are interested in when the differential equation (1.2) has such reflecting function, and how to apply these reflecting functions to study the centre-focus problem of the cubic system (1.1) and their equivalent systems.

In the following, all the differential systems have been discussed which have a continuously differentiable right-hand side and have a unique solution for their initial value problem in a neighborhood of the origin.

2 Main results

Now, let us consider the general cubic system

$$\begin{cases} x'=-y+a_{20}x^2+a_{11}xy+a_{02}y^2+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3,\\ y'=x+b_{20}x^2+b_{11}xy+b_{02}y^2+b_{30}x^3+b_{21}x^2y+b_{12}xy^2+b_{03}y^3, \end{cases} \tag{2.1}$$

where a_{ij}, b_{ij} $(i,j=0,1,2,3)$ are constants.

Setting $x = r\cos\theta$, $y = r\sin\theta$ in (2.1), we obtain

$$\frac{dr}{d\theta} = \frac{b_1 + b_2 r}{1 + a_1 r + a_2 r^2} r^2 = R(\theta, r), \tag{2.2}$$

where

$$\begin{cases} a_1 = b_{20}C^3 + (b_{11} - a_{20})C^2 S + (b_{02} - a_{11})CS^2 - a_{02}S^3, \\ b_1 = a_{20}C^3 + (a_{11} + b_{20})C^2 S + (a_{02} + b_{11})CS^2 + b_{02}S^3, \\ a_2 = b_{30}C^4 + (b_{21} - a_{30})C^3 S + (b_{12} - a_{21})C^2 S^2 + (b_{03} - a_{12})CS^3 - a_{03}S^4, \\ b_2 = a_{30}C^4 + (a_{21} + b_{30})C^3 S + (a_{12} + b_{21})C^2 S^2 + (a_{03} + b_{12})CS^3 + b_{03}S^4, \end{cases} \tag{2.3}$$

where $C := \cos\theta$, $S := \sin\theta$.

In the following, we will denote $\bar{a}_i = a_i(-\theta)$, $\bar{b}_i = b_i(-\theta)$, $\overline{\phi(\theta)} = \phi(-\theta)$.

Theorem 2.1 *$F = f_0(\theta) + f_1(\theta)r$ is the reflecting function of* (2.2), *if and only if*

$$\begin{cases} b_1 + \overline{b_1} = 0, \\ b_2\bar{a}_2 + \overline{b_2\bar{a}_2} = 0, \\ b_2 + b_1\bar{a}_1 + \overline{b_2 + b_1\bar{a}_1} = 0, \\ b_1\bar{a}_2 + b_2\bar{a}_1 + \overline{b_1\bar{a}_2 + b_2\bar{a}_1} = 0. \end{cases} \tag{2.4}$$

and $f_0 = 0, f_1 = 1$. Therefore, $r = 0$ is a center of (2.2).

Proof By equation (1.4), we know $F = f_0(\theta) + f_1(\theta)r$ is the reflecting function of (2.2), if and only if

$$\begin{aligned} &f_0' + f_1' r + f_1 R(\theta, r) + R(-\theta, f_0 + f_1 r) = 0, \\ &f_0(0) = 0, \qquad f_1(0) = 1. \end{aligned} \tag{2.5}$$

Taking $r = 0$, it implies that

$$f_0' = -\frac{\bar{b}_1 + \bar{b}_2 f_0}{1 + \bar{a}_1 f_0 + \bar{a}_2 f_0^2} f_0^2, \qquad f_0(0) = 0.$$

By the uniqueness of solutions for initial value problems of the above differential equation, it yields $f_0(\theta) = 0$. Substituting it into (2.5) we have

$$f_1' r + f_1 R(\theta, r) + R(-\theta, f_1 r) = 0, \qquad f_1(\theta) = 1. \tag{2.6}$$

Equating the coefficients of the same power of r on both sides of (2.6), we obtain (2.4). So, $F = r$ is the reflecting function of (2.2). By Theorem 1.2, all the solutions of (2.2) nearby $r = 0$ are 2π-periodic, thus, the $r = 0$ is a center. □

Applying Theorem 1.4 we have the following.

Corollary 2.2 *If all the conditions of Theorem* 2.1 *are satisfied, then for the equation*

$$\frac{dr}{d\theta} = \frac{b_1 + b_2 r}{1 + a_1 r + a_2 r^2} r^2 + G(\theta, r) - G(-\theta, r),$$

$r = 0$ is a center. Here $G(\theta, r)$ is an arbitrary continuously differentiable vector function and such that $G(\theta, 0) - G(-\theta, 0) = 0$.

It is not difficult to prove the following lemma by using the method of mathematical induction.

Lemma 2.3 *If $\sum_{i+j=n} a_{ij} \cos^i \theta \sin^j \theta \equiv 0$, $\theta \in R$, then $a_{ij} = 0$ $(i, j = 0, 1, 2, \ldots, n)$. Here a_{ij} are constants.*

Theorem 2.4 *If*

$$\begin{cases} a_{20} = a_{30} = a_{03}b_{03} = a_{02} + b_{11} = 0, \\ a_{12}b_{20} - a_{11}b_{21} + a_{02}a_{21} + a_{02}b_{30} = 0, \\ a_{12}b_{02} + a_{02}a_{03} + a_{02}b_{12} - a_{11}b_{03} = 0, \\ a_{12} + b_{21} - b_{11}(a_{11} + b_{20}) = b_{03} + a_{02}b_{02} = 0, \\ a_{12}b_{12} - a_{03}b_{21} - a_{21}b_{03} = a_{12}b_{30} - a_{21}b_{21} = 0, \end{cases} \tag{2.7}$$

then the cubic system (2.1) *has a center at* $(0, 0)$.

