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Existence of solutions for fractional stochastic impulsive neutral functional differential equations with infinite delay

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Abstract
This paper addresses a class of fractional stochastic impulsive neutral functional differential equations with infinite delay which arise from many practical applications such as viscoelasticity and electrochemistry. Using fractional calculations, fixed point theorems and the stochastic analysis technique, sufficient conditions are derived to ensure the existence of solutions. An example is provided to prove the main result.

Keywords: fractional stochastic functional differential equations; existence; neutral; impulsive; infinite delay; fixed point theorem

1 Introduction

It is commonly believed that fractional calculus dates back to 1695. Fractional derivatives supply a powerful tool in describing the memory and hereditary properties of many materials and processes [1, 2]. Many researchers have focused their attention on fractional differential equations. For example, robust stability and stabilization of fractional-order interval systems were investigated in [3]. Li et al. presented a stability theorem for fractional-order nonlinear dynamic systems [4].

Dynamical behaviors such as existence and stability are basic problems of fractional differential equations [5–10]. Shen and Lam proved that for fractional-order nonlinear system described by Caputo's or Riemann-Liouville's definition, any equilibrium cannot be finite-time stable as long as the continuous solution corresponding to the initial value problem globally exists [5]. Song and Cao gave some sufficient conditions ensuring the existence and uniqueness of the nontrivial solution [6]. In recent years, scholars have paid more attention to impulsive differential equations. This is mainly because of many processes in which their state is changed suddenly at some instants. These phenomena can be described by impulsive differential equations. So far, there have been several interesting results that studied the existence of solutions for fractional impulsive differential equations, see [11–20] and the references therein.

It is well known that time delays exist in different technical systems which may cause unpredictable system behaviors. There are some results about integer-order and fractional-order functional differential equations with infinite delay [11, 12, 21–24]. Sakthivel et al. studied the existence of solutions for a class of nonlinear fractional differential equations with infinite delays by utilizing fractional calculations and a fixed point technique [11]. Another kind of time-delay, called neutral-type time-delay, has received considerable attention [25–27]. Actually, many real delayed systems can be described as neutral differential equations. The differential expression includes the derivative terms of current state and past state. In [12], Liao et al. gave the existence theorem of solutions for fractional impulsive neutral functional differential equations with infinite delay by using the Caputo fractional derivative, Hausdorff's measure of noncompactness and the theory of Mönch.

Since the real environment is influenced by noise inevitably, it is significant to consider the dynamical properties for a fractional stochastic impulsive neutral functional differential equation with infinite delay, especially for the existence of solutions. To the best of our knowledge, few results have studied this problem, and the aim of this paper is to shorten this gap.

Motivated by the above discussions, in this paper we aim to study the existence of solutions for fractional stochastic impulsive neutral functional differential equations with infinite delays. In the established model, the stochastic disturbances are described in terms of a Brownian motion. By using fixed point theorems, we derive sufficient criteria to ensure the existence of solutions. Moreover, our results take some well-studied models, such as integer-order functional differential equations with infinite delay, as special cases.

This paper is organized as follows. In Section 2, we introduce some useful preliminaries. In Section 3, we prove the existence of solutions for the fractional-order system under investigation. In Section 4, an example is given to demonstrate the correctness of the main theorems. Conclusions are made in Section 5.

2 Preliminaries

In this paper, we adopt the symbols as follows: K_1 and K_2 are separable Hilbert spaces. $\mathcal{L}(K_2,K_1)$ is the space which contains all the bounded linear operators from K_2 into K_1. $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner product in K_1 and K_2. $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ is a complete filtered probability space satisfying the fact that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. $W = \{W(t)\}_{t\geq 0}$ is a Q-Wiener process defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ with the covariance operator Q such that $TrQ < \infty$. $E\{\cdot\}$ denotes the expectation. It is assumed that $Q\delta_k = \gamma_k\delta_k$, $k = 1,2,\ldots$, and $(w(u),\delta)_{K_2} = \sum_{k=1}^{\infty}\sqrt{\gamma_k}(\delta_k,\delta)_{K_2}\beta_k(u)$, $\delta \in K_2$, $e \geq 0$, where $\{\delta_k\}_{k\geq 1}$ in K_2 is a complete orthonormal system, γ_k is a bounded sequence of nonnegative real numbers, $\{\beta_k\}_{k\geq 1}$ are independent Brownian motions.

We discuss the following fractional functional differential equations:

$$\begin{cases} {}^{c}D_u^{\alpha}[x(u)+g(u,x_u)] = f(u,x_u) + \sigma(u,x_u)\frac{dW(u)}{du}, & u \in H = [0,T], u \neq u_k, \\ \Delta x(u_k) = I_k(x(u_k)), & u = u_k, k = 1,2,\ldots,m, \\ x(u) = \xi(u) \in \mathcal{B}_h, & u \in (-\infty,0], \end{cases} \tag{2.1}$$

where ${}^{c}D_u^{\alpha}$ denotes α-order Caputo fractional derivative, $\alpha > \frac{1}{2}$; $x(\cdot) \in K_1$. The history $x_u : (-\infty,0] \to K_1, x_u(v) = x(u+v) \in \mathcal{B}_h, v \leq 0. f : H \times \mathcal{B}_h \to K_1, g : H \times \mathcal{B}_h \to K_1, \sigma : H \times \mathcal{B}_h \to \mathcal{L}_2^0$, $I_k : \mathcal{B}_h \to K_1$ $(k = 1,2,\ldots m)$ are appropriate functions. Here $0 = u_0 \leq u_1 \leq \cdots \leq u_m \leq u_{m+1} = T$, $\Delta x(u_k) = x(u_k^+) - x(u_k^-)$, $x(u_k^+) = \lim_{\epsilon\to 0^+} x(u_k+\epsilon)$ and $x(u_k^-) = \lim_{\epsilon\to 0^+} x(u_k-\epsilon)$. $\xi = \{\xi(u), u \in (-\infty,0]\}$ denotes the initial condition, and it is an $\mathcal{F}_0$-measurable $\mathcal{B}_h$-values random variable which is independent of ω.

We adopt the following symbols in [22].

Suppose $h:(-\infty,0]\to(0,\infty)$ is a continuous function satisfying $l=\int_{-\infty}^{0}h(u)\,du<\infty$. Define the space $\mathcal{B}_h$ by

$$\begin{aligned}\mathcal{B}_h=\biggl\{&\xi:(-\infty,0]\to K_1,\text{for any } a>0, \bigl(E\bigl|\xi(\theta)\bigr|^2\bigr)^{\frac{1}{2}} \text{ is a bounded and measurable}\\ &\text{function on } [-a,0] \text{ with } \xi(0)=0 \text{ and } \int_{-\infty}^{0}h(v)\sup_{v\le\theta\le0}\bigl(E\bigl|\xi(\theta)\bigr|^2\bigr)^{\frac{1}{2}}\,dv<\infty\biggr\}.\end{aligned}$$

Let $\|\xi\|_{\mathcal{B}_h}=\int_{-\infty}^{0}h(v)\sup_{v\le\theta\le0}(E|\xi(\theta)|^2)^{\frac{1}{2}}\,dv$, $\xi\in\mathcal{B}_h$, then $(\mathcal{B}_h,\|\cdot\|_{\mathcal{B}_h})$ is a Banach space. Define the space

$$\begin{aligned}\mathcal{B}_b=\bigl\{&x:(-\infty,T]\to K_1 \text{ such that } x|_{H_k}\in C(H_k,K_1) \text{ and there exist } x\bigl(u_k^+\bigr),\\ &x\bigl(u_k^-\bigr),x(u_k)=x\bigl(u_k^-\bigr),x_0=\xi\in\mathcal{B}_h,k=1,2,\ldots,m\bigr\},\end{aligned}$$

where $x|_{H_k}$ is the restriction of x to $H_k=(u_k,u_{k+1}]$, $k=0,1,2,\ldots,m$. Define $\|x\|_{\mathcal{B}_b}=\|\xi\|_{\mathcal{B}_h}+\sup_{v\in[0,T]}(E\|x(v)\|_{K_1}^2)^{\frac{1}{2}}$, $x\in\mathcal{B}_b$, then $\|\cdot\|_{\mathcal{B}_b}$ is a seminorm in $\mathcal{B}_b$.

