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Stability and traveling waves in a Beddington-DeAngelis type stage-structured predator-prey reaction-diffusion systems with nonlocal delays and harvesting

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Abstract

The goal of this paper is to study the stability and traveling waves of stage-structured predator-prey reaction-diffusion systems of Beddington-DeAngelis functional response with both nonlocal delays and harvesting. By analyzing the corresponding characteristic equations, the local stability of various equilibria is discussed. We reduce the existence of traveling waves to the existence of a pair of upper-lower solutions by using the cross iteration method and the Schauder's fixed point theorem. The existence of traveling waves connecting the zero equilibrium and the positive equilibrium is then established by constructing a pair of upper-lower solutions.

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Keywords: stability; traveling wave; predator-prey system; Beddington-DeAngelis functional response; nonlocal delay; stage structure; Schauder's fixed point theorem

1 Introduction

The study of the dynamics of predator-prey systems is one of the most popular areas in applied mathematics and theoretical ecology. Over the years, a great deal of predator-prey models have been proposed and investigated extensively since the pioneering work of Lotka [1] in the context of chemical reactions and Volterra [2] in predator-prey dynamics. One crucial ingredient of the predator-prey relationship is the predator's functional response, a function that describes consumption rate of prey by a unit number of predators. There have been several popular functional response types: Holling I-IV types [3, 4], ratio-dependent type [5], Beddington-DeAngelis type [6, 7], and Hassell-Varley type [8]. The functional response has a strong impact on the dynamical behaviors of the systems such as stability, persistence, permanence, bifurcation, periodic oscillation, and so on. In particular, the study of traveling wave solutions has received significant attention in the last few decades.

It is generally recognized that each species' natural tendency is to migrate toward the lower density, so spatial diffusion makes an important contribution to population dynamics, especially to the species invasion and wave propagation. To include spatial variation into our consideration, reaction-diffusion systems have been considered for predator-prey models. The general predator-prey models are given by reaction-diffusion systems and read as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + g(u) - p(u,v)\xi(v), \\ \frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + \eta(v)(p(u,v) - d), \end{cases} \tag{1.1}$$

where the functions $u(t,x)$, $v(t,x)$ denote the density of the prey and the predator at time t and position x, respectively; the constants $d_1 \geq 0$, $d_2 \geq 0$ are the spatial diffusion rates of the two species; the function $g(u)$ is the net growth rate of the prey in the absence of predator; the function $p(u,v)$ is the predator functional response; the constant $d > 0$ is the natural death rate of predator species; the function $\eta(v)p(u,v)$ represents the predation contributes toward the growth of predator species.

The pioneering contributions were by Dunbar [9, 10] who gave a complete analysis of the existence of traveling waves for predator-prey systems (1.1) with Holling type I functional response $p(u,v) = Bu$. Traveling wave solutions of predator-prey systems with Holling type I have been widely studied since [9, 10], see [11, 12] and the references therein. For the situation of Holling type II functional response $p(u,v) = \frac{u}{1+Eu}$, Dunbar [13] considered the traveling wave solutions including the point-to-point orbits, periodic orbits (Hopf bifurcation), and point-to-periodic heteroclinic orbits as $d_1 = 0$. Owen and Lewis [14] showed that system (1.1) with $d_1 \neq 0$ and $d_2 \neq 0$ possesses traveling wave solutions numerically. Huang *et al.* [15] proved theoretically that the numerical simulation in [14] is true. Gardner [16] proved the existence of traveling wave solutions by using the connection index. Later, for $d_1 = 0$, Li and Wu [17], Ding and Huang [18] investigated the existence of traveling waves for system (1.1) but with simplified Holling type III functional response $p(u,v) = \frac{u^2}{1+Eu^2}$. Lin *et al.* [19] extended the result in [17] to a more general Holling type III functional response $p(u,v) = \frac{au^2}{1+bu+u^2}$. Hsu *et al.* [20] also generalized the results of [17] to the case $d_1 \neq 0$ and obtained the existence results for traveling wave solutions of system (1.1) with Ivlev type functional response $p(u,v) = E(1 - e^{-Mu})$. When $d_1 = 0$, diffusive predator-prey systems with Ivlev type scheme is investigated in [21]. There is some work on the traveling waves of diffusive predator-prey systems with Leslie-Gower functional response [22] and Holling type IV functional response $p(u,v) = \frac{u}{1+Eu^2}$ [23]. In particular, some authors studied traveling waves for diffusive predator-prey systems with general functional response, we can refer to [20, 24–31] and the references therein.

In order to reflect time delays that occur frequently in nature, a large body of work has been carried out the traveling waves of delayed predator-prey systems. By constructing upper and lower solutions, Li and Li [32] and Lin *et al.* [33] investigated the following predator-prey systems with Holling type I functional response:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + u(x,t)[a_1 - b_1 u(x,t-\tau_1) - c_1 v(x,t-\tau_2)], \\ \frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + v(x,t)[a_2 + b_2 u(x,t-\tau_3) - c_2 v(x,t-\tau_4)], \end{cases} \tag{1.2}$$

and established the existence of traveling wave solutions connecting $(0,0)$ with the positive steady state, we also refer to Huang and Zou [34]. Furthermore, Liang *et al.* [35] established the existence of the point-to-periodic and the point-to-point heteroclinic traveling wave solutions for the delayed Holling type II predator-prey systems. In realistic ecological models, individuals may not necessarily be at the same spatial location at previous times, that is, the delay affects both the temporal and the spatial variables. Such delays are called spatio-temporal delays or nonlocal delays. The existence of traveling waves for the predator-prey system with nonlocal delays has been taken into account. In [36], Li and Xu studied the effect of nonlocal delays on the existence of traveling wave solutions in reaction-diffusion predator-prey systems (1.2).

As is well known, almost all species have the stage structure of immature and mature stages, performance thus being of different kinds of characteristics at each stage of growth. Therefore, the ecological models with stage structure are more reasonable than the ones without stage structure. Recently, Zhang and Xu [37] and Ge and He [38] studied the existence of traveling waves of predator-prey systems with stage structure for the prey, Holling type I functional response and nonlocal spatial impact. On the other hand, the exploitation of biological resources has generally a strong impact on population dynamics of a harvested species. In particular, stage-structured predator-prey models with harvesting have received many attention. Lv *et al.* [39] investigated the existence of traveling waves for a Holling type II functional response predator-prey model with harvesting and stage structure for predator. Hong and Weng [40] considered the existence of traveling waves for the following predator-prey model with Holling type-II functional response:

$$\begin{cases} \frac{\partial u_1}{\partial t} = D_1\frac{\partial^2 u_1}{\partial x^2} + au_2(x,t) - d_1u_1(x,t) - a_{11}u_1^2(x,t) \\ \qquad - ae^{-d_1\tau}\int_{\mathbb{R}} G(\tau,x-y)u_2(y,t-\tau)\,dy, \\ \frac{\partial u_2}{\partial t} = D_2\frac{\partial^2 u_2}{\partial x^2} + ae^{-d_1\tau}\int_{\mathbb{R}} G(\tau,x-y)u_2(y,t-\tau)\,dy - (d_2+q_2e_2)u_2(x,t) \\ \qquad - a_{22}u_2^2(x,t) - \frac{a_{23}u_2(x,t)v(x,t)}{1+mu_2(x,t)}, \\ \frac{\partial v}{\partial t} = D_3\frac{\partial^2 v}{\partial x^2} + [a_1 - bv(x,t)]v(x,t) - q_3e_3v(x,t) + \frac{a_{32}u_2(x,t)v(x,t)}{1+mu_2(x,t)}, \end{cases} \tag{1.3}$$

where $G(\tau,x-y) = \frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{(x-y)^2}{4D_1\tau}}$.