Proof Since the even part of an odd function is equal to zero, using (2.3) and the first relation of (2.4) and Lemma 2.3 we get

$$a_{20} = 0, \qquad b_{11} + a_{02} = 0. \tag{2.8}$$

Using the second relation of (2.4) and simplifying we obtain

$$a_{30} = 0, \qquad b_{03} + b_{02}a_{02} = 0, \qquad a_{12} + b_{21} = b_{11}(a_{11} + b_{20}). \tag{2.9}$$

Applying the third relation of (2.4) and Lemma 2.3 we have

$$a_{12}b_{20} - a_{11}b_{21} + a_{02}a_{21} + a_{02}b_{30} = 0, \qquad a_{12}b_{02} + a_{02}a_{03} + a_{02}b_{12} - a_{11}b_{03} = 0. \tag{2.10}$$

By the fourth relation of (2.4) and simplifying, we get

$$a_{12}b_{30} - a_{21}b_{21} = 0, \qquad a_{12}b_{12} - a_{03}b_{21} - a_{21}b_{03} = 0, \qquad b_{03}a_{03} = 0. \tag{2.11}$$

It follows from (2.8)-(2.11) that equation (2.7) is true. Thus, it follows that equation (2.7) holds. Due to Theorem 2.1, $F = r$ is the reflecting function of (2.2), by Theorem 1.2, the origin point $(0, 0)$ of (2.1) is a center. Thus, the proof is finished. □

Simplifying equation (2.7), we can get the equivalent theorem as following.

Theorem 2.4′ *The origin point* $(0, 0)$ *of the cubic system* (2.1) *is a center, if* $a_{20} = a_{30} = 0$ *and one of the following conditions is satisfied*:

(I) $b_{02} = b_{03} = 0, \qquad b_{20} = -a_{11}, \qquad b_{11} = -a_{02},$

$b_{30} = -a_{21}, \qquad b_{21} = -a_{12}, \qquad b_{12} = -a_{03};$

(II) $a_{02} = a_{12} = 0, \qquad b_{03} = b_{11} = b_{21} = 0;$

$$\text{(III)} \quad \begin{cases} a_{03} = 0, \qquad a_{12} \neq 0, \qquad a_{12}^2 - a_{21}a_{02}^2 + a_{02}a_{12}a_{11} = 0, \\ b_{11} = -a_{02}, \qquad b_{03} = -a_{02}b_{02}, \\ b_{12} = -\frac{a_{21}a_{02}}{a_{12}}b_{02}, \qquad b_{21} = -\frac{a_{02}^2 a_{21}}{a_{12}} - a_{02}b_{20}, \qquad b_{30} = -\frac{a_{21}^2 a_{02}^2}{a_{12}^2} - \frac{a_{02}a_{21}}{a_{12}}b_{20}; \end{cases}$$

$$\text{(IV)} \quad \begin{cases} a_{03} = a_{12} = a_{21} = 0, \qquad b_{12} = -a_{11}b_{02}, \qquad b_{03} = -a_{02}b_{02}, \qquad b_{11} = -a_{02}, \\ b_{21} = -a_{02}(a_{11} + b_{20}), \qquad b_{30} = -a_{11}(a_{11} + b_{20}). \end{cases}$$

From Theorem 2.4′, the origin point $(0,0)$ is a center of system (2.1), if it can be expressed in one of the following forms:

$$\begin{cases} x' = y(-1 + a_{11}x + a_{02}y + a_{21}x^2 + a_{12}xy + a_{03}y^2), \\ y' = -x(-1 + a_{11}x + a_{02}y + a_{21}x^2 + a_{12}xy + a_{03}y^2); \end{cases}$$

$$\begin{cases} x' = y(-1 + a_{11}x + a_{21}x^2 + a_{03}y^2), \\ y' = x(1 + b_{20}x + b_{30}x^2 + b_{12}y^2) + b_{02}y^2; \end{cases}$$

$$\begin{cases} x' = y(-1 + a_{11}x + a_{02}y + a_{21}x^2 + a_{12}xy), \\ y' = (1 - a_{02}y - \frac{a_{02}a_{21}}{a_{12}}x)(x + (b_{20} + \frac{a_{02}a_{21}}{a_{12}})x^2 + b_{02}y^2); \end{cases}$$

$$\begin{cases} x' = y(-1 + a_{11}x + a_{02}y), \\ y' = (1 - a_{11}x - a_{02}y)(x + (b_{20} + a_{11})x^2 + b_{02}y^2). \end{cases}$$