The following definitions and lemmas are needed to ensure the existence of solutions of (2.1).

Definition 2.1 ([1, 2]) The fractional integral of order α for a function f is defined as

$$I^{\alpha}f(u)=\frac{1}{\Gamma(\alpha)}\int_{u_0}^{u}(u-v)^{\alpha-1}f(v)\,dv,$$

where $u\ge u_0$ and $\alpha>0$.

Definition 2.2 ([1, 2]) Caputo's derivative of order α for a function $f\in C^n([u_0,+\infty),R)$ is defined by

$${}^{\mathrm{c}}D_{\alpha}^{u}f(u)=\frac{1}{\Gamma(n-\alpha)}\int_{u_0}^{u}(u-v)^{n-\alpha-1}f^{(n)}(v)\,dv,$$

where $u\ge u_0$ and n is a positive integer such that $n-1<\alpha<n$.

Particularly, when $0<\alpha<1$, ${}^{\mathrm{c}}D_{\alpha}^{u}f(u)=\frac{1}{\Gamma(1-\alpha)}\int_{u_0}^{u}(u-v)^{-\alpha}f'(v)\,dv$.

Definition 2.3 An $\mathcal{F}_t$-adapted stochastic process $x:(-\infty,T]\to K_1$ is called a solution of (2.1) if $x_0=\xi\in\mathcal{B}_h$ satisfying $x_0\in\mathcal{L}_2^0(\Omega,K_1)$ and the following conditions hold:

(i) $x(u)$ is $\mathcal{B}_h$-valued and the restriction of $x(\cdot)$ to the interval $(u_k,u_{k+1}]$ $(k=1,2,\ldots,m)$ is continuous.

(ii)

$$x(u)=\begin{cases}\xi(u), & u\in(-\infty,0],\\ \xi(0)+g(0,\xi(0))-g(u,x_u) & \\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^u(u-v)^{\alpha-1}f(v,x_v)\,dv & \\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^u(u-v)^{\alpha-1}\sigma(v,x_v)\,dW(v), & u\in(0,u_1],\\ \xi(0)+g(0,\xi(0))-g(u,x_u) & \\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^u(u-v)^{\alpha-1}f(v,x_v)\,dv & \\ \quad+\frac{1}{\Gamma(\alpha)}\int_0^u(u-v)^{\alpha-1}\sigma(v,x_v)\,dW(v) & \\ \quad+\sum_{i=1}^{k}I_i(x(u_i)), & u\in(u_k,u_{k+1}],k=1,2,\ldots,m.\end{cases}\tag{2.2}$$

(iii) $\Delta x|_{u=u_k} = I_k(x(u_k))$, $k = 1, 2, \ldots, m$, the restriction of $x(\cdot)$ to the interval $[0, T] \setminus \{u_1, \ldots, u_m\}$ is continuous.

Lemma 2.4 *Assume, for all $u \in H = [0, T]$, $x_u \in \mathcal{B}_h$, $x_0 = \xi \in \mathcal{B}_h$, then*

$$\|x_u\|_{\mathcal{B}_h} \le l \sup_{u\in[0,T]} \bigl(E\bigl\|x(u)\bigr\|_{K_1}^2\bigr)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h}.$$

Proof For all $u \in [0, T]$,

$$\begin{aligned}
\sup_{v\le\theta\le 0} \bigl(E\bigl\|x(u+\theta)\bigr\|^2\bigr)^{\frac{1}{2}} &\le \max\Bigl\{\sup_{v\le\theta\le -u} \bigl(E\bigl\|x(u+\theta)\bigr\|^2\bigr)^{\frac{1}{2}}, \sup_{-u\le\theta\le 0} \bigl(E\bigl\|x(u+\theta)\bigr\|^2\bigr)^{\frac{1}{2}}\Bigr\} \\
&\le \sup_{v\le\theta\le -u} \bigl(E\bigl\|x(u+\theta)\bigr\|^2\bigr)^{\frac{1}{2}} + \sup_{-u\le\theta\le 0} \bigl(E\bigl\|x(u+\theta)\bigr\|^2\bigr)^{\frac{1}{2}} \\
&= \sup_{v+u\le\theta\le 0} \bigl(E\bigl\|x(\theta)\bigr\|^2\bigr)^{\frac{1}{2}} + \sup_{0\le v\le u} \bigl(E\bigl\|x(v)\bigr\|^2\bigr)^{\frac{1}{2}} \\
&\le \sup_{v\le\theta\le 0} \bigl(E\bigl\|x(\theta)\bigr\|^2\bigr)^{\frac{1}{2}} + \sup_{0\le u\le T} \bigl(E\bigl\|x(u)\bigr\|^2\bigr)^{\frac{1}{2}}.
\end{aligned}$$

So

$$\begin{aligned}
\|x_u\|_{\mathcal{B}_h} &= \int_{-\infty}^0 h(v) \sup_{v\le\theta\le 0} \bigl(E\bigl|x_u(\theta)\bigr|^2\bigr)^{\frac{1}{2}}\,dv \\
&= \int_{-\infty}^0 h(v) \sup_{v\le\theta\le 0} \bigl(E\bigl|x(u+\theta)\bigr|^2\bigr)^{\frac{1}{2}}\,dv \\
&\le \int_{-\infty}^0 h(v)\Bigl(\sup_{v\le\theta\le 0} \bigl(E\bigl\|x(\theta)\bigr\|^2\bigr)^{\frac{1}{2}} + \sup_{0\le u\le T} \bigl(E\bigl\|x(u)\bigr\|^2\bigr)^{\frac{1}{2}}\Bigr)\,dv \\
&= \int_{-\infty}^0 h(v) \sup_{v\le\theta\le 0} \bigl(E\bigl\|x(\theta)\bigr\|^2\bigr)^{\frac{1}{2}}\,dv + \int_{-\infty}^0 h(v) \sup_{0\le u\le T} \bigl(E\bigl\|x(u)\bigr\|^2\bigr)^{\frac{1}{2}}\,dv \\
&= \int_{-\infty}^0 h(v)\,dv \sup_{0\le u\le T} \bigl(E\bigl\|x(u)\bigr\|^2\bigr)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h} \\
&= l \sup_{0\le u\le T} \bigl(E\bigl\|x(u)\bigr\|^2\bigr)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h}.
\end{aligned}$$

This completes the proof. □

Lemma 2.5 (Krasnoselskii's fixed point theorem [19]) *Let B be a nonempty closed convex set of a Banach space $(X, \|\cdot\|)$. Suppose that P and Q map B into X such that*

(i) *$Px + Qy \in B$ whenever $x, y \in B$;*
(ii) *P is compact and continuous;*
(iii) *Q is a contraction mapping;*

then there exists $z \in B$ such that $z = Pz + Qz$.