We note that in the models mentioned above, predator functional response functions used are of prey-dependent type, which neglect the competition and share between predators in the process of searching for limited foods. There is much significant evidence to suggest that predator dependence in the functional response occurs quite frequently in natural systems and laboratory (see *e.g.* [41]). The ratio-dependent functional function incorporates mutual interference by predator, Ge *et al.* [42] considered the existence of traveling waves for a two-species ratio-dependent predator-prey system with diffusion terms and stage structure

$$\begin{cases} \frac{\partial u_1}{\partial t} = D_1\frac{\partial^2 u_1}{\partial x^2} + \alpha u_2(x,t) - \gamma u_1(x,t) - \alpha e^{-\gamma\tau}\int_{\mathbb{R}} G(\tau,x-y)u_2(y,t-\tau)\,dy, \\ \frac{\partial u_2}{\partial t} = D_2\frac{\partial^2 u_2}{\partial x^2} + \alpha e^{-\gamma\tau}\int_{\mathbb{R}} G(\tau,x-y)u_2(y,t-\tau)\,dy - \beta u_2^2(x,t) - \frac{C_0u_2(x,t)v(x,t)}{u_2(x,t)+mv(x,t)}, \\ \frac{\partial v}{\partial t} = D_3\frac{\partial^2 v}{\partial x^2} + v(x,t)(-d + \frac{fu_2(x,t)}{u_2(x,t)+mv(x,t)}). \end{cases} \tag{1.4}$$

However, the ratio-dependent response function has somewhat singular behaviors at low densities, which has been the source of controversy and has been criticized on the other grounds [43, 44]. The Beddington-DeAngelis functional functions $p(u,v) = \frac{mu}{a+bu+cv}$ were introduced independently by Beddington [6] and DeAngelis *et al.* [7], which provide a better description of predator feeding over a range of prey-predator abundances [41]. Beddington-DeAngelis functional function is similar to the well-known Holling type II functional response but it contains an extra term bv in the denominator describing mutual interference by predators. It has some of the same qualitative features as the ratio-dependent models but avoids some of the singular behavior of ratio-dependent models at low densities [45]. Hence it is worthy to further study the existence of traveling waves of the Beddington-DeAngelis model. Ding and Huang [18] investigated the traveling wave solutions of Beddington-DeAngelis type predator-prey systems without delay. Liao *et al.* [46] introduced the maturation delay τ into the predator and proposed the following stage-structured Beddington-DeAngelis models with harvesting:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \frac{\partial^2 u}{\partial x^2} + Au(1 - \frac{u}{k}) - \frac{\alpha uv}{1+au+bv} - q_1 E_1 u, \\ \frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + b_0 e^{-\gamma\tau} v(x, t-\tau) - (d_0 - \frac{\beta u}{1+au+bv})v - q_2 E_2 v. \end{cases} \tag{1.5}$$

They studied the stability of the nonnegative constant equilibria and traveling wavefront connecting the zero solution to the positive equilibrium of the system (1.5).

Motivated by the above, in the present work, we investigate the following predator-prey system:

$$\begin{cases} \frac{\partial u_1}{\partial t} = D_1 \frac{\partial^2 u_1}{\partial x^2} + mu_2(x,t) - d_1 u_1(x,t) - a_{11} u_1^2(x,t) \\ \qquad - me^{-d_1\tau} \int_{\mathbb{R}} G(\tau, x-y) u_2(y, t-\tau)\, dy, \\ \frac{\partial u_2}{\partial t} = D_2 \frac{\partial^2 u_2}{\partial x^2} + me^{-d_1\tau} \int_{\mathbb{R}} G(\tau, x-y) u_2(y, t-\tau)\, dy - (d_2 + q_2 e_2) u_2(x,t) \\ \qquad - a_{22} u_2^2(x,t) - \frac{a_{23} u_2(x,t) v(x,t)}{1+au_2(x,t)+bv(x,t)}, \\ \frac{\partial v}{\partial t} = D_3 \frac{\partial^2 v}{\partial x^2} + a_1 v(x,t) - a_{33} v^2(x,t) - q_3 e_3 v(x,t) + \frac{a_{32} u_2(x,t) v(x,t)}{1+au_2(x,t)+bv(x,t)} \end{cases} \tag{1.6}$$

for $x \in (-\infty, \infty)$ with initial condition

$$u_1(x,0) = \delta_1(x) > 0, \qquad u_2(x,t) = \delta_2(x,t) \geq 0,$$
$$\delta_2(x,0) > 0, \qquad v(x,0) = \delta_3(x) > 0, \quad -\tau \leq t \leq 0,$$

where $G(\tau, x-y) = \frac{1}{\sqrt{4\pi D_1 \tau}} e^{-\frac{(x-y)^2}{4D_1\tau}}$.

The biological meanings for model (1.6) are as follows.

- The variables $u_1(x,t)$, $u_2(x,t)$, $v(x,t)$ represent the densities of the immature prey, mature prey, predator population at time t and position x, respectively. The parameters D_1, D_2, and D_3 are diffusion coefficients.
- The birth rate of immature prey population is proportional to the existing mature prey population with proportionality $m > 0$. $d_1 > 0$ and $a_{11} > 0$ are the death rate and the overcrowding rate of immature prey population, respectively. The term $me^{-d_1\tau} \int_{\mathbb{R}} G(\tau, x-y) u_2(y, t-\tau)\, dy$ represents the total number of prey population which leave juveniles to adults at time t and position x.
- $d_2 > 0$ and $a_{22} > 0$ are the death rate and the overcrowding rate of mature prey population, respectively. The term $\frac{a_{23} u_2(x,t) v(x,t)}{1+au_2(x,t)+bv(x,t)}$ represents the functional response known as the Beddington-DeAngelis response, where a, b stand for mature prey saturation constant and predator interference, respectively. The constants $q_2 > 0$ and $e_2 > 0$ are the catch-ability coefficients and harvesting effort for the mature prey species.
- The predator species not only feed on the given mature prey species, but also feed on other preys. The predator grows logistically with growth rate $a_1 > 0$ and $\frac{a_{33}}{a_1}$ is the

environmental carrying capacity of predator population. The constants $q_3 > 0$ and $e_3 > 0$ are the catch-ability coefficients and harvesting effort of the predator population, respectively. The term $\frac{a_{32}u_2(x,t)v(x,t)}{1+au_2(x,t)+bv(x,t)}$ stands for the growth rate due to predation.

The model (1.6) is similar to the model (1.3), but it has Beddington-DeAngelis functional response $p(u,v) = \frac{hu}{c+au+bv}$. If $b = 0$, then system (1.6) reduces to system (1.3). If $c = 0$, then it gives a ratio-dependent functional response. In this paper, the stability of the equilibria is firstly investigated, and the existence of traveling wave solutions is then established by constructing a pair of upper-lower solutions and using the cross iteration method and Schauder's fixed point theorem. These methods can be found in [47–49] and some references therein.

The remaining parts of this paper are organized as follows. In Section 2, the local stability of equilibria for system (1.6) is discussed by using the linearized method. In Section 3, by applying the cross iteration method and Schauder's fixed point theorem, we reduce the existence of traveling waves connecting $(0,0)$ with the positive equilibrium to the existence of a pair of upper-lower solutions.

2 Local stability of equilibria

In this section, we discuss the local stability of the equilibria of system (1.6). We find that $u_2(x,t)$ and $v(x,t)$ of system (1.6) are independent of $u_1(x,t)$ but determine the behavior of $u_1(x,t)$. Hence, it is sufficient to consider the last two equations. For simplicity of notation, we denote $u_2(x,t)$ by $u(x,t)$ to get the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = D_2\frac{\partial^2 u}{\partial x^2} + me^{-d_1\tau}\int_{\mathbb{R}} G(\tau, x-y)u(y,t-\tau)\,dy - (d_2+q_2e_2)u(x,t) \\ \qquad - a_{22}u^2(x,t) - \frac{a_{23}u(x,t)v(x,t)}{1+au(x,t)+bv(x,t)}, \\ \frac{\partial v}{\partial t} = D_3\frac{\partial^2 v}{\partial x^2} + a_1v(x,t) - a_{33}v^2(x,t) - q_3e_3v(x,t) + \frac{a_{32}u(x,t)v(x,t)}{1+au(x,t)+bv(x,t)}. \end{cases} \tag{2.1}$$

Let $m_1 = me^{-d_1\tau} - d_2 - q_2e_2$, $m_2 = a_1 - q_3e_3$. It is easy to check that system (2.1) possesses three constant equilibria, denoted by $E_0(0,0)$, $E_1(\frac{m_1}{a_{22}},0)$, $E_2(0,\frac{m_2}{a_{33}})$ as $m_1 > 0$ and $m_2 > 0$.