Theorem 2.5 *The fractional function* $F = \frac{\beta_0(\theta)+\beta_1(\theta)r}{1+\alpha(\theta)r}$ ($\alpha(0) = 0$, $\beta_0(0) = 0$, $\beta_1(0) = 1$) *is a reflecting function of* (2.2), *if and only if*,

$$\begin{cases} b_2 - b_1a_1 + \overline{b_2 - b_1a_1} = 0; \\ (\frac{1}{2}\alpha + \bar{a}_1)(b_2 - b_1a_1) - b_1a_2 + \overline{(\frac{1}{2}\alpha + \bar{a}_1)(b_2 - b_1a_1) - b_1a_2} = 0; \\ b_2\bar{a}_2 - b_1a_2\alpha - (b_1 + \bar{b}_1)a_1\bar{a}_2 + \overline{b_2\bar{a}_2 - b_1a_2\alpha - (b_1 + \bar{b}_1)a_1\bar{a}_2} = 0; \\ b_1\alpha^2 + \bar{a}_2(b_1 + \bar{b}_1) - (\bar{b}_2 - (b_1 + \bar{b}_1)\bar{a}_1)\alpha = 0, \end{cases} \tag{2.12}$$

and $\beta_0 = 0$, $\beta_1 = 1$, $\alpha = \int_0^\theta (b_1 + \bar{b}_1)\,d\theta$. *Besides, if* $\int_0^\pi (b_1 + \bar{b}_1)\,d\theta = 0$, *then* $r = 0$ *is a center of* (2.2); *If* $\int_0^\pi (b_1 + \bar{b}_1)\,d\theta \neq 0$, *then equation* (2.2) *has only one* 2π*-periodic solution, i.e.*, $r = 0$.

Proof By (1.4), $F = \frac{\beta_0(\theta)+\beta_1(\theta)r}{1+\alpha(\theta)r}$ is the reflecting function of (2.2), if and only if

$$\begin{aligned} &\beta_0' + (\beta_0'\alpha + \beta_1 - \alpha'\beta_0)r - \alpha'\beta_1r^2 + (\beta_1 - \alpha\beta_0)\frac{b_1 + b_2r}{1 + a_1r + a_2r^2}r^2 \\ &\quad + \frac{(\bar{b}_1(1 + \alpha r) + \bar{b}_2(\beta_0 + \beta_1r))(1 + \alpha r)(\beta_0 + \beta_1r)^2}{(1 + \alpha r)^2 + \bar{a}_1(\beta_0 + \beta_1r)(1 + \alpha r) + \bar{a}_2(\beta_0 + \beta_1r)^2} = 0, \\ &\alpha(0) = 0, \qquad \beta_0(0) = 0, \qquad \beta_1(0) = 1. \end{aligned} \tag{2.13}$$

Taking $r = 0$ in (2.13) we get

$$\beta_0' + \frac{\bar{b}_1 + \bar{b}_2\beta_0}{1 + \bar{a}_1\beta_0 + \bar{a}_2\beta_0^2}\beta_0^2 = 0, \qquad \beta_0(0) = 0.$$

By the uniqueness of solutions for initial value problems of the above differential equation, it yields $\beta_0(\theta) = 0$. Substituting it into (2.13) and equating the coefficients of the same power of r on both sides of (2.13), we obtain

$$\beta_1'(\theta) = 0, \qquad \beta_1(0) = 1; \tag{2.14}$$

$$\alpha' = b_1 + \bar{b}_1, \qquad \alpha(0) = 0; \tag{2.15}$$

$$\alpha'(a_1 + \bar{a}_1 + 2\alpha) = b_1(2\alpha + \bar{a}_1) + b_2 + \bar{b}_1 a_1 + 2\alpha\bar{b}_1 + \bar{b}_2; \tag{2.16}$$

$$\begin{aligned}&\alpha'\bigl(\alpha^2 + \alpha\bar{a}_1 + \bar{a}_2 + a_2 + 2a_1\alpha + a_1\bar{a}_1\bigr)\\&\quad = b_1\bigl(\alpha^2 + \bar{a}_1\alpha + \bar{a}_2\bigr) + b_2(2\alpha + \bar{a}_1) + 2\alpha\bar{b}_1 a_1 + \bar{b}_2 a_1 + \bar{b}_1 a_2 + \bar{b}_1\alpha^2 + \bar{b}_2\alpha;\end{aligned} \tag{2.17}$$

$$\begin{aligned}&\alpha'\bigl(a_1\alpha^2 + a_1\bar{a}_1\alpha + a_1\bar{a}_2 + 2\alpha a_2 + \bar{a}_1 a_2\bigr)\\&\quad = b_2\bigl(\alpha^2 + \bar{a}_1\alpha + \bar{a}_2\bigr) + 2\alpha\bar{b}_1 a_2 + \bar{b}_2 a_2 + \bar{b}_1\alpha^2 a_1 + \bar{b}_2 a_1\alpha;\end{aligned} \tag{2.18}$$

$$\alpha'\bigl(\alpha^2 + \bar{a}_1\alpha + \bar{a}_2\bigr) = \bar{b}_1\alpha^2 + \bar{b}_2\alpha. \tag{2.19}$$

From (2.14) follows $\beta_1(\theta) = 1$. By (2.15) we get $\alpha(\theta) = \int_0^\theta (b_1 + \bar{b}_1)\,d\theta$. Substituting $\alpha' = b_1 + \bar{b}_1$ into (2.16)-(2.19) and simplifying, it follows equation (2.12) is true. From above we know $F = \frac{r}{1+\alpha(\theta)r}$ is the reflecting function of (2.2), so $F(-\pi, r) = r$ is equivalent to $\alpha(-\pi)r = 0$, it implies if $\alpha(-\pi) = 0$, $r = 0$ is a center, if $\alpha(-\pi) \neq 0$, $r = 0$ is an unique 2π-periodic solution of (2.2).