3 Main results

To obtain the existence of solutions of (2.1), we need the following assumptions:

(H1) There exists $L_1 > 0$ such that

$$E\bigl\|f(u,x) - f(u,y)\bigr\|_{K_1}^2 \le L_1\|x - y\|_{\mathcal{B}_h}^2, \quad \forall x, y \in \mathcal{B}_h.$$

(H2) There exists $L_2 > 0$ such that

$$E\|g(u,x) - g(u,y)\|_{K_1}^2 \le L_2\|x-y\|_{\mathcal{B}_h}^2, \quad \forall x,y \in \mathcal{B}_h.$$

(H3) There exists $L_3 > 0$ such that

$$E\|\sigma(u,x) - \sigma(u,y)\|_{\mathcal{L}_2^0}^2 \le L_3\|x-y\|_{\mathcal{B}_h}^2, \quad \forall x,y \in \mathcal{B}_h.$$

(H4) There exists $L_4 > 0$ such that

$$E\|I_k(x) - I_k(y)\|_{K_1}^2 \le L_4\|x-y\|_{\mathcal{B}_h}^2, \quad \forall x,y \in \mathcal{B}_h \text{ and } k = 1,2,\ldots,m.$$

Now we will use the Banach fixed point theorem to prove the existence theorem for (2.1).

Theorem 3.1 *Assume that conditions* (H1)-(H4) *hold, then* (2.1) *has a unique solution if the following condition holds*:

$$4l^2\left(L_2 + \frac{1}{\Gamma(\alpha)}L_1\frac{T^{2\alpha}}{\alpha^2} + \frac{1}{\Gamma(\alpha)}L_3\frac{T^{2\alpha-1}}{2\alpha-1} + m^2L_4\right) < 1. \tag{3.1}$$

Proof Define the operator $\Pi : \mathcal{B}_b \to \mathcal{B}_b$ by

$$\Pi x(t) = \begin{cases} \xi(u), & u \in (-\infty,0], \\ \xi(0) + g(0,\phi(0)) - g(u,x_u) & \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,x_v)\,dv & \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,x_v)\,dW(v), & u \in (0,u_1], \\ \xi(0) + g(0,\phi(0)) - g(u,x_u) & \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,x_v)\,dv & \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,x_v)\,dW(v) & \\ \quad + \sum_{i=1}^k I_i(x(u_i)), & u \in (u_k,u_{k+1}], k = 1,2,\ldots,m. \end{cases} \tag{3.2}$$

For $\xi \in \mathcal{B}_b$, define

$$\bar{\xi}(t) = \begin{cases} \xi(u), & u \in (-\infty,0], \\ 0, & u \in H, \end{cases}$$

then $\bar{\xi}_0 = \xi$. Next, define the function

$$\bar{\eta}(t) = \begin{cases} 0, & u \in (-\infty,0], \\ \eta(u), & u \in H, \end{cases}$$

for each $\eta \in C(H,R)$, with $\eta(0) = 0$.

If $x(\cdot)$ satisfies (2.2), then $x(u) = \bar{\xi}(u) + \bar{\eta}(u)$ for $u \in H$, which implies $x_u = \bar{\xi}_u + \bar{\eta}_u$ for $u \in H$, and the function $\eta(\cdot)$ satisfies

$$\eta(u) = \begin{cases} g(0,\xi(0)) - g(u,x_u) + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,x_v)\,dv & \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,x_v)\,dW(v), & u \in (0,u_1], \\ g(0,\xi(0)) - g(u,x_u) + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,x_v)\,dv & \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,x_v)\,dW(v) & \\ \quad + \sum_{i=1}^k I_i(x(u_i)), & u \in (u_k,u_{k+1}], \\ & k = 1,2,\ldots,m \end{cases}$$

$$= \begin{cases} g(0,\xi(0)) - g(u,\bar{\xi}_u + \bar{\eta}_u) + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,\bar{\xi}_v + \bar{\eta}_v)\,dv \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,\bar{\phi}_v + \bar{\eta}_v)\,dW(v), & u \in (0,u_1], \\ g(0,\xi(0)) - g(u,\bar{\xi}_u + \bar{\eta}_u) + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,\bar{\xi}_v + \bar{\eta}_v)\,dv \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,\bar{\xi}_v + \bar{\eta}_v)\,dW(v) \\ \quad + \sum_{i=1}^k I_i(\bar{\xi}(u_i) + \bar{\eta}(u_i)), & u \in (u_k,u_{k+1}], \\ & k = 1,2,\ldots,m. \end{cases}$$

Set $\mathcal{B}_b^0 = \{\eta \in \mathcal{B}_b, \text{such that } \eta_0 = 0\}$ and for any $\eta \in \mathcal{B}_b^0$, one has

$$\|\eta\|_{\mathcal{B}_b^0} = \|\eta_0\|_{\mathcal{B}_h} + \sup_{u\in H}\bigl(E\|\eta(u)\|_{K_1}^2\bigr)^{\frac{1}{2}} = \sup_{u\in H}\bigl(E\|\eta(u)\|_{K_1}^2\bigr)^{\frac{1}{2}},$$

thus $(\mathcal{B}_b^0, \|\cdot\|_{\mathcal{B}_b^0})$ is a Banach space.

Define the operator $\Psi : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ by

$$(\Psi\eta)(u) = \begin{cases} g(0,\xi(0)) - g(u,\bar{\xi}_u + \bar{\eta}_u) + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,\bar{\xi}_v + \bar{\eta}_v)\,dv \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,\bar{\xi}_v + \bar{\eta}_v)\,dW(v), & u \in (0,u_1], \\ g(0,\xi(0)) - g(u,\bar{\xi}_u + \bar{\eta}_u) + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1} f(v,\bar{\xi}_v + \bar{\eta}_v)\,dv \\ \quad + \frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\sigma(v,\bar{\xi}_v + \bar{\eta}_v)\,dW(v) \\ \quad + \sum_{i=1}^k I_i(\bar{\xi}(u_i) + \bar{\eta}(u_i)), & u \in (u_k,u_{k+1}], \\ & k = 1,2,\ldots,m. \end{cases}$$