The linearized system of system (2.1) at a constant equilibrium (u^*,v^*) is

$$\begin{cases} \frac{\partial u}{\partial t} = D_2\frac{\partial^2 u}{\partial x^2} + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{(x-y)^2}{4D_1\tau}}u(y,t-\tau)\,dy - (d_2+q_2e_2)u(x,t) \\ \qquad - 2a_{22}u^*u(x,t) - \frac{a_{23}v^*+a_{23}bv^{*2}}{(1+au^*+bv^*)^2}u(x,t) - \frac{a_{23}u^*+a_{23}au^{*2}}{(1+au^*+bv^*)^2}v(x,t), \\ \frac{\partial v}{\partial t} = D_3\frac{\partial^2 v}{\partial x^2} + (a_1-q_3e_3)v(x,t) - 2a_{33}v^*v(x,t) + \frac{a_{32}v^*+a_{32}bv^{*2}}{(1+au^*+bv^*)^2}u(x,t) \\ \qquad + \frac{a_{32}u^*+a_{32}au^{*2}}{(1+au^*+bv^*)^2}v(x,t). \end{cases} \tag{2.2}$$

System (2.2) has nontrivial solutions of the form $\binom{c_1}{c_2}e^{\lambda t+i\sigma x}$ if and only if

$$\begin{aligned} &\left[\lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2 + 2a_{22}u^* + \frac{a_{23}v^*(1+bv^*)}{(1+au^*+bv^*)^2}\right] \\ &\quad\times\left[\lambda + D_3\sigma^2 - a_1 + q_3e_3 + 2a_{33}v^* - \frac{a_{32}u^*(1+au^*)}{(1+au^*+bv^*)^2}\right] \\ &\quad+ \frac{a_{23}a_{32}u^*v^*(1+au^*)(1+bv^*)}{(1+au^*+bv^*)^4} = 0, \end{aligned} \tag{2.3}$$

where λ is a complex number and σ is a real number.

2.1 Asymptotical stability of $E_0(0,0)$

From (2.3), it follows that at the equilibrium $E_0(0,0)$:

$$\bigl(\lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2\bigr)\bigl(\lambda + D_3\sigma^2 - a_1 + q_3e_3\bigr) = 0.$$

Thus either

$$\lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2 = 0 \tag{2.4}$$

or

$$\lambda + D_3\sigma^2 - a_1 + q_3e_3 = 0 \tag{2.5}$$

Let

$$f_\sigma(\lambda) = \lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2.$$

We assume that $m_1 = me^{-d_1\tau} - d_2 - q_2e_2 > 0$. Then $f_\sigma(0) = D_2\sigma^2 - me^{-d_1\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2$ and hence there exists a $\sigma_1 > 0$ such that $f_{\sigma_1}(0) < 0$. Further, we have $f_{\sigma_1}(\lambda) \to +\infty$ as $\lambda \to +\infty$, we can see that there exists a $\lambda_1 > 0$ such that (λ_1, σ_1) satisfying (2.4). If $m_1 < 0$, we claim that $\lambda < 0$ for all (λ, σ) satisfying (2.4). Otherwise, we suppose that there exists a (λ_2, σ_2) satisfying (2.4) such that $\operatorname{Re}\lambda_2 \geq 0$. Then

$$\operatorname{Re}\lambda_2 + D_2\sigma_2^2 + d_2 + q_2e_2 \leq \bigl|\lambda_2 + D_2\sigma_2^2 + d_2 + q_2e_2\bigr| = \bigl|me^{-d_1\tau-\lambda_2\tau}e^{-D_1\tau\sigma_2^2}\bigr| \leq me^{-d_1\tau},$$

which contradicts the fact that $m_1 < 0$.

By a similar method, we can show that there exists a $\lambda_3 > 0$ such that (λ_3, σ_3) satisfies (2.5) when $m_2 = a_1 - q_3e_3 > 0$ and $\lambda < 0$ for all (λ, σ) satisfying (2.5) if $m_2 < 0$.

From the above discussion, we obtain the following result.

Theorem 2.1 *Assume that either $m_1 = me^{-d_1\tau} - d_2 - q_2e_2 > 0$ or $m_2 = a_1 - q_3e_3 > 0$. Then the equilibrium $E_0(0,0)$ is linearly unstable.*

2.2 Asymptotical stability of $E_1(\frac{m_1}{a_{22}}, 0)$

From (2.3), it follows that at $E_1(\frac{m_1}{a_{22}}, 0)$, either

$$\lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2 + 2m_1 = 0 \tag{2.6}$$

or

$$\lambda + D_3\sigma^2 - a_1 + q_3e_3 - \frac{\frac{a_{32}m_1}{a_{22}}}{1 + \frac{am_1}{a_{22}}} = 0. \tag{2.7}$$

We have the following conclusion.

Theorem 2.2 *Assume that $m_1 > 0$, then*

(1) *when $m_2 + \frac{\frac{a_{32}m_1}{a_{22}}}{1+\frac{am_1}{a_{22}}} < 0$, the equilibrium $E_1(\frac{m_1}{a_{22}}, 0)$ is locally asymptotically stable*;

(2) *when* $m_2 + \frac{\frac{a_{32}m_1}{a_{22}}}{1+\frac{am_1}{a_{22}}} > 0$, *the equilibrium* $E_1(\frac{m_1}{a_{22}}, 0)$ *is unstable.*

Proof (1) If $m_2 + \frac{\frac{a_{32}m_1}{a_{22}}}{1+\frac{am_1}{a_{22}}} < 0$, then from (2.7), we know that

$$\lambda = -D_3\sigma^2 + a_1 - q_3e_3 + \frac{\frac{a_{32}m_1}{a_{22}}}{1+\frac{am_1}{a_{22}}} < 0 \quad \text{for any } \sigma,$$

i.e., all roots of equation (2.7) are negative.

Using (2.6), we have

$$\lambda + D_2\sigma^2 - d_2 - q_2e_2 + 2me^{-d_1\tau} = me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2}. \tag{2.8}$$

We claim that all roots of (2.8) satisfy $\operatorname{Re}\lambda < 0$. Otherwise, there exists a root λ_4 satisfying $\operatorname{Re}\lambda_4 \geq 0$. Then

$$\begin{aligned} \left|me^{-d_1\tau}\right| &< \left|\operatorname{Re}\lambda_4 + i\operatorname{Im}\lambda_4 + me^{-d_1\tau} + m_1\right| \\ &< \left|\lambda_4 + D_2\sigma^2 + 2me^{-d_1\tau} - d_2 - q_2e_2\right| \\ &= \left|me^{-d_1\tau-\lambda_4\tau}e^{-D_1\tau\sigma^2}\right| \leq me^{-d_1\tau}, \end{aligned}$$

a contradiction. Therefore, all roots of (2.8) satisfy $\operatorname{Re}\lambda < 0$. Thus, $E_1(\frac{m_1}{a_{22}}, 0)$ is locally asymptotically stable.