Thus, the proof is completed. □

Applying Theorem 1.4, we get the following corollary.

Corollary 2.6 *If all the conditions of Theorem* 2.5 *are satisfied, then for the equation*

$$\frac{dr}{d\theta} = \frac{b_1 + b_2 r}{1 + a_1 r + a_2 r^2} r^2 + (1 + \alpha r)^2 G(\theta, r) - G\left(-\theta, \frac{r}{1 + \alpha r}\right),$$

$r = 0$ *is a center. Here* $G(\theta, r)$ *is an arbitrary continuously differentiable vector function and such that* $G(\theta, 0) - G(-\theta, 0) = 0$.

Theorem 2.7 *Suppose that* $a_{20} \neq 0$ *and one of the following conditions is satisfied*

$$\begin{cases} a_{02} = -2a_{20}, \\ b_{20} = \frac{a_{30}}{a_{20}}, \quad b_{11} = 3a_{20}, \quad b_{02} = \frac{1}{a_{20}}(a_{12} - a_{11}a_{20}), \\ b_{30} = 0, \quad b_{21} = 2a_{30}, \quad b_{12} = 2a_{20}^2, \quad b_{03} = 2(a_{12} - a_{11}a_{20}), \end{cases} \tag{2.20}$$

$$\begin{cases} a_{03} = a_{20}(a_{20} + a_{02}), \\ a_{12} + 2a_{30} + \frac{1}{a_{20}}(a_{03}a_{11} + a_{30}a_{02}) = 0, \\ a_{21} + a_{03} - \frac{a_{30}}{a_{20}^2}(a_{30} + a_{20}a_{11}) = 0, \\ b_{20} = \frac{a_{30}}{a_{20}}, \quad b_{11} = a_{20} - a_{02}, \quad b_{02} = 0, \\ b_{30} = 0, \quad b_{21} = 2a_{30}, \quad b_{12} = -2a_{03}, \quad b_{03} = 0. \end{cases} \tag{2.21}$$

Then the origin point $(0,0)$ *is a center of the cubic system* (2.1).

Proof Applying Theorem 2.5 and equation (2.3), we know

$$\alpha(\theta) = \int_0^\theta (b_1 + \bar{b}_1)\,d\theta = \frac{2}{3}(a_{02} + b_{11} - a_{20})\sin^3\theta + 2a_{20}\sin\theta. \tag{2.22}$$

As the even part of the odd function is equal to zero, using (2.3) and Lemma 2.3, from the first relation of (2.12) we get

$$a_{30} - a_{20}b_{20} = 0; \tag{2.23}$$

$$a_{12} + b_{21} + a_{30} - a_{20}(b_{02} - a_{11}) - (a_{11} + b_{20})(b_{11} - a_{20}) - b_{20}(a_{02} + b_{11}) = 0; \tag{2.24}$$

$$b_{03} + a_{12} + b_{21} + (a_{11} + b_{20})a_{02} - (a_{02} + b_{11})(b_{02} - a_{11}) - b_{02}(b_{11} - a_{02}) = 0; \tag{2.25}$$

$$b_{03} + b_{02}a_{02} = 0. \tag{2.26}$$

Using the second relation of (2.12) and Lemma 2.3, we obtain

$$a_{30}b_{20} - a_{20}b_{20}^2 - a_{20}b_{30} = 0; \tag{2.27}$$

$$\begin{aligned} &a_{20}\big(2a_{21} - b_{12} + 2a_{20}^2 - 3a_{20}b_{11} - b_{20}b_{02} + b_{11}^2\big) - a_{21}(b_{11} - a_{20}) \\ &\quad + b_{20}(a_{12} + b_{21}) - b_{20}^2(a_{02} + b_{11}) - (a_{11} + b_{20})(b_{21} - a_{30}) = 0; \end{aligned} \tag{2.28}$$

$$\begin{aligned} &a_{20}\big(a_{21} + a_{20}^2 - b_{20}^2 - a_{20}b_{11} - a_{11}b_{20}\big) + a_{20}\big(a_{03} + b_{12} + a_{21} + 2a_{20}a_{02} + a_{11}^2 \\ &\quad - 2b_{20}b_{02} - b_{11}^2 + a_{20}b_{11} + a_{11}b_{20} - a_{11}b_{02} - a_{02}b_{11}\big) + a_{20}b_{20}(b_{02} - a_{11}) \\ &\quad - a_{21}(b_{11} - a_{20}) + b_{20}(a_{12} + b_{21}) + a_{21}a_{02} + (a_{12} + b_{21})(b_{02} - a_{11}) \\ &\quad - (a_{03} + b_{12})(b_{11} - a_{20}) + b_{03}b_{20} - a_{20}(b_{12} - a_{21}) - (a_{11} + b_{20})(b_{21} - a_{30}) \\ &\quad + a_{20}a_{03} - (a_{11} + b_{20})(b_{03} - a_{12}) - (a_{02} + b_{11})(b_{12} - a_{21}) - b_{02}(b_{21} - a_{30}) \\ &\quad - a_{20}\big((b_{02} - a_{11})^2 + 2a_{02}(b_{11} - a_{20})\big) \\ &\quad - (a_{02} + b_{11})\big(2b_{20}(b_{02} - a_{11}) - (b_{11} - a_{20})^2\big) = 0; \end{aligned} \tag{2.29}$$