In order to prove the existence result, it is enough to show that Ψ has a unique fixed point. Let $\eta, \eta^* \in \mathcal{B}_b^0$, then for all $u \in (0,u_1]$, we have

$$\begin{aligned} &E\bigl\|(\Psi\eta)(u) - \bigl(\Psi\eta^*\bigr)(u)\bigr\|_{K_1}^2 \\ &\quad \le 3E\bigl\|g(u,\bar{\xi}_u + \bar{\eta}_u) - g\bigl(u,\bar{\xi}_u + \bar{\eta}_u^*\bigr)\bigr\|_{K_1}^2 \\ &\qquad + 3E\biggl\|\frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\bigl[f(v,\bar{\xi}_v + \bar{\eta}_v) - f\bigl(v,\bar{\xi}_v + \bar{\eta}_v^*\bigr)\bigr]\,dv\biggr\|_{K_1}^2 \\ &\qquad + 3E\biggl\|\frac{1}{\Gamma(\alpha)}\int_0^u (u-v)^{\alpha-1}\bigl[\sigma(v,\bar{\xi}_v + \bar{\eta}_v) - \sigma\bigl(v,\bar{\xi}_v + \bar{\eta}_v^*\bigr)\bigr]\,dW(v)\biggr\|_{K_1}^2 \\ &\quad \le 3L_2\bigl\|\bar{\eta}_u - \bar{\eta}_u^*\bigr\|_{\mathcal{B}_h}^2 + 3\biggl(\frac{1}{\Gamma(\alpha)}\biggr)^2 \int_0^u (u-v)^{\alpha-1}\,dv \int_0^u (u-v)^{\alpha-1} \\ &\qquad \times E\bigl\|f(v,\bar{\xi}_v + \bar{\eta}_v) - f\bigl(v,\bar{\xi}_v + \bar{\eta}_v^*\bigr)\bigr\|_{K_1}^2\,dv \\ &\qquad + 3\biggl(\frac{1}{\Gamma(\alpha)}\biggr)^2 \int_0^u (u-v)^{2(\alpha-1)} E\bigl\|\sigma(v,\bar{\xi}_v + \bar{\eta}_v) - \sigma\bigl(v,\bar{\xi}_v + \bar{\eta}_v^*\bigr)\bigr\|_{\mathcal{L}_2^0}^2\,dv \\ &\quad \le 3L_2\bigl\|\bar{\eta}_u - \bar{\eta}_u^*\bigr\|_{\mathcal{B}_h}^2 + 3\biggl(\frac{1}{\Gamma(\alpha)}\biggr)^2 \frac{T^\alpha}{\alpha}\int_0^u (u-v)^{\alpha-1} L_1\bigl\|\bar{\eta}_v - \bar{\eta}_v^*\bigr\|_{\mathcal{B}_h}^2\,dv \\ &\qquad + 3\biggl(\frac{1}{\Gamma(\alpha)}\biggr)^2 \int_0^u (u-v)^{2(\alpha-1)} L_3\bigl\|\bar{\eta}_v - \bar{\eta}_v^*\bigr\|_{\mathcal{B}_h}^2\,dv \\ &\quad \le 3L_2 l^2 \sup_{v\in[0,T]}\bigl\|\eta(v) - \eta^*(v)\bigr\|_{K_1}^2 \\ &\qquad + 3\biggl(\frac{1}{\Gamma(\alpha)}\biggr)^2 \frac{T^\alpha}{\alpha} L_1 \int_0^u (u-v)^{\alpha-1} l^2 \sup_{v\in[0,T]}\bigl\|\eta(v) - \eta^*(v)\bigr\|_{K_1}^2\,dv \end{aligned}$$

$$
\begin{aligned}
&+3\left(\frac{1}{\Gamma(\alpha)}\right)^2 L_3\int_0^u (u-v)^{2(\alpha-1)} l^2 \sup_{v\in[0,T]}\|\eta(v)-\eta^*(v)\|_{K_1}^2\,dv \\
&\leq 3L_2 l^2\|\eta-\eta^*\|_{\mathcal{B}_b^0}^2 + 3\left(\frac{1}{\Gamma(\alpha)}\right)^2 l^2\frac{T^{2\alpha}}{\alpha^2}L_1\|\eta-\eta^*\|_{\mathcal{B}_b^0}^2 \\
&\quad+3\left(\frac{1}{\Gamma(\alpha)}\right)^2 L_3\frac{T^{2\alpha-1}}{2\alpha-1}l^2\|\eta-\eta^*\|_{\mathcal{B}_b^0}^2 \\
&=3l^2\left[L_2+\left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{T^{2\alpha}}{\alpha^2}L_1+\left(\frac{1}{\Gamma(\alpha)}\right)^2 L_3\frac{T^{2\alpha-1}}{2\alpha-1}\right]\|\eta-\eta^*\|_{\mathcal{B}_b^0}^2.
\end{aligned}
$$

For $u\in(u_k,u_{k+1}]$, $k=1,2,\ldots,m$, one can obtain

$$
\begin{aligned}
&E\|(\Psi\eta)(u)-(\Psi\eta^*)(u)\|_{K_1}^2 \\
&\leq 4E\|g(u,\bar{\xi}_u+\bar{\eta}_u)-g(u,\bar{\xi}_u+\bar{\eta}_u^*)\|_{K_1}^2 \\
&\quad+4E\left\|\sum_{i=1}^k\left(I_i(\bar{\eta}(u_1)+\bar{\xi}(u_1))-I_i(\bar{\eta}^*(u_1)+\bar{\xi}(u_1))\right)\right\|_{K_1}^2 \\
&\quad+4E\left\|\frac{1}{\Gamma(\alpha)}\int_0^u(u-v)^{\alpha-1}\left[f(v,\bar{\xi}_v+\bar{\eta}_v)-f(v,\bar{\xi}_v+\bar{\eta}_v^*)\right]dv\right\|_{K_1}^2 \\
&\quad+4E\left\|\frac{1}{\Gamma(\alpha)}\int_0^u(u-v)^{\alpha-1}\left[\sigma(v,\bar{\xi}_v+\bar{\eta}_v)-\sigma(v,\bar{\xi}_v+\bar{\eta}_v^*)\right]dW(v)\right\|_{K_1}^2 \\
&\leq 4L_2\|\bar{\eta}_u-\bar{\eta}_u^*\|_{\mathcal{B}_h}^2+4m^2l^2L_4E\|\bar{\eta}(u_1)-\bar{\eta}^*(u_1)\|_{K_1}^2 \\
&\quad+4\left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{T^\alpha}{\alpha}\int_0^u(u-v)^{\alpha-1}L_1\|\bar{\eta}_v-\bar{\eta}_v^*\|_{\mathcal{B}_h}^2\,dv \\
&\quad+4\left(\frac{1}{\Gamma(\alpha)}\right)^2\int_0^u(u-v)^{2(\alpha-1)}L_3\|\bar{\eta}_v-\bar{\eta}_v^*\|_{\mathcal{B}_h}^2\,dv \\
&\leq 4L_2l^2\sup_{v\in[0,T]}\|\eta(v)-\eta^*(v)\|_{K_1}^2+4m^2l^2L_4\sup_{v\in[0,T]}\|\eta(v)-\eta^*(v)\|_{K_1}^2 \\
&\quad+4\left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{T^\alpha}{\alpha}L_1\int_0^u(u-v)^{\alpha-1}l^2\sup_{v\in[0,T]}\|\eta(v)-\eta^*(v)\|_{K_1}^2\,dv \\
&\quad+4\left(\frac{1}{\Gamma(\alpha)}\right)^2L_3\int_0^u(u-v)^{2(\alpha-1)}l^2\sup_{v\in[0,T]}\|\eta(v)-\eta^*(v)\|_{K_1}^2\,dv \\
&\leq\left(4l^2\left[L_2+\left(\frac{1}{\Gamma(\alpha)}\right)^2L_1\frac{T^{2\alpha}}{\alpha^2}+\left(\frac{1}{\Gamma(\alpha)}\right)^2L_3\frac{T^{2\alpha-1}}{2\alpha-1}\right]+4m^2l^2L_4\right)\|\eta-\eta^*\|_{\mathcal{B}_b^0}^2.
\end{aligned}
$$