(2) If $m_2 + \frac{\frac{a_{32}m_1}{a_{22}}}{1+\frac{am_1}{a_{22}}} > 0$, we can easily see that there exists at least a (λ_5, σ_5) such that $\lambda_5 > 0$ satisfying (2.7). Hence, $E_1(\frac{m_1}{a_{22}}, 0)$ is unstable. The proof is complete. □

2.3 Asymptotical stability of $E_2(0, \frac{m_2}{a_{33}})$

In this subsection, we discuss the local stability of the equilibrium $E_2(0, \frac{m_2}{a_{33}})$. We have the following result.

Theorem 2.3 *Assume that* $m_2 > 0$, *then*

(1) *when* $m_1 < \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}}$, *the equilibrium* $E_2(0, \frac{m_2}{a_{33}})$ *is locally asymptotically stable*;

(2) *when* $m_1 > \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}}$, *the equilibrium* $E_2(0, \frac{m_2}{a_{33}})$ *is unstable.*

Proof In this case, we get from (2.3) that

$$\left(\lambda + D_2\sigma^2 + d_2 + q_2e_2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}}\right)\left(\lambda + D_3\sigma^2 + m_2\right) = 0. \tag{2.9}$$

From the second factor of (2.9), we have

$$\lambda = -D_3\sigma^2 - m_2 < 0.$$

(1) If $m_1 < \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}}$, we claim that all roots of (2.9) satisfy $\operatorname{Re}\lambda < 0$. Otherwise, we suppose that there exists a root λ_5 of (2.9) satisfying $\operatorname{Re}\lambda_5 \geq 0$. From the first factor of (2.9), we

directly calculate the real part of λ_5 to yield

$$\begin{aligned}\operatorname{Re}\lambda_5 &= -D_2\sigma^2 - d_2 - q_2e_2 + me^{-d_1\tau-\tau\operatorname{Re}\lambda_5}e^{-D_1\tau\sigma^2}\cos(\tau\operatorname{Im}\lambda_5) - \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}}\\ &\le me^{-d_1\tau} - d_2 - q_2e_2 - \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}} = m_1 - \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}} < 0,\end{aligned}$$

which contradicts with the above assumption $\operatorname{Re}\lambda_5 \ge 0$. Thus, the equilibrium $E_2(0, \frac{m_2}{a_{33}})$ is locally asymptotically stable.

(2) If $m_1 > \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}}$, we claim that there exists at least one (λ_6, σ_6) satisfying (2.9) such that $\operatorname{Re}\lambda_6 > 0$. Let

$$f_\sigma(\lambda) = -D_2\sigma^2 - d_2 - q_2e_2 + me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} - \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}}.$$

Then we have

$$f_0(0) = me^{-d_1\tau} - d_2 - q_2e_2 - \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}} > 0.$$

Hence, there exists a sufficiently small σ_6 such that

$$f_{\sigma_6}(0) = -D_2\sigma_6^2 - d_2 - q_2e_2 + me^{-d_1\tau}e^{-D_1\tau\sigma_6^2} - \frac{\frac{a_{23}m_2}{a_{33}}}{1+\frac{bm_2}{a_{33}}} > 0.$$

Using the fact $f_{\sigma_6}(\infty) < 0$, we deduce that $\lambda = f_{\sigma_6}(\lambda)$ has positive root λ_6. Thus, the equilibrium $E_2(0, \frac{m_2}{a_{33}})$ is unstable. The proof is complete. □

2.4 Asymptotical stability of the positive equilibrium $E^*(k_1, k_2)$

In this subsection, we consider the existence and stability of the positive equilibrium $E^*(k_1, k_2)$. The positive equilibrium $E^*(k_1, k_2)$ is given by the solution of the following system:

$$m_1 - a_{22}k_1 - \frac{a_{23}k_2}{1+ak_1+bk_2} = 0, \tag{2.10}$$

$$m_2 - a_{33}k_2 + \frac{a_{32}k_1}{1+ak_1+bk_2} = 0. \tag{2.11}$$

Substituting the value of $k_2 = \frac{(m_1-a_{22}k_1)(1+ak_1)}{a_{23}-bm_1+ba_{22}k_1}$ from equation (2.10) into (2.11), we have the following equation in k_1:

$$A_0k_1^3 + A_1k_1^2 + A_2k_1 + A_3 = 0, \tag{2.12}$$

where

$$A_0 = a_{22}a_{23}a_{33}a^2 + a_{22}^2a_{32}b^2,$$

$$A_1 = aba_{22}m_2a_{23} + 2aa_{22}a_{23}a_{33} - a^2a_{23}a_{33}m_1 + 2ba_{22}a_{23}a_{32} - 2b^2a_{22}a_{32}m_1,$$

$$\begin{aligned}A_2 &= ba_{22}a_{23}m_2 + aa_{23}^2m_2 - aba_{23}m_1m_2 + a_{22}a_{23}a_{33}\\&\quad - 2aa_{23}a_{33}m_1 + a_{23}^2a_{32} + b^2m_1^2a_{32} - 2a_{23}a_{32}bm_1,\\A_3 &= a_{23}^2m_2 - a_{23}bm_1m_2 - a_{23}a_{33}m_1.\end{aligned}$$

The constant term of equation (2.12) is negative if $bm_1 > a_{23}$ and $A_0 > 0$, it follows that the cubic equation (2.12) possesses at least one positive root k_1. For this value of k_1, the corresponding value of k_2 will be $k_2 = \frac{(m_1-a_{22}k_1)(1+ak_1)}{a_{23}-bm_1+ba_{22}k_1}$.

At the equilibrium $E^*(k_1,k_2)$, (2.3) becomes

$$\begin{aligned}&\left[\lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2 + 2a_{22}k_1 + \frac{a_{23}k_2(1+bk_2)}{(1+ak_1+bk_2)^2}\right]\\&\quad\times\left[\lambda + D_3\sigma^2 - a_1 + q_3e_3 + 2a_{33}k_2 - \frac{a_{32}k_1(1+ak_1)}{(1+ak_1+bk_2)^2}\right]\\&\quad+\frac{a_{23}a_{32}k_1k_2(1+ak_1)(1+bk_2)}{(1+ak_1+bk_2)^4} = 0.\end{aligned}\tag{2.13}$$

Let $\gamma_1 = \frac{a_{32}k_1}{(1+ak_1+bk_2)^2}$, $\gamma_2 = \frac{a_{23}k_2}{(1+ak_1+bk_2)^2}$. Obviously, if (λ,σ) satisfies (2.12), then $\lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2 + 2a_{22}k_1 + \gamma_2(1+bk_2) \neq 0$. Thus, we can rewrite (2.13) as

$$\begin{aligned}\lambda &= -\frac{\gamma_1\gamma_2(1+ak_1)(1+bk_2)}{\lambda + D_2\sigma^2 - me^{-d_1\tau-\lambda\tau}e^{-D_1\tau\sigma^2} + d_2 + q_2e_2 + 2a_{22}k_1 + \gamma_2(1+bk_2)}\\&\quad-\big(D_3\sigma^2 + a_{33}k_2 + bk_2\gamma_1\big).\end{aligned}\tag{2.14}$$