$$\begin{aligned} &a_{20}\big(a_{03} + b_{12} + a_{21} + 2a_{20}a_{02} + a_{11}^2 - 2b_{20}b_{02} - b_{11}^2 + a_{20}b_{11} + a_{11}b_{20} - a_{11}b_{02} - a_{02}b_{11}\big) \\ &\quad + a_{20}\big(a_{03} + b_{12} + a_{02}^2 - b_{02}^2 + a_{02}b_{11} + a_{11}b_{02}\big) + a_{21}a_{02} + (a_{03} + b_{12})a_{02} \\ &\quad + (a_{12} + b_{21})(b_{02} - a_{11}) - (a_{03} + b_{12})(b_{11} - a_{20}) + b_{03}b_{20} + b_{03}(b_{02} - a_{11}) \\ &\quad + a_{02}^2a_{20} + a_{20}a_{03} - (a_{02} + b_{11})\big((b_{02} - a_{11})^2 + 2a_{02}(b_{11} - a_{20})\big) \\ &\quad - (a_{11} + b_{20})(b_{03} - a_{12}) - (a_{02} + b_{11})(b_{12} - a_{21}) \\ &\quad - b_{02}(b_{21} - a_{30}) + (a_{02} + b_{11})a_{03} - b_{02}(b_{03} - a_{12}) = 0; \end{aligned} \tag{2.30}$$

$$\begin{aligned} &a_{20}\big(a_{03} + b_{12} + a_{02}^2 - b_{02}^2 + a_{02}b_{11} + a_{11}b_{02}\big) + (a_{03} + b_{12})a_{02} \\ &\quad + b_{03}(b_{02} - a_{11}) + a_{02}^2(a_{02} + b_{11}) + (a_{02} + b_{11})a_{03} - b_{02}(b_{03} - a_{12}) = 0. \end{aligned} \tag{2.31}$$

Substituting (2.23) into (2.27), we have

$$a_{20}b_{30} = 0. \tag{2.32}$$

Equation (2.29) plus (2.31) minus (2.28) minus (2.30) gives us

$$(a_{02}+b_{11}-a_{20})\big((a_{11}-b_{02}+b_{20})^2+(a_{02}+b_{11}-a_{20})^2\big)=0,$$

which implies

$$a_{02}+b_{11}-a_{20}=0. \tag{2.33}$$

Thus, $\alpha(\theta)=2a_{20}\sin\theta$. As $a_{20}\neq 0$, by (2.32) we get $b_{30}=0$. Substituting $b_{30}=0$ and (2.33) into (2.23), (2.24) and (2.25) we obtain

$$a_{30}=a_{20}b_{20}, \tag{2.34}$$

$$a_{12}+b_{21}=-a_{20}(a_{11}-b_{02})-a_{02}(a_{11}+b_{20}). \tag{2.35}$$

Substituting (2.33) and (2.26) into (2.35), we obtain (2.35), too. Using the above relations and simplifying (2.28)-(2.31) we have

$$a_{20}\big(2a_{21}+a_{20}a_{02}+a_{02}^2-b_{12}\big)+a_{21}a_{02}-(a_{11}+b_{20})(b_{21}+b_{20}a_{02})=0, \tag{2.36}$$

$$\begin{aligned}&(2a_{21}+a_{03})(2a_{20}+a_{02})+(3a_{20}a_{02}+a_{11}b_{02})(a_{20}+a_{02})+2a_{20}b_{20}(a_{11}-b_{02})\\&\quad+2b_{20}a_{12}+a_{12}b_{02}-2a_{11}b_{21}+a_{02}b_{12}-a_{20}b_{02}^2-a_{20}b_{12}=0,\end{aligned} \tag{2.37}$$

$$\begin{aligned}&\big(2a_{03}+a_{21}-b_{02}^2\big)(2a_{20}+a_{02})+(2b_{12}+3a_{20}a_{02}+2b_{02}a_{12})(a_{20}+a_{02})+2a_{12}b_{02}\\&\quad+a_{20}b_{20}(a_{11}-b_{02})-b_{21}a_{11}+b_{20}a_{12}-a_{20}b_{12}+a_{02}b_{02}^2=0,\end{aligned} \tag{2.38}$$

$$\begin{aligned}&a_{03}(2a_{20}+a_{02})+(b_{12}+a_{20}a_{02})(a_{20}+a_{02})\\&\quad+b_{02}(a_{12}-a_{20}b_{02}+a_{20}a_{11}+a_{02}a_{11})=0.\end{aligned} \tag{2.39}$$