Therefore, for all $u\in[0,T]$,

$$
\begin{aligned}
&E\|(\Psi\eta)(u)-(\Psi\eta^*)(u)\|_{K_1}^2 \\
&\leq 4l^2\left(L_2+\left(\frac{1}{\Gamma(\alpha)}\right)^2L_1\frac{T^{2\alpha}}{\alpha^2}+\left(\frac{1}{\Gamma(\alpha)}\right)^2L_3\frac{T^{2\alpha-1}}{2\alpha-1}+m^2L_4\right)\|\eta-\eta^*\|_{\mathcal{B}_b^0}^2.
\end{aligned}
$$

From (3.1), we conclude that Ψ is a contraction mapping. This implies that (2.1) has a unique solution on $(-\infty,T]$. The proof is complete. □

The next result is established by using Krasnoselskii's fixed point theorem. We need the following assumptions.

(H5) $f : H \times \mathcal{B}_h \to K_1$ is continuous, and there exists a continuous function $\mu_1 : H \to (0, +\infty)$ such that

$$E\|f(u,x)\|_{K_1}^2 \le \mu_1(u)\|x\|_{\mathcal{B}_h}^2, \quad \forall (u,x) \in H \times \mathcal{B}_h,$$

where $\mu_1^* = \sup_{0\le v\le u} \mu_1(v)$.

(H6) $g : H \times \mathcal{B}_h \to K_1$ is continuous, and there exists a continuous function $\mu_2 : H \to (0, +\infty)$ such that

$$E\|g(u,x)\|_{K_1}^2 \le \mu_2(u)\|x\|_{\mathcal{B}_h}^2, \quad \forall (u,x) \in H \times \mathcal{B}_h,$$

where $\mu_2^* = \sup_{0\le v\le u} \mu_2(v)$.

(H7) $\sigma : H \times \mathcal{B}_h \to \mathcal{L}_2^0$ is continuous, and there exists a continuous function $\mu_3 : H \to (0, +\infty)$ such that

$$E\|\sigma(u,x)\|_{\mathcal{L}_2^0}^2 \le \mu_3(u)\|x\|_{\mathcal{B}_h}^2, \quad \forall (u,x) \in H \times \mathcal{B}_h,$$

where $\mu_3^* = \sup_{0\le v\le u} \mu_3(v)$.

(H8) There exists $K > 0$ such that $I_k : \mathcal{B}_h \to K_1$, $k = 1, 2, \ldots, m$, $E\|I_k(x)\|_{K_1}^2 \le K$.

Let $B_q = \{y \in \mathcal{B}_b^0, \|y\|_{\mathcal{B}_b^0}^2 \le q, q > 0\}$, then B_q is a bounded closed convex set in $\mathcal{B}_b^0$, $\forall y \in B_q$. From Lemma 2.4, we get

$$\begin{aligned}
\|y_u + \bar{\eta}_u\|_{\mathcal{B}_h}^2 &\le 2\left(\|y_u\|_{\mathcal{B}_h}^2 + \|\bar{\eta}_u\|_{\mathcal{B}_h}^2\right) \\
&\le 4\Big(l^2 \sup_{v\in[0,u]} E\|y(v)\|_{K_1}^2 + \|y_0\|_{\mathcal{B}_h}^2\Big) + 4\Big(l^2 \sup_{v\in[0,u]} E\|\bar{\eta}(v)\|_{K_1}^2 + \|\bar{\eta}_0\|_{\mathcal{B}_h}^2\Big) \\
&\le 8\left(\|\xi\|_{\mathcal{B}_h}^2 + l^2 q\right) \triangleq M.
\end{aligned}$$

Theorem 3.2 *Assume that conditions* (H1)-(H8) *hold, then* (2.1) *has at least one solution if the following conditions hold*:

$$40l^2\left[\mu_2^* + \left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_1^* \frac{T^{2\alpha}}{\alpha^2} + \left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_3^* \frac{T^{2\alpha-1}}{2\alpha-1}\right] < 1, \tag{3.3}$$

and

$$2\left(\frac{1}{\Gamma(\alpha)}\right)^2 \left(\frac{T^{2\alpha}}{\alpha^2} L_1 + \frac{T^{2\alpha-1}}{2\alpha-1} L_3\right) l^2 < 1. \tag{3.4}$$

Proof See Appendix. □

4 Example

The existence, uniqueness and stability of integer-order Volterra integro-differential equation have been investigated for its wide and important application in the fields of financial mathematics, physics, biology, medicine, automatic control, demography, dynamics etc.

But there are few results about fractional stochastic Volterra integro-differential equations. In this section, we provide an example for which there is at least one solution due to the fact that the conditions in Theorem 3.2 are satisfied.

Example 4.1 Consider the following fractional stochastic impulsive neural functional differential equations with infinite delay:

$$\begin{cases} {}^{c}D_{u}^{\frac{2}{3}}[x(u) - \frac{1}{8}\int_{-\infty}^{0} e^{2v}x(u+v)\,dv] \\ \quad = \frac{1}{8}\int_{-\infty}^{0} e^{2v}x(u+v)\,dv + \frac{1}{8}\int_{-\infty}^{0} e^{2v}x(u+v)\,dv\frac{dW(v)}{dv}, & u \in (0,\frac{1}{2}) \cup (\frac{1}{2},1], \\ \Delta x(\frac{1}{2}) = \frac{1}{4}, \\ x(u) = 0, & u \in (-\infty, 0], \end{cases} \tag{4.1}$$

where $g(u,x_u) = -\frac{1}{8}\int_{-\infty}^{0} e^{2v}x(u+v)\,dv, f(u,x_u) = \frac{1}{8}\int_{-\infty}^{0} e^{2v}x(u+v)\,dv$, $\sigma(u,x_u) = \frac{1}{8}\int_{-\infty}^{0} e^{2v} \times x(u+v)\,dv$, $H = [0,\frac{1}{2}) \cup (\frac{1}{2},1]$, $T = 1$, $m = 1$.

For $\xi \in \mathcal{B}_h$, define $\|\xi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(v)\sup_{v\le\theta\le 0}(E|\xi(\theta)|^2)^{\frac{1}{2}}\,dv$, $h(v) = e^{2v}$, $l = \frac{1}{2}$, then $\xi \in \mathcal{B}_h$, and $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space which has the following properties.

A1. If $x(u): (-\infty, T] \to R$ is continuous on H, and $x_0 \in \mathcal{B}_h$, then $x_u \in \mathcal{B}_h$, and x_u is continuous on H.

A2. $\mathcal{B}_h$ is a Banach space.

A3. $\|x_u\|_{\mathcal{B}_h} \le \frac{1}{2}\sup_{u\in[0,T]}(E\|x(u)\|_{K_1}^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h}$.