We assume that $a_{22} \geq \frac{aa_{23}k_2}{(1+ak_1+bk_2)^2}$ holds. Substituting $\lambda = \mu + i\omega$ into (2.14), we claim that $\mu < 0$. Otherwise, we suppose that there exists a $(\mu_7 + i\omega_7, \sigma_7)$ such that $\mu_7 \geq 0$. Thus,

$$\begin{aligned}0 &\leq \mu_7\\&= -\big(\gamma_1\gamma_2(1+ak_1)(1+bk_2)\big[\mu_7 + D_2\sigma_7^2 - me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\cos(\omega_7\tau) + d_2 + q_2e_2\\&\quad+ 2a_{22}k_1 + \gamma_2(1+bk_2)\big]/\big(\big[\mu_7 + D_2\sigma_7^2 - me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\cos(\omega_7\tau) + d_2 + q_2e_2\\&\quad+ 2a_{22}k_1 + \gamma_2(1+bk_2)\big]^2 + \big[\omega_7 + me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\sin(\omega_7\tau)\big]^2\big)\big)\\&\quad-\big(D_3\sigma^2 + a_{33}k_2 + bk_2\gamma_1\big)\\&\leq -\big(\gamma_1\gamma_2(1+ak_1)(1+bk_2)\big[-me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\cos(\omega_7\tau) + d_2 + q_2e_2\\&\quad+ 2a_{22}k_1 + \gamma_2(1+bk_2)\big]/\big(\big[\mu_7 + D_2\sigma^2 - me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\cos(\omega_7\tau) + d_2 + q_2e_2\\&\quad+ 2a_{22}k_1 + \gamma_2(1+bk_2)\big]^2 + \big[\omega_7 + me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\sin(\omega_7\tau)\big]^2\big)\big)\\&\quad-\big(D_3\sigma^2 + a_{33}k_2 + bk_2\gamma_1\big)\\&= -\big(\gamma_1\gamma_2(1+ak_1)(1+bk_2)\big[-me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\cos(\omega_7\tau) + me^{-d_1\tau} + a_{22}k_1 - \gamma_2ak_1\big]\\&\quad/\big(\big[\mu_7 + D_2\sigma^2 - me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\cos(\omega_7\tau) + d_2 + q_2e_2 + 2a_{22}k_1 + \gamma_2(1+bk_2)\big]^2\\&\quad+\big[\omega_7 + me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\sin(\omega_7\tau)\big]^2\big)\big)\\&\quad-\big(D_3\sigma^2 + a_{33}k_2 + bk_2\gamma_1\big)\\&\leq -\big(\gamma_1\gamma_2(1+ak_1)(1+bk_2)[a_{22}k_1 - \gamma_2ak_1]/\big(\big[\mu_7 + D_2\sigma^2 - me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\cos(\omega_7\tau)\end{aligned}$$

$$+ d_2 + q_2e_2 + 2a_{22}k_1 + \gamma_2(1 + bk_2)\big]^2 + \big[\omega_7 + me^{-d_1\tau-\mu_7\tau}e^{-D_1\tau\sigma_7^2}\sin(\omega_7\tau)\big]^2\big)\big)$$
$$- \big(D_3\sigma^2 + a_{33}k_2 + bk_2\gamma_1\big) < 0,$$

a contradiction.

In summary, we can obtain the following theorem.

Theorem 2.4 *Assume that $E^*(k_1,k_2)$ exists. If $a_{22} \geq \frac{aa_{23}k_2}{(1+ak_1+bk_2)^2}$ holds, then the positive equilibrium $E^*(k_1,k_2)$ is locally asymptotically stable.*

3 Existence of traveling wave solutions

In this section, we use the Schauder fixed point theorem, and a cross iteration scheme associated with upper-lower solutions to establish the existence of traveling waves connecting the zero equilibrium and positive equilibrium $E^*(k_1,k_2)$.

A traveling wave solution of (2.1) is a special translation invariant solution of the form $u(x,t) = \varphi(x + ct)$, $v(x,t) = \psi(x + ct)$, where $c > 0$ is the wave speed and φ, ψ are the wave profile functions. Substituting $u(x,t) = \varphi(x + ct)$, $v(x,t) = \psi(x + ct)$ into (2.1), and denoting the traveling wave coordinate $x + ct$ still by t, we obtain the corresponding wave equations

$$\begin{cases} D_2\varphi''(t) - c\varphi'(t) + f_2(\varphi,\psi)(t) = 0, \\ D_3\psi''(t) - c\psi'(t) + f_3(\varphi,\psi)(t) = 0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f_2(\varphi,\psi)(t) &= me^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\varphi(t-y-c\tau)\,dy - (d_2+q_2e_2)\varphi(t) \\ &\quad - a_{22}\varphi^2(t) - \frac{a_{23}\varphi(t)\psi(t)}{1+a\varphi(t)+b\psi(t)}, \\ f_3(\varphi,\psi)(t) &= a_1\psi(t) - q_3e_3\psi(t) - a_{33}\psi^2(t) + \frac{a_{32}\varphi(t)\psi(t)}{1+a\varphi(t)+b\psi(t)}. \end{aligned} \tag{3.2}$$

In this paper, we are interested in traveling wave solutions satisfying the following asymptotic boundary conditions:

$$\lim_{t\to-\infty}\big(\varphi(t),\psi(t)\big) = (0,0), \qquad \lim_{t\to+\infty}\big(\varphi(t),\psi(t)\big) = (k_1,k_2). \tag{3.3}$$

We also need the following definition of upper and lower solutions to system (2.1).

Definition 3.1 A pair of continuous functions $\overline{\Phi} = (\overline{\varphi},\overline{\psi})$ and $\underline{\Phi} = (\underline{\varphi},\underline{\psi})$ are called a pair of upper-lower solutions of (2.1), respectively, if there exists $T = \{T_i|i = 1,\dots,m\}$ such that $\overline{\Phi}$ and $\underline{\Phi}$ are continuously differentiable in $\mathbb{R}\backslash T$ and satisfy

$$\begin{cases} D_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + f_2(\overline{\varphi},\underline{\psi})(t) \leq 0, \\ D_3\overline{\psi}''(t) - c\overline{\psi}'(t) + f_3(\overline{\phi},\overline{\psi})(t) \leq 0, \end{cases}$$

and

$$\begin{cases} D_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + f_2(\underline{\varphi},\overline{\psi})(t) \geq 0, \\ D_3\underline{\psi}''(t) - c\underline{\psi}'(t) + f_3(\underline{\phi},\underline{\psi})(t) \geq 0, \end{cases}$$

respectively.

Let

$$M_2 \geq \max\left\{k_1, \frac{me^{-d_1\tau} - d_2 - q_2e_2}{a_{22}}\right\}, \qquad M_3 \geq \max\left\{k_2, \frac{a_1 - q_3e_3 + a_{32}M_2}{a_{33}}\right\}.$$

Define

$$C_{[\mathbf{0},\mathbf{M}]}\big(\mathbb{R},\mathbb{R}^2\big) = \big\{(\varphi,\psi) \in C\big(\mathbb{R},\mathbb{R}^2\big) : 0 \leq \varphi(s) \leq M_2, 0 \leq \psi(s) \leq M_3\big\},$$

where $\mathbf{M} := (M_2, M_3)$.

Take

$$\beta_2 \geq d_2 + q_2e_2 + 2a_{22}M_2 + a_{23}M_3(1 + bM_3), \qquad \beta_3 \geq q_3e_3 + 2a_{33}M_3 - a_1. \tag{3.4}$$

Define the operator $H = (H_1, H_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^2) \to C(\mathbb{R},\mathbb{R}^2)$ by

$$H_2(\varphi,\psi)(t) = f_2(\varphi,\psi)(t) + \beta_2\varphi(t), \qquad H_3(\varphi,\psi)(t) = f_3(\varphi,\psi)(t) + \beta_3\psi(t)$$

and the operator $\mathbf{F} = (F_1, F_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^2) \to C(\mathbb{R},\mathbb{R}^2)$ by

$$F_2(\varphi,\psi)(t) = \frac{1}{D_2(\lambda_{22} - \lambda_{21})}\left[\int_{-\infty}^{t} e^{\lambda_{21}(t-s)}H_2(\varphi,\psi)(s)\,ds + \int_{t}^{\infty} e^{\lambda_{22}(t-s)}H_2(\varphi,\psi)(s)\,ds\right],$$

$$F_3(\varphi,\psi)(t) = \frac{1}{D_3(\lambda_{32} - \lambda_{31})}\left[\int_{-\infty}^{t} e^{\lambda_{31}(t-s)}H_3(\varphi,\psi)(s)\,ds + \int_{t}^{\infty} e^{\lambda_{32}(t-s)}H_3(\varphi,\psi)(s)\,ds\right],$$

where

$$\lambda_{21} = \frac{c - \sqrt{c^2 + 4\beta_2D_2}}{2D_2} < 0, \qquad \lambda_{22} = \frac{c + \sqrt{c^2 + 4\beta_2D_2}}{2D_2} > 0,$$

$$\lambda_{31} = \frac{c - \sqrt{c^2 + 4\beta_3D_3}}{2D_3} < 0, \qquad \lambda_{32} = \frac{c + \sqrt{c^2 + 4\beta_3D_3}}{2D_3} > 0.$$

It is obvious that F is well defined and a fixed point of F is a solution of (3.1), which is a traveling solution of (2.1) connecting $(0,0)$ and (k_1,k_2) if it satisfies (3.3).