Computing the third relation of (2.12) and using the above relations we get

$$a_{03}b_{02}(2a_{20}+a_{02})=0, \tag{2.40}$$

$$\begin{aligned}&(2a_{20}+a_{02})(a_{11}a_{03}+b_{20}a_{03}-b_{02}b_{12}+b_{02}a_{21})+a_{20}(b_{02}-a_{11})a_{03}\\&\quad=(b_{03}-a_{12})\big(2a_{20}^2+2a_{20}a_{02}+a_{03}+b_{12}\big),\end{aligned} \tag{2.41}$$

$$\begin{aligned}&\big(2a_{20}^2+2a_{20}a_{02}+a_{03}+b_{12}\big)(a_{30}-b_{21})+a_{20}b_{20}a_{03}-\big(2a_{20}^2+2a_{20}a_{02}+a_{21}\big)(b_{03}-a_{12})\\&\quad+(a_{21}-b_{12})\big((2a_{20}+a_{02})(a_{11}+b_{20})+a_{20}(b_{02}-a_{11})\big)=0,\end{aligned} \tag{2.42}$$

$$\big(2a_{20}^2+2a_{20}a_{02}+a_{21}\big)(b_{21}-a_{30})+a_{20}b_{20}(b_{12}-a_{21})=0. \tag{2.43}$$

Applying the fourth relation of (2.12) and Lemma 2.3, we get

$$b_{02}(2a_{20}+a_{02})=0, \tag{2.44}$$

$$b_{12}+2a_{20}(a_{02}+a_{20})=0, \tag{2.45}$$

$$b_{03}+b_{21}-2a_{20}(b_{02}+b_{20})=0, \tag{2.46}$$

$$2a_{20}^2+2a_{20}a_{02}+b_{12}=0, \tag{2.47}$$

$$a_{20}b_{20} + a_{30} - b_{21} = 0. \tag{2.48}$$

Using (2.23) and (2.48) we obtain

$$b_{21} = 2a_{30}. \tag{2.49}$$

Obviously, equation (2.47) is the same as (2.45). Using (2.49) and (2.26) and (2.34), (2.46) becomes

$$b_{02}(2a_{20} + a_{02}) = 0. \tag{2.50}$$

Thus, from the fourth relation of (2.12) we have

$$\begin{cases} b_{12} + 2a_{20}(a_{20} + a_{02}) = 0, \\ b_{02}(2a_{20} + a_{02}) = 0, \\ b_{21} = 2a_{30}. \end{cases} \tag{2.51}$$

Using (2.35) and (2.23) and (2.49) we derive

$$b_{21} = 2a_{20}b_{20} = -a_{12} - a_{20}(a_{11} - b_{02}) - a_{02}(a_{11} + b_{20}). \tag{2.52}$$

Substituting (2.52) and (2.51) into (2.41),(2.42) and (2.43), it shows that these equations are identical. Substituting (2.51) and (2.52) into (2.36) we have

$$\bigl(a_{21} + a_{20}^2 - b_{20}^2 + a_{20}a_{02} - b_{20}a_{11}\bigr)(2a_{20} + a_{02}) = 0. \tag{2.53}$$

Substituting (2.51) and (2.52) into (2.39) we obtain

$$\bigl(a_{03} - a_{20}^2 - a_{20}a_{02}\bigr)(2a_{20} + a_{02}) = 0. \tag{2.54}$$

Substituting (2.33), (2.53), (2.35), (2.52) into (2.37) and (2.38), it shows that these equations are identical. Thus from above discussion we know equation (2.12) is equivalent to the following relations:

$$\begin{cases} b_{03} + a_{02}b_{02} = a_{02} + b_{11} - a_{20} = b_{02}(2a_{20} + a_{02}) = 0, \\ b_{12} + 2a_{20}(a_{20} + a_{02}) = b_{30} = b_{21} - 2a_{30} = 0, \\ a_{12} + b_{21} + a_{20}(a_{11} - b_{02}) + a_{02}(a_{11} + b_{20}) = 0, \\ (a_{21} + a_{20}^2 - b_{20}^2 + a_{20}a_{02} - a_{11}b_{20})(2a_{20} + a_{02}) = 0, \\ (a_{03} - a_{20}^2 - a_{20}a_{02})(a_{02} + 2a_{20}) = a_{30} - a_{20}b_{20} = 0. \end{cases} \tag{2.55}$$

Considering (2.55) and using Theorem 2.5 and Theorem 1.2, the conclusion of the present theorem is true and the proof is finished. □

Remark 1 Under the conditions (2.20), the system (2.1) with $a_{20} \neq 0$ can be expressed as follows:

$$\begin{cases} x' = -y + a_{20}x^2 + a_{11}xy - 2a_{20}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ y' = x + \frac{a_{30}}{a_{20}}x^2 + 3a_{20}xy + (\frac{a_{12}}{a_{20}} - a_{11})y^2 + 2a_{30}x^2y + 2a_{20}^2xy^2 + 2(a_{12} - a_{20}a_{11})y^3, \end{cases}$$

and this cubic system has a center at $(0,0)$. Here a_{11}, a_{30}, a_{21}, a_{12}, a_{03} are arbitrary constants.