In addition, let $\mu_1(u) = \frac{1}{64}$, $\mu_2(u) = \frac{1}{64}$, $\mu_3(u) = \frac{1}{64}$, $\mu_1^* = \frac{1}{64}$, $\mu_2^* = \frac{1}{64}$, $\mu_3^* = \frac{1}{64}$. $L_1 = \frac{1}{64}$, $L_2 = \frac{1}{64}$, $L_3 = \frac{1}{64}$, $K = \frac{1}{16}$, $L_4 = 0$, we have that conditions (H1)-(H8) are satisfied and (3.3), (3.4) hold, i.e.,

$$\begin{aligned} &40l^2\left[\mu_2^* + \left(\frac{1}{\Gamma(\alpha)}\right)^2\mu_1^*\frac{T^{2\alpha}}{\alpha^2} + \left(\frac{1}{\Gamma(\alpha)}\right)^2\mu_3^*\frac{T^{2\alpha-1}}{2\alpha-1}\right] \\ &\quad\approx 40 * \frac{1}{4} * \left[\frac{1}{64} + \frac{1}{1.3541^2} * \frac{1}{64} * \frac{9}{4} + \frac{1}{64} * \frac{1}{1.3541^2} * 3\right] = 0.6036 < 1, \end{aligned}$$

and

$$\begin{aligned} &2\left(\frac{1}{\Gamma(\alpha)}\right)^2\left(\frac{T^{2\alpha}}{\alpha^2}L_1 + \frac{T^{2\alpha-1}}{2\alpha-1}L_3\right)l^2 \\ &\quad\approx 2 * \frac{1}{1.3541^2}\left(\frac{1}{64} * \frac{9}{4} + \frac{1}{64} * 3\right) * \left(\frac{1}{2}\right)^2 = 0.0224 < 1, \end{aligned}$$

then (4.1) has at least one solution by Theorem 3.2.

5 Conclusions

Fractional stochastic impulsive neutral functional differential equations are very useful in viscoelasticity, electrochemistry, automatic control etc. In this paper, based on fractional calculation, fixed point theorems and the stochastic analysis technique, new existence theorems of solutions for these equations are given. Moreover, our results take some well-studied models, such as integer-order functional differential equations with infinite delay, as special cases. The main result is verified by an example.

Appendix

Proof of Theorem 3.2 Define the operator $\Pi_1 : B_q \to B_q$ and $\Pi_2 : B_q \to B_q$, where

$$(\Pi_1\eta)(u) = \begin{cases} 0, & u \in (-\infty, 0], \\ g(0,\xi(0)) - g(u, \bar{\xi}_u + \bar{\eta}_u), & u \in (0, u_1], \\ g(0,\xi(0)) - g(u, \bar{\xi}_u + \bar{\eta}_u) + \sum_{i=1}^{k} I_i(\bar{\xi}(u_i) + \bar{\eta}(u_i)), & u \in (u_k, u_{k+1}], \\ & k = 1, 2, \ldots, m, \end{cases} \tag{A.1}$$

$$(\Pi_2\eta)(u) = \begin{cases} 0, & u \in (-\infty, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} f(v, \bar{\xi}_v + \bar{\eta}_v)\,dv & \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} \sigma(v, \bar{\xi}_v + \bar{\eta}_v)\,dW(v), & u \in (0, u_1], \\ \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} f(v, \bar{\xi}_v + \bar{\eta}_v)\,dv & \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^u (u-v)^{\alpha-1} \sigma(v, \bar{\xi}_v + \bar{\eta}_v)\,dW(v), & u \in (u_k, u_{k+1}], \\ & k = 1, 2, \ldots, m. \end{cases} \tag{A.2}$$

If Π_1 is compact and continuous and Π_2 is a contraction operator, from Lemma 2.5, (2.1) has at least one solution. We will prove them according to the following five steps.

Step 1: We use contradiction to prove that there is $q^* \in N$ such that $\Pi_1\eta + \Pi_2\eta^* \in B_{q^*}$ for $\eta, \eta^* \in B_{q^*}$. Otherwise, for each $q \in N$, there would exist $\eta^q \in B_q$, $\eta^{*q} \in B_q$ and $u_q \in [0, T]$ such that

$$E\left\| \Pi_1\eta^q + \Pi_2\eta^{*q} \right\|_{K_1}^2 > q. \tag{A.3}$$

Without losing generality, we assume $\lim_{q\to\infty} u_q = T$.

For $u_q \in (0, u_1]$, we have

$$\begin{aligned} q &< E\left\| \left(\Pi_1\eta^q\right)(u_q) + \left(\Pi_2\eta^{*q}\right)(u_q) \right\|_{K_1}^2 \\ &\leq 4E\left\| g\left(0, \xi(0)\right) \right\|_{K_1}^2 + 4E\left\| g\left(u_q, \bar{\eta}_{u_q}^q + \bar{\xi}_{u_q}^q\right) \right\|_{K_1}^2 \\ &\quad + 4E\left\| \frac{1}{\Gamma(\alpha)} \int_0^{u_q} (u_q - v)^{\alpha-1} f\left(v, \bar{\eta}_v^* + \bar{\xi}_v\right) dv \right\|_{K_1}^2 \\ &\quad + 4E\left\| \frac{1}{\Gamma(\alpha)} \int_0^{u_q} (u_q - v)^{\alpha-1} \sigma\left(v, \bar{\eta}_v^* + \bar{\xi}_v\right) dW(v) \right\|_{K_1}^2 \\ &\leq 4E\left\| g\left(0, \xi(0)\right) \right\|_{K_1}^2 + 4\mu_2^* \left\| \bar{\eta}_{u_q}^q + \bar{\xi}_{u_q}^q \right\|_{\mathcal{B}_h}^2 \\ &\quad + 4\left(\frac{1}{\Gamma(\alpha)}\right)^2 \int_0^{u_q} (u_q - v)^{\alpha-1}\,dv \int_0^{u_q} (u_q - v)^{\alpha-1} E\left\| f\left(v, \bar{\eta}_v^* + \bar{\xi}_v\right) \right\|_{K_1}^2 dv \\ &\quad + 4\left(\frac{1}{\Gamma(\alpha)}\right)^2 \int_0^{u_q} (u_q - v)^{2(\alpha-1)} E\left\| \sigma\left(v, \bar{\eta}_v^* + \bar{\xi}_v\right) \right\|_{\mathcal{L}_2^0}^2 dv \\ &\leq 4E\left\| g\left(0, \xi(0)\right) \right\|_{K_1}^2 + 4\mu_2^* M + 4\left(\frac{1}{\Gamma(\alpha)}\right)^2 \frac{u_q^\alpha}{\alpha} \int_0^{u_q} (u_q - v)^{\alpha-1} \mu_1^* M\,dv \\ &\quad + 4\left(\frac{1}{\Gamma(\alpha)}\right)^2 \int_0^{u_q} (u_q - s)^{2(\alpha-1)} \mu_3^* M\,dv \\ &\leq 4E\left\| g\left(0, \xi(0)\right) \right\|_{K_1}^2 + 4\mu_2^* M + 4\mu_1^* M \left(\frac{1}{\Gamma(\alpha)}\right)^2 \frac{T^{2\alpha}}{\alpha^2} + 4\left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_3^* M \frac{T^{2\alpha-1}}{2\alpha - 1}. \end{aligned}$$