Throughout this paper, we use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^2$ and $\|\cdot\|$ to denote the supremum norm in $C([-\tau,0],\mathbb{R}^2)$.

For $0 < \mu < \min\{-\lambda_{21}, \lambda_{22}, -\lambda_{31}, \lambda_{32}\}$, we equipped $C(\mathbb{R},\mathbb{R}^2)$ with the exponential decay norm

$$\big|(\varphi,\psi)\big|_\mu = \sup_{t\in\mathbb{R}}\big|(\varphi,\psi)(t)\big|e^{-\mu|t|}.$$

Denote

$$B_\mu\big(\mathbb{R},\mathbb{R}^2\big) = \left\{(\varphi,\psi) \in C\big(\mathbb{R},\mathbb{R}^2\big), \sup_{t\in\mathbb{R}}\big|(\varphi,\psi)(t)\big|e^{-\mu|t|} < \infty\right\}.$$

Then it is easy to check that $(B_\mu(\mathbb{R},\mathbb{R}^n), |\cdot|_\mu)$ is a Banach space.

Now, we can follow the method of [40] to prove the properties of H and F.

Lemma 3.1 *Assume that β_2, β_3 satisfy* (3.4). *Then*

$$\begin{aligned} &H_2(\varphi_2,\psi_1)(t)\le H_2(\varphi_1,\psi_1)(t), \qquad H_2(\varphi_1,\psi_1)(t)\le H_2(\varphi_1,\psi_2)(t),\\ &H_3(\varphi_2,\psi_1)(t)\le H_3(\varphi_1,\psi_1)(t), \qquad H_3(\varphi_1,\psi_2)(t)\le H_3(\varphi_1,\psi_1)(t) \end{aligned}$$

for $t\in\mathbb{R}$ and $\varphi_i,\psi_i\in C(\mathbb{R},\mathbb{R})$, $i=1,2$, with $0\le\varphi_2(t)\le\varphi_1(t)\le M_2$, $0\le\psi_2(t)\le\psi_1(t)\le M_3$.

Lemma 3.2 *Assume that β_2, β_3 satisfy* (3.4). *Then*

$$\begin{aligned} &F_2(\varphi_2,\psi_1)(t)\le F_2(\varphi_1,\psi_1)(t), \qquad F_2(\varphi_1,\psi_1)(t)\le F_2(\varphi_1,\psi_2)(t),\\ &F_3(\varphi_2,\psi_1)(t)\le F_3(\varphi_1,\psi_1)(t), \qquad F_3(\varphi_1,\psi_2)(t)\le F_3(\varphi_1,\psi_1)(t) \end{aligned}$$

for $t\in\mathbb{R}$ and $\varphi_i,\psi_i\in C(\mathbb{R},\mathbb{R})$, $i=1,2$, with $0\le\varphi_2(t)\le\varphi_1(t)\le M_2$, $0\le\psi_2(t)\le\psi_1(t)\le M_3$.

In the following, we assume that a desired pair of upper-lower solutions $(\overline{\varphi},\overline{\psi})$ and $(\underline{\varphi},\underline{\psi})$ of (3.1) are given so that

(P1) $(0,0)\le(\underline{\varphi}(t),\underline{\psi}(t))\le(\overline{\varphi}(t),\overline{\psi}(t))\le(M_2,M_3)$, $t\in\mathbb{R}$;
(P2) $\lim_{t\to-\infty}(\overline{\varphi}(t),\overline{\psi}(t))=(0,0)$, $\lim_{t\to\infty}(\underline{\varphi}(t),\underline{\psi}(t))=\lim_{t\to\infty}(\overline{\varphi}(t),\overline{\psi}(t))=(k_1,k_2)$;
(P3) $\overline{\phi}'(t+)\le\overline{\phi}'(t-)$, $\underline{\phi}'(t+)\ge\underline{\phi}'(t-)$ for all $t\in\mathbb{R}$.

Define the wave profile set Γ by

$$\Gamma=\bigl\{(\varphi,\psi)\in C_{[0,M]}\bigl(\mathbb{R},\mathbb{R}^2\bigr)\big|\bigl(\underline{\varphi}(t),\underline{\psi}(t)\bigr)\le\bigl(\varphi(t),\psi(t)\bigr)\le\bigl(\overline{\varphi}(t),\overline{\psi}(t)\bigr),t\in\mathbb{R}\bigr\}.$$

Obviously, Γ is a nonempty, closed, and bounded convex set.

Similar to the proof in [40], we can also obtain the following conclusions.

Lemma 3.3 *$F=(F_2,F_3)$ is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R},\mathbb{R}^2)$.*

Lemma 3.4 *$F:\Gamma\to\Gamma$.*

Lemma 3.5 *$F:\Gamma\to\Gamma$ is compact with respect to the decay norm $|\cdot|_\mu$.*

Now we formulate our main result as follows.

Theorem 3.1 *Suppose that there is a desirable pair of upper-lower solutions $(\overline{\varphi},\overline{\psi})$ and $(\underline{\varphi},\underline{\psi})$ for* (3.1) *satisfying* (P1)-(P3). *Then system* (1.6) *has a traveling wave solution.*

In order to apply Theorem 3.1, we need to construct a pair of upper and lower solutions for (3.1). Define

$$\begin{aligned} &\Delta_1(\lambda,c)=D_2\lambda^2-c\lambda+me^{-d_1\tau}e^{(D_1\lambda^2-c\lambda)\tau}-d_2-q_2e_2,\\ &\Delta_2(\lambda,c)=D_3\lambda^2-c\lambda+a_1-q_3e_3. \end{aligned} \tag{3.5}$$

Throughout the paper, we assume that $m_1>0$, $m_2>0$. We can easily obtain the following results.

Lemma 3.6 *There exist $c_1 > 0$, $c_2 > 0$ such that $\Delta_1(\lambda, c)$, $\Delta_2(\lambda, c)$ have two distinct positive roots $\lambda_1(c)$, $\lambda_2(c)$ and $\lambda_3(c)$, $\lambda_4(c)$ with $\lambda_1(c) < \lambda_2(c)$, $\lambda_3(c) < \lambda_4(c)$ for any $c > c_1$ and $c > c_2$, respectively. Moreover,*

$$\Delta_1(\lambda, c)\begin{cases} > 0 & \text{for } 0 < \lambda < \lambda_1(c), \\ < 0 & \text{for } \lambda_1(c) < \lambda < \lambda_2(c), \\ > 0 & \text{for } \lambda > \lambda_2(c), \end{cases} \qquad \Delta_2(\lambda, c)\begin{cases} > 0 & \text{for } 0 < \lambda < \lambda_3(c), \\ < 0 & \text{for } \lambda_3(c) < \lambda < \lambda_4(c), \\ > 0 & \text{for } \lambda > \lambda_4(c). \end{cases}$$