Remark 2 Under the conditions (2.21), the system (2.1) with $a_{20} \neq 0$ can be expressed as follows:

$$\begin{cases} x' = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{20}(a_{20} + a_{02})y^3, \\ y' = x + \frac{a_{30}}{a_{20}}x^2 + (a_{20} - a_{02})xy + 2a_{30}x^2y - 2a_{20}(a_{20} + a_{02})xy^2, \end{cases}$$

where

$$a_{20}\bigl(a_{12} + a_{11}(a_{20} + a_{02})\bigr) + a_{30}(2a_{20} + a_{02}) = 0,$$
$$a_{20}^2\bigl(a_{21} + a_{20}(a_{20} + a_{02})\bigr) - a_{30}(a_{30} + a_{11}a_{20}) = 0.$$

and this cubic system has a center at $(0,0)$.

Remark 3 In the literature [1], the authors have studied the center-focus problem of the cubic system in the Kukles form, *i.e.*, $a_{ij} = 0$ $(i,j = 0,1,2,3)$ and in the literature [2], they have discussed the cubic system with $a_{03} = 0$ and $b_{30} = 0$. In the other literature [3–5], the center-focus problem of some special cubic system has been studied. In this paper, we use the Mironenko's method to give the sufficient conditions of the center of the cubic system in the general form (2.1). Just this advantage is not enough, by the equivalence [7] of differential system, if the reflecting function of one differential system is given, at the same time, we know the reflecting function of its infinite equivalent differential systems. Thus, if we solve the center-focus problem of one system, at the same time, we open a class of differential systems with the same character at the critical point. The following example will illustrate this advantage.

Example 1 Consider the cubic system

$$\begin{cases} x' = -y + \frac{1}{2}x^2 - y^2 - \frac{1}{2}y^3, \\ y' = x + \frac{3}{2}xy + \frac{1}{2}xy^2. \end{cases} \tag{2.56}$$

It is easy to check that, for this system the condition (2.20) is satisfied, so the point $(0,0)$ is a center.

Taking $x = r\cos\theta$, $y = r\sin\theta$, (2.56) becomes

$$\frac{dr}{d\theta} = \frac{\cos\theta}{2 + 2r\sin\theta + r^2\sin^2\theta}r^2 \tag{2.57}$$

and it has a reflecting function $F = \frac{r}{1+r\sin\theta}$.

By Theorem 1.4, equation (2.57) is equivalent to the equation

$$\frac{dr}{d\theta} = \frac{\cos\theta}{2 + 2r\sin\theta + r^2\sin^2\theta}r^2 + (1 + r\sin\theta)^2G(\theta,r) - G\left(-\theta, \frac{r}{1 + r\sin\theta}\right) \tag{2.58}$$

and which has a center at $r = 0$. Here $G(\theta,r)$ nearby $r = 0$ is an arbitrary continuously differentiable function and such that $G(\theta,0) - G(-\theta,0) = 0$.

By [17], the first integral of (2.58) is $H(x,y) = \Phi(\theta,u)$, where $u = \frac{2+r\sin\theta}{1+r\sin\theta}r$ and $\Phi(\theta,u)$ is the first integral of the following first order equation

$$\frac{du}{d\theta} = \alpha(u,\theta),$$

where $\alpha(\theta,u) = (1+(1+r\sin\theta)^2)(G(\theta,r) - \frac{1}{(1+r\sin\theta)^2}G(-\theta,\frac{r}{1+r\sin\theta}))$.

Taking

$$G(\theta,r) = \frac{r^{2k-1}}{(1+r\sin\theta)(1+(1+r\sin\theta)^2)^k},$$

where $k \geq 1$ is an arbitrary constant, then equation (2.58) becomes

$$\frac{dr}{d\theta} = \frac{r^2\cos\theta(1+(1+r\sin\theta)^2)^{k-1} + r^{2k-1}(1+r\sin\theta-(1+r\sin\theta)^2)}{(1+(1+r\sin\theta)^2)^k}, \tag{2.59}$$

by Corollary 2.6, its equivalent system,

$$\begin{cases} \frac{dx}{dt} = (-y+\frac{1}{2}x^2-y^2-\frac{1}{2}y^3)(1+y+\frac{1}{2}y^2)^{k-1} - \frac{1}{2^k}xy(x^2+y^2)^{k-1}(1+y), \\ \frac{dy}{dt} = (x+\frac{3}{2}xy+\frac{1}{2}xy^2)(1+y+\frac{1}{2}y^2)^{k-1} - \frac{1}{2^k}y^2(x^2+y^2)^{k-1}(1+y), \end{cases} \tag{2.60}$$

has a center at $(0,0)$, too.

Taking $k=1$ in (2.60), which shows the cubic system

$$\begin{cases} x' = -y+\frac{1}{2}x^2-\frac{1}{2}xy-y^2-\frac{1}{2}xy^2-\frac{1}{2}y^3, \\ y' = x+\frac{3}{2}xy-\frac{1}{2}y^2+\frac{1}{2}xy^2-\frac{1}{2}y^3, \end{cases} \tag{2.61}$$

has a center at $(0,0)$. On the other hand, it is easy to check that system (2.61) satisfies the condition (2.20) of Theorem 2.7.