Similarly, for $u_q \in (u_k, u_{k+1}]$, $k = 1, \dots, m$, we can obtain

$$\begin{aligned}
q &< E\left\|(\Pi_1\eta^q)(u_q) + (\Pi_2\eta^{*q})(u_q)\right\|_{K_1}^2 \\
&\le 5E\left\|g(0,\xi(0))\right\|_{K_1}^2 + 5E\left\|g(u_q, \bar{\eta}_{u_q}^q + \bar{\xi}_{u_q}^q)\right\|_{K_1}^2 + 5E\left\|\sum_{i=1}^{k} I_i(\bar{\eta}(u_i) + \bar{\xi}(u_i))\right\|_{K_1}^2 \\
&\quad + 5E\left\|\frac{1}{\Gamma(\alpha)}\int_0^{u_q}(u_q - \nu)^{\alpha-1} f(\nu, \bar{\eta}_\nu^* + \bar{\xi}_\nu)\,d\nu\right\|_{K_1}^2 \\
&\quad + 5E\left\|\frac{1}{\Gamma(\alpha)}\int_0^{u_q}(u_q - \nu)^{\alpha-1}\sigma(\nu, \bar{\eta}_\nu^* + \bar{\xi}_s)\,dW(\nu)\right\|_{K_1}^2 \\
&\le 5E\left\|g(0,\xi(0))\right\|_{K_1}^2 + 5\mu_2^*\left\|\bar{\eta}_{u_q}^q + \bar{\xi}_{u_q}^{*q}\right\|_{\mathcal{B}_h}^2 + 5m\sum_{i=1}^{m} E\left\|I_i(\bar{\eta}(u_i) + \bar{\xi}(u_i))\right\|_{K_1}^2 \\
&\quad + 5\left(\frac{1}{\Gamma(\alpha)}\right)^2 \int_0^{u_q}(u_q - \nu)^{\alpha-1}\,d\nu \int_0^{u_q}(u_q - \nu)^{\alpha-1} E\left\|f(\nu, \bar{\eta}_\nu^* + \bar{\xi}_\nu)\right\|_{K_1}^2\,d\nu \\
&\quad + 5\left(\frac{1}{\Gamma(\alpha)}\right)^2 \int_0^{u_q}(u_q - \nu)^{2(\alpha-1)} E\left\|\sigma(\nu, \bar{\eta}_\nu^* + \bar{\xi}_\nu)\right\|_{\mathcal{L}_2^0}^2\,d\nu \\
&\le 5E\left\|g(0,\xi(0))\right\|_{K_1}^2 + 5\mu_2^* M + 5\mu_1^* M\left(\frac{1}{\Gamma(\alpha)}\right)^2 \frac{T^{2\alpha}}{\alpha^2} + 5\left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_3^* M \frac{T^{2\alpha-1}}{2\alpha - 1} \\
&\quad + 5m\sum_{i=1}^{m} E\left\|I_i(\bar{\eta}(u_i) + \bar{\xi}(u_i))\right\|_{K_1}^2 \\
&\le 5E\left\|g(0,\xi(0))\right\|_{K_1}^2 + 5\mu_2^* M + 5\mu_1^* M\left(\frac{1}{\Gamma(\alpha)}\right)^2 \frac{T^{2\alpha}}{\alpha^2} \\
&\quad + 5\left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_3^* M \frac{T^{2\alpha-1}}{2\alpha - 1} + 5m^2 K.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
q &< 5E\left\|g(0,\xi(0))\right\|_H^2 + 5\mu_2^* M + 5\mu_1^* M\left(\frac{1}{\Gamma(\alpha)}\right)^2 \frac{T^{2\alpha}}{\alpha^2} \\
&\quad + 5\left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_3^* M \frac{T^{2\alpha-1}}{2\alpha - 1} + 5m^2 K.
\end{aligned} \tag{A.4}$$

Dividing by q and taking the lower limit on both sides of inequality (A.4), one can get

$$1 \le \left(\lim_{q\to+\infty}\inf\frac{M}{q}\right) 5\left(\mu_2^* M + \mu_1^*\left(\frac{1}{\Gamma(\alpha)}\right)^2 \frac{T^{2\alpha}}{\alpha^2} + \left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_3^* \frac{T^{2\alpha-1}}{2\alpha - 1}\right). \tag{A.5}$$

On the other hand, according to $\lim_{q\to+\infty}\inf\frac{M}{q} = 8l^2$ and inequality (A.5),

$$1 \le 40l^2\left[\mu_2^* + \left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_1^* \frac{T^{2\alpha}}{\alpha^2} + \left(\frac{1}{\Gamma(\alpha)}\right)^2 \mu_3^* \frac{T^{2\alpha-1}}{2\alpha - 1}\right]$$

are easily obtained. This is a contradiction with inequality (3.3). Therefore, there exists $q^* \in N$ such that $\Pi_1\eta + \Pi_2\eta^* \in B_{q^*}$ for $\eta, \eta^* \in B_{q^*}$.

Step 2: We need to prove Π_1 is continuous on B_{q^*}.

Suppose $\{\eta^n\}_{n=1}^{\infty}$ is a sequence in B_{q^*} with $\lim_{n\to\infty}\eta^n = \eta \in B_{q^*}$. Then, for $u \in (0, u_1]$, we have

$$\begin{aligned}
&E\big\|\big(\Pi_1\eta^n\big)(u) - (\Pi_1\eta)(u)\big\|_{K_1}^2\\
&\quad\le E\big\|g\big(u, \bar{\eta}_u^n + \bar{\xi}_u\big) - g(u, \bar{\eta}_u + \bar{\xi}_u)\big\|_{K_1}^2\\
&\quad\le L_2\big\|\eta_u^n - \eta_u\big\|_{\mathcal{B}_h}^2 \le L_2 l^2\big\|\eta^n - \eta\big\|_{\mathcal{B}_b^0}^2.
\end{aligned}$$

Similarly, for $u \in (u_k, u_{k+1}]$, $k = 1, 2, \ldots, m$,

$$\begin{aligned}
&E\big\|\big(\Pi_1\eta^n\big)(u) - (\Pi_1\eta)(u)\big\|_{K_1}^2\\
&\quad\le 2E\big\|g\big(u, \bar{\eta}_u^n + \bar{\xi}_u\big) - g(u, \bar{\eta}_u + \bar{\eta}_u)\big\|_{K_1}^2\\
&\qquad + 2E\left\|\sum_{i=1}^{k} I_i\big(\bar{\eta}^n(u_i) + \bar{\xi}(u_i)\big) - \sum_{i=1}^{k} I_i\big(\bar{\eta}(u_i) + \bar{\xi}(u_i)\big)\right\|_{K_1}^2\\
&\quad\le 2L_2\big\|\eta_u^n - \eta_u\big\|_{\mathcal{B}_h}^2 + 2m^2L_4\big\|\eta_u^n - \eta_u\big\|_{\mathcal{B}_h}^2\\
&\quad\le 2\big(L_2 + m^2L_4\big)l^2\big\|\eta^n - \eta\big\|_{\mathcal{B}_b^0}^2.
\end{aligned}$$

So, $\lim_{n\to\infty} E\|\Pi_1\eta^n - \Pi_1\eta\|_{K_1}^2 = 0$, which implies that the mapping Π_1 is continuous on B_{q^*}.