For convenience, we denote $\lambda_i = \lambda_i(c)$ $(i = 1, 2, 3, 4)$. For fixed

$$\eta \in \left(1, \min\left\{2, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_4}{\lambda_3}, \frac{\lambda_1 + \lambda_3}{\lambda_1}, \frac{\lambda_1 + \lambda_3}{\lambda_3}\right\}\right)$$

and large constant $q > 0$, we define the functions $l_1(t) = e^{\lambda_1 t} - qe^{\eta\lambda_1 t}$, $l_2(t) = e^{\lambda_3 t} - qe^{\eta\lambda_3 t}$. We can easily see that $l_1(t)$ and $l_2(t)$ have global maxima $h_1 > 0$, $h_2 > 0$, respectively. Define $t_3 = \max\{t : l_1(t) = h_1\}$, $t_4 = \max\{t : l_2(t) = h_2\}$. Then, for any given small $\lambda > 0$, there exist ε_1 $(0 < \varepsilon_1 < (\sqrt{2} - 1)k_1)$, ε_2 $(0 < \varepsilon_2 < \frac{k_2}{2})$ such that

$$k_1 - \varepsilon_1 e^{-\lambda t_3} = h_1, \qquad k_2 - \varepsilon_2 e^{-\lambda t_4} = h_2.$$

For the above constants, we define the continuous functions as follows:

$$\overline{\varphi}(t) = \begin{cases} e^{\lambda_1 t}, & t \le t_1, \\ \min\{M_2, k_1 + k_1 e^{-\lambda t}\}, & t \ge t_1, \end{cases}$$

$$\overline{\psi}(t) = \begin{cases} e^{\lambda_3 t} + qe^{\eta\lambda_3 t}, & t \le t_2, \\ \min\{M_3, k_2 + k_2 e^{-\lambda t}\}, & t \ge t_2, \end{cases}$$

$$\underline{\varphi}(t) = \begin{cases} e^{\lambda_1 t} - qe^{\eta\lambda_1 t}, & t \le t_3, \\ k_1 - \varepsilon_1 e^{-\lambda t}, & t \ge t_3, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} e^{\lambda_3 t} - qe^{\eta\lambda_3 t}, & t \le t_4, \\ k_2 - \varepsilon_2 e^{-\lambda t}, & t \ge t_4, \end{cases}$$

where $q > 0$ is large enough and $\lambda > 0$ is small enough. If $q > 0$ is large enough, we can easily see that $t_1 \ge \max\{t_2, t_3, t_4\}$.

It is easy to check that $(\overline{\varphi}(t), \overline{\psi}(t))$, $(\underline{\varphi}(t), \underline{\psi}(t))$ satisfy (P1)-(P3). We now prove that the continuous functions $(\overline{\varphi}(t), \overline{\psi}(t))$ and $(\underline{\varphi}(t), \underline{\psi}(t))$ are an upper solution and a lower solution of (3.1), respectively.

Lemma 3.7 *Assume that $a_{22}k_1 \ge \frac{(3+2\sqrt{2})a_{23}M_3}{1+ak_1+bk_2}$ and $a_{33}k_2 \ge \frac{4a_{32}k_1}{1+ak_1+bk_2}$ hold. Then $(\overline{\varphi}(t), \overline{\psi}(t))$ is an upper solution of* (3.1).

Proof For $(\overline{\varphi}(t), \overline{\psi}(t)) \in C(\mathbb{R}, \mathbb{R}^2)$, if $t \le t_1$, then $\overline{\varphi}(t) = e^{\lambda_1 t}$. If $t - y - c\tau \le t_1$, then $\overline{\varphi}(t - y - c\tau) = e^{\lambda_1(t-y-c\tau)}$; if $t - y - c\tau > t_1$, then $\overline{\varphi}(t - y - c\tau) = \min\{M_2, k_2 + k_2 e^{-\lambda(t-y-c\tau)}\} \le e^{\lambda_1(t-y-c\tau)}$. Thus,

$$D_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} \overline{\varphi}(t - y - c\tau)\, dy - (d_2 + q_2 e_2)\overline{\varphi}(t)$$
$$- a_{22}\overline{\varphi}^2(t) - \frac{a_{23}\overline{\varphi}(t)\underline{\psi}(t)}{1 + a\overline{\varphi}(t) + b\underline{\psi}(t)}$$

$$
\begin{aligned}
&\le D_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\overline{\varphi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\overline{\varphi}(t) \\
&= e^{\lambda_1 t}\bigl[D_2\lambda_1^2 - c\lambda_1 + me^{-d_1\tau}e^{(D_1\lambda_1^2-c\lambda_1)\tau} - d_2 - q_2e_2\bigr] = e^{\lambda_1 t}\Delta_1(\lambda_1,c) = 0.
\end{aligned}
$$

If $t > t_1$, then $\overline{\varphi}(t) = M_2$, and if $t-y-c\tau \le t_1$, then $\overline{\varphi}(t-y-c\tau) = e^{\lambda_1(t-y-c\tau)} \le M_2$; if $t-y-c\tau > t_1$, then $\overline{\varphi}(t-y-c\tau) = M_2$. Since $M_2 \ge \max\{k_1, \frac{me^{-d_1\tau}-d_2-q_2e_2}{a_{22}}\}$, we know that

$$
\begin{aligned}
&D_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\overline{\varphi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\overline{\varphi}(t) \\
&\quad - a_{22}\overline{\varphi}^2(t) - \frac{a_{23}\overline{\varphi}(t)\underline{\psi}(t)}{1+a\overline{\varphi}(t)+b\underline{\psi}(t)} \\
&\le me^{-d_1\tau}M_2 - (d_2+q_2e_2)M_2 - a_{22}M_2^2 \le 0.
\end{aligned}
$$

Otherwise, $\overline{\varphi}(t) = k_1 + k_1e^{-\lambda t}$, $\underline{\psi}(t) = k_2 - \varepsilon_2e^{-\lambda t}$. If $t-y-c\tau \le t_1$, then $\overline{\varphi}(t-y-c\tau) = e^{\lambda_1(t-y-c\tau)} \le e^{\lambda_1t_1} = k_1 + k_1e^{-\lambda t_1} \le k_1 + k_1e^{-\lambda(t-y-c\tau)}$; if $t-y-c\tau > t_1$, then $\overline{\varphi}(t-y-c\tau) = k_1 + k_1e^{-\lambda(t-y-c\tau)}$. Therefore, we have

$$
\begin{aligned}
&D_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\overline{\varphi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\overline{\varphi}(t) \\
&\quad - a_{22}\overline{\varphi}^2(t) - \frac{a_{23}\overline{\varphi}(t)\underline{\psi}(t)}{1+a\overline{\varphi}(t)+b\underline{\psi}(t)} \\
&\le D_2k_1\lambda^2e^{-\lambda t} + c\lambda k_1e^{-\lambda t} + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\bigl(k_1 + k_1e^{-\lambda(t-y-c\tau)}\bigr)\,dy \\
&\quad - (d_2+q_2e_2)\bigl(k_1+k_1e^{-\lambda t}\bigr) - a_{22}\bigl(k_1+k_1e^{-\lambda t}\bigr)^2 - \frac{a_{23}(k_1+k_1e^{-\lambda t})(k_2-\varepsilon_2e^{-\lambda t})}{1+ak_1(1+e^{-\lambda t})+b(k_2-\varepsilon_2e^{-\lambda t})} \\
&= D_2k_1\lambda^2e^{-\lambda t} + c\lambda k_1e^{-\lambda t} + me^{-d_1\tau}k_1 + me^{-d_1\tau}k_1e^{-\lambda(t-c\tau)}e^{\lambda^2D_1\tau} - (d_2+q_2e_2)k_1 \\
&\quad - (d_2+q_2e_2)k_1e^{-\lambda t} - a_{22}k_1^2 - 2a_{22}k_1^2e^{-\lambda t} - a_{22}k_1^2e^{-2\lambda t} \\
&\quad - \frac{a_{23}(k_1+k_1e^{-\lambda t})(k_2-\varepsilon_2e^{-\lambda t})}{1+ak_1(1+e^{-\lambda t})+b(k_2-\varepsilon_2e^{-\lambda t})} \\
&= k_1e^{-\lambda t}\Delta_1(-\lambda,c) - 2a_{22}k_1^2e^{-\lambda t} - a_{22}k_1^2e^{-2\lambda t} + \frac{a_{23}k_1k_2}{1+ak_1+bk_2} \\
&\quad - \frac{a_{23}(k_1+k_1e^{-\lambda t})(k_2-\varepsilon_2e^{-\lambda t})}{1+ak_1(1+e^{-\lambda t})+b(k_2-\varepsilon_2e^{-\lambda t})} \\
&= k_1e^{-\lambda t}\left\{\Delta_1(-\lambda,c) - \frac{3}{2}a_{22}k_1\right\} - k_1\left\{\frac{1}{2}a_{22}k_1e^{-\lambda t} + a_{22}k_1e^{-2\lambda t} - \frac{a_{23}k_2}{1+ak_1+bk_2}\right. \\
&\quad \left. + \frac{a_{23}(1+e^{-\lambda t})(k_2-\varepsilon_2e^{-\lambda t})}{1+ak_1(1+e^{-\lambda t})+b(k_2-\varepsilon_2e^{-\lambda t})}\right\}.
\end{aligned}
$$

Since $a_{22}k_1 \ge \frac{(3+2\sqrt{2})a_{23}M_3}{1+ak_1+bk_2}$, we have

$$
\begin{aligned}
\Delta_1(0,c) - \frac{3}{2}a_{22}k_1 = m_1 - \frac{3}{2}a_{22}k_1 &= \frac{a_{23}k_2}{1+ak_1+bk_2} - \frac{1}{2}a_{22}k_1 \\
&\le \frac{a_{23}k_2}{1+ak_1+bk_2} - \frac{(3+2\sqrt{2})a_{23}M_3}{2(1+ak_1+bk_2)} < 0,
\end{aligned}
$$

and there exists a constant λ_1^* such that $\Delta_1(-\lambda, c) - \frac{3}{2}a_{22}k_1 < 0$ for $\lambda \in (0, \lambda_1^*)$. On the other hand, using $\varepsilon_2 \in (0, \frac{k_2}{2})$, we obtain

$$\begin{aligned} I_1(\lambda, t) &:= \frac{1}{2}a_{22}k_1e^{-\lambda t} + a_{22}k_1e^{-2\lambda t} - \frac{a_{23}k_2}{1 + ak_1 + bk_2} + \frac{a_{23}(1 + e^{-\lambda t})(k_2 - \varepsilon_2e^{-\lambda t})}{1 + ak_1(1 + e^{-\lambda t}) + b(k_2 - \varepsilon_2e^{-\lambda t})} \\ &\geq \frac{1}{2}a_{22}k_1e^{-\lambda t} - \frac{a_{23}k_2}{1 + ak_1 + bk_2} + \frac{a_{23}(1 + e^{-\lambda t})(k_2 - \varepsilon_2e^{-\lambda t})}{1 + ak_1(1 + e^{-\lambda t}) + bk_2} \\ &\geq \frac{1}{2}a_{22}k_1e^{-\lambda t} - \frac{a_{23}k_2}{1 + ak_1 + bk_2} + \frac{a_{23}(k_2 - \varepsilon_2e^{-\lambda t})}{1 + ak_1 + bk_2} \\ &= \frac{1}{2}a_{22}k_1e^{-\lambda t} - \frac{a_{23}\varepsilon_2e^{-\lambda t}}{1 + ak_1 + bk_2} \\ &= e^{-\lambda t}\left(\frac{1}{2}a_{22}k_1 - \frac{a_{23}\varepsilon_2}{1 + ak_1 + bk_2}\right) \\ &> e^{-\lambda t}\left(\frac{1}{2}a_{22}k_1 - \frac{1}{2}\frac{a_{23}k_2}{1 + ak_1 + bk_2}\right) > 0. \end{aligned}$$

Hence, we can get

$$\begin{aligned} &D_2\overline{\varphi}''(t) - c\overline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\overline{\varphi}(t - y - c\tau)\,dy - (d_2 + q_2e_2)\overline{\varphi}(t) \\ &\quad - a_{22}\overline{\varphi}^2(t) - \frac{a_{23}\overline{\varphi}(t)\underline{\psi}(t)}{1 + a\overline{\varphi}(t) + b\underline{\psi}(t)} \leq 0 \end{aligned}$$

for any $\lambda \leq \lambda_1^*$.

We now consider $\overline{\psi}(t)$. If $t \leq t_2$, then $\overline{\psi}(t) = e^{\lambda_3 t} + qe^{\eta\lambda_3 t}$, $\overline{\varphi}(t) = e^{\lambda_1 t}$. We have

$$\begin{aligned} &D_3\overline{\psi}''(t) - c\overline{\psi}'(t) + (a_1 - q_3e_3)\overline{\psi}(t) - a_{33}\overline{\psi}^2(t) + \frac{a_{32}\overline{\varphi}(t)\overline{\psi}(t)}{1 + a\overline{\varphi}(t) + b\overline{\psi}(t)} \\ &\quad = D_3\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)'' - c\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)' + (a_1 - q_3e_3)\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right) \\ &\qquad - a_{33}\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)^2 + \frac{a_{32}e^{\lambda_1 t}(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1 + ae^{\lambda_1 t} + b(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})} \\ &\quad \leq D_3\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)'' - c\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)' + (a_1 - q_3e_3)\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right) \\ &\qquad + \frac{a_{32}e^{\lambda_1 t}(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1 + ae^{\lambda_1 t} + b(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})} \\ &\quad = e^{\lambda_3 t}\Delta_2(\lambda_3, c) + qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) + \frac{a_{32}e^{\lambda_1 t}(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1 + ae^{\lambda_1 t} + b(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})} \\ &\quad = qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) + \frac{a_{32}e^{\lambda_1 t}(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1 + ae^{\lambda_1 t} + b(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})} \\ &\quad \leq qe^{(\lambda_1+\lambda_3)t}\Delta_2(\eta\lambda_3, c) + a_{32}e^{(\lambda_1+\lambda_3)t} + a_{32}qe^{(\lambda_1+\eta\lambda_3)t} \\ &\quad = e^{(\lambda_1+\lambda_3)t}\left[q\Delta_2(\eta\lambda_3, c) + a_{32} + a_{32}qe^{(\eta-1)\lambda_3 t}\right] \\ &\quad \leq e^{(\lambda_1+\lambda_3)t}\left[q\Delta_2(\eta\lambda_3, c) + a_{32} + a_{32}qe^{(\eta-1)\lambda_3 t_2}\right] \\ &\quad = e^{(\lambda_1+\lambda_3)t}\left\{q\left[\Delta_2(\eta\lambda_3, c) + a_{32}e^{(\eta-1)\lambda_3 t_2}\right] + a_{32}\right\} < 0 \quad \text{for enough large } q. \end{aligned}$$

When $t > t_2$, if $\overline{\psi}(t) = M_3$, $\overline{\varphi}(t) \le M_2$, using $M_3 \ge \frac{a_1 - q_3 e_3 + a_{32} M_2}{a_{33}}$, we have

$$\begin{aligned}
&D_3\overline{\psi}''(t) - c\overline{\psi}'(t) + (a_1 - q_3e_3)\overline{\psi}(t) - a_{33}\overline{\psi}^2(t) + \frac{a_{32}\overline{\varphi}(t)\overline{\psi}(t)}{1 + a\overline{\varphi}(t) + b\overline{\psi}(t)} \\
&\quad = (a_1 - q_3e_3)M_3 - a_{33}M_3^2 + \frac{a_{33}\overline{\varphi}(t)M_3}{1 + a\overline{\varphi}(t) + bM_3} \\
&\quad \le M_3(a_1 - q_3e_3 - a_{33}M_3 + a_{32}M_2) \le 0.
\end{aligned}$$

If $\overline{\psi}(t) = k_2 + k_2e^{-\lambda t}$, $\overline{\varphi}(t) \le k_1 + k_1e^{-\lambda t}$, then

$$\begin{aligned}
&D_3\overline{\psi}''(t) - c\overline{\psi}'(t) + (a_1 - q_3e_3)\overline{\psi}