The Darboux first integral [18] of (2.61) is $H(x,y) = \Phi(\theta,u)$ ($u = \frac{2+r\sin\theta}{1+r\sin\theta}r$), $\Phi(\theta,u)$ is the first integral of the following first order equation:

$$\frac{du}{d\theta} = -\frac{\sin\theta u^2}{2+\sqrt{4+u^2\sin^2\theta}}.$$

Taking $G = \frac{r^2\sin\theta(2+r\sin\theta)^2}{2(1+r\sin\theta)^2(1+(1+r\sin\theta)^2)}$, then (2.58) becomes

$$\frac{dr}{d\theta} = \frac{\cos\theta}{2+2r\sin\theta+r^2\sin^2\theta}r^2 + \sin\theta\frac{(2+r\sin\theta)^2}{1+(1+r\sin\theta)^2}r^2$$

and its equivalent system

$$\begin{cases} x' = -y+\frac{1}{2}x^2-y^2-\frac{1}{2}y^3+\frac{1}{2}xy(2+y)^2, \\ y' = x+\frac{3}{2}xy+\frac{1}{2}xy^2+\frac{1}{2}y^2(2+y)^2, \end{cases} \tag{2.62}$$

has a center at $(0,0)$.

Solving the equation $\frac{du}{d\theta} = u^2 \sin\theta$, we get the Darboux first integral of (2.62),

$$H = \frac{(2+y)\sqrt{x^2+y^2}}{1+y-x(2+y)}$$

and its inverse integrating factor is

$$V = \big(1+y-x(2+y)\big)^2\sqrt{x^2+y^2}.$$

By this, the system (2.62) does not exist a limit cycle nearby $(0,0)$ ($|x| < 1$).

Remark 4 If in the system (2.1) taking $a_{20} = \frac{1}{2}$, then under the assumption of Theorem 2.7, equation (2.2) has a center at $r = 0$ which can be expressed as (2.58).

Remark 5 From the above, we see that using the method of Mironenko we not only solve a center-focus problem, but also at the same time, we open a class of differential systems, which do not have to be polynomial differential systems, with the same character at the critical point $(0,0)$. Therefore, we can say, sometimes, the method of Mironenko is more effective than Poincaré and Lyapunov's method.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZZ gave the proof of all theorems, finishing the article. FM gave the calculations and verifications of some theorems. YY gave some constructive comments. All authors have read and approved the final manuscript.

Acknowledgments
This work is supported by the National Natural Science Foundation of China (61374010, 11571301), and the National Natural Science Foundation of province Jiangsu (BK20161327), and the advantage subject Fund Project of High Education Institutions of Jiangsu Province (PPZY2015B109).

Received: 19 July 2016 Accepted: 6 February 2017 Published online: 16 February 2017

References

1. Lloyd, NG, Pearson, JM: Computing centre conditions for certain cubic systems. J. Comput. Appl. Math. **40**, 323-336 (1992)
2. Devlin, J, Lloyd, NG, Pearson, JM: Cubic systems and Abel equations. J. Differ. Equ. **147**, 435-454 (1998)
3. Christopher, CJ: Invariant algebraic curves in polynomial differential systems. PhD thesis, Univ. College Wales, Aberystwyth (1989)
4. Cherkas, IA: Center conditions of a equation. Differ. Uravn. **13**(2), 271-275 (1977)
5. Li, C, Liu, C, Yang, J: A cubic system with thirteen limit cycles. J. Differ. Equ. **246**(9), 3609-3619 (2009)
6. Sadowskii, AP: Polynomial Ideals and Manifold. University Press, Minsk (2008)
7. Mironenko, VI: Analysis of Reflective Function and Multivariate Differential System. University Press, Gomel (2004)
8. Mironenko, VV: Time symmetry preserving perturbations of differential systems. Differ. Uravn. **40**(20), 1395-1403 (2004)
9. Musafirov, EV: Differential systems, the mapping over period for which is represented by a product of three exponential matrixes. J. Math. Anal. Appl. **329**, 647-654 (2007)
10. Belsky, VA: On quadratic differential systems with equal reflecting functions. Differ. Uravn. **49**(12), 1639-1644 (2013)
11. Verecovich, PP: Nonautonomous second order quadric system equivalent to linear system. Differ. Uravn. **34**(12), 2257-2259 (1998)
12. Maiorovskaya, SV: Quadratic systems with a linear reflecting function. Differ. Uravn. **45**(2), 271-273 (2009)
13. Zhou, Z: A new method for research on the center-focus problem of differential systems. Abstr. Appl. Anal. **2014**, Article ID 926538 (2014)
14. Zhou, Z: The structure of reflective function of polynomial differential systems. Nonlinear Anal. **71**, 391-398 (2009)
15. Zhou, Z: On the symmetry and periodicity of solutions of differential systems. Nonlinear Anal., Real World Appl. **17**, 64-70 (2014)
16. Zhou, Z: The Theory of Reflecting Function and Application, pp. 200-205. China Machine Press, Beijing (2014)
17. Zhou, Z: On the structure of the equivalent differential systems and their reflecting integrals. Bull. Braz. Math. Soc. (2016). doi:10.1007/s00574-016-0026-4
18. Chavarriga, J, Giacomini, H, Gine, J, Llibre, J: Darboux integrability and the inverse integrating factor. J. Differ. Equ. **194**, 116-139 (2003)