Step 3: We prove that Π_1 maps bounded sets into bounded sets in B_{q^*}.

For $u \in (0, u_1]$, we have

$$\begin{aligned}
E\big\|(\Pi_1\eta)(t)\big\|_{K_1}^2 &\le 2E\big\|g\big(0, \xi(0)\big)\big\|_{K_1}^2 + 2E\big\|g(u, \bar{\eta}_u + \bar{\xi}_u)\big\|_{K_1}^2\\
&\le 2E\big\|g\big(0, \xi(0)\big)\big\|_{\eta}^2 + 2\mu_2^*\|\bar{\eta}_u + \bar{\xi}_u\|_{\mathcal{B}_h}^2\\
&\le 2E\big\|g\big(0, \xi(0)\big)\big\|_{K_1}^2 + 2\mu_2^* M.
\end{aligned}$$

If $u \in (u_k, u_{k+1}]$, $k = 1, 2, \ldots, m$,

$$\begin{aligned}
E\big\|(\Pi_1\eta)(u)\big\|_{K_1}^2 &\le 3E\big\|g\big(0, \xi(0)\big)\big\|_{K_1}^2 + 3E\big\|g(u, \bar{\eta}_u + \bar{\xi}_u)\big\|_{K_1}^2\\
&\quad + 3E\left\|\sum_{i=1}^{k} I_i\big(\bar{\eta}(u_i) + \bar{\xi}(u_i)\big)\right\|_{K_1}^2\\
&\le 3E\big\|g\big(0, \xi(0)\big)\big\|_{K_1}^2 + 3\mu_2^* M + 3m^2K \triangleq \hat{r}.
\end{aligned}$$

Hence, for $q^* > 0$, there exists $\hat{r} > 0$ such that $E\|(\Pi_1\eta)(u)\|_{K_1}^2 \le \hat{r}$, $\forall \eta \in B_{q^*}$, $u \in (u_i, u_{i+1}]$, $i = 0, 1, \ldots, m$.

Step 4: The map Π_1 is equicontinuous.

Let $0 < t < s \le u_1$, $\eta \in B_{q^*}$, we obtain

$$E\big\|(\Pi_1\eta)(t) - (\Pi_1\eta)(s)\big\|_{K_1}^2 = E\big\|g(t, \bar{\eta}_t + \bar{\xi}_t) - g(s, \bar{\eta}_s + \bar{\xi}_s)\big\|_{K_1}^2,$$

for $u_k < t < s \le u_{k+1}$, $k = 1, \ldots, m$,

$$\begin{aligned}
&E\|(\Pi_1\eta)(t) - (\Pi_1\eta)(s)\|_{K_1}^2 \\
&\quad = E\left\| g(t, \bar{\eta}_t + \bar{\xi}_t) - g(s, \bar{\eta}_s + \bar{\xi}_s) + \sum_{i=1}^{k} I_i\big(\bar{\eta}(u_i) + \bar{\xi}(u_i)\big) - \sum_{i=1}^{k} I_i\big(\bar{\eta}(u_i) + \bar{\xi}(u_i)\big)\right\|_{K_1}^2 \\
&\quad = E\|g(t, \bar{\eta}_t + \bar{\xi}_t) - g(s, \bar{\eta}_s + \bar{\xi}_s)\|_{K_1}^2.
\end{aligned}$$

Combining g is continuous and the definition of $\bar{\eta}, \bar{\xi}$, we conclude that $\lim_{t\to s}\|g(t, \bar{\eta}_t + \bar{\xi}_t) - g(s, \bar{\eta}_s + \bar{\xi}_s)\|_{K_1}^2 = 0$, so $\Pi_1(B_{q^*})$ is equicontinuous. From Steps 1-4 and Ascoli's theorem, Π_1 is compact.

Step 5: Π_2 is a contraction mapping.

Let $\eta, \eta^* \in B_{q^*}$ and $u \in (u_k, u_{k+1}]$, $k = 0, 1, \ldots, m$,

$$\begin{aligned}
&E\|(\Pi_2\eta)(u) - \big(\Pi_2\eta^*\big)(u)\|_{K_1}^2 \\
&\quad \le 2E\left\|\frac{1}{\Gamma(\alpha)}\int_{u_i}^{u}(u-v)^{\alpha-1}\big[f(v, \bar{\eta}_v + \bar{\xi}_v) - f\big(v, \bar{\eta}_v^* + \bar{\xi}_v\big)\big]\,dv\right\|_{K_1}^2 \\
&\qquad + 2E\left\|\frac{1}{\Gamma(\alpha)}\int_{u_i}^{u}(u-v)^{\alpha-1}\big[\sigma(v, \bar{\eta}_v + \bar{\xi}_v) - \sigma\big(v, \bar{\eta}_v^* + \bar{\xi}_v\big)\big]\,dW(v)\right\|_{K_1}^2 \\
&\quad \le 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\int_{u_i}^{u}(u-v)^{\alpha-1}\,dv\int_{u_i}^{u}(u-v)^{\alpha-1}E\|f(v, \bar{\eta}_v + \bar{\xi}_v) - f\big(v, \bar{\eta}_v^* + \bar{\xi}_v\big)\|_{K_1}^2\,du \\
&\qquad + 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\int_{t_i}^{u}(u-v)^{2(\alpha-1)}E\|\sigma(v, \bar{\eta}_v + \bar{\xi}_v) - \sigma\big(v, \bar{\eta}_v^* + \bar{\xi}_v\big)\|_{\mathcal{L}_2^0}^2\,du \\
&\quad \le 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{(u-u_i)^\alpha}{\alpha}\int_{u_i}^{u}(u-v)^{\alpha-1}L_1\|\bar{\eta}_v - \bar{\eta}_v^*\|_{\mathcal{B}_h}^2\,dv \\
&\qquad + 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\int_{u_i}^{u}(u-v)^{2(\alpha-1)}L_3\|\bar{\eta}_v - \bar{\eta}_v^*\|_{\mathcal{B}_h}^2\,dv \\
&\quad \le 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{T^{2\alpha}}{\alpha^2}L_1\|\bar{\eta}_v - \bar{\eta}_v^*\|_{\mathcal{B}_h}^2 + 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\frac{T^{2\alpha-1}}{2\alpha-1}L_3\|\bar{\eta}_v - \bar{\eta}_v^*\|_{\mathcal{B}_h}^2 \\
&\quad \le 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\left(\frac{T^{2\alpha}}{\alpha^2}L_1 + \frac{T^{2\alpha-1}}{2\alpha-1}L_3\right)l^2\sup_{v\in[0,T]}\|\eta(v) - \eta^*(v)\|_{K_1}^2 \\
&\quad \le 2\left(\frac{1}{\Gamma(\alpha)}\right)^2\left(\frac{T^{2\alpha}}{\alpha^2}L_1 + \frac{T^{2\alpha-1}}{2\alpha-1}L_3\right)l^2\|\eta - \eta^*\|_{\mathcal{B}_b^0}^2.
\end{aligned}$$

From (3.4), Π_2 is a contraction mapping. Therefore, according to Krasnoselskii's fixed point theorem, (2.1) has at least one solution on $(-\infty, T]$. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HB carried out the main results of this paper and drafted the manuscript. JC directed the study and helped to inspect the manuscript. All authors read and approved the final manuscript.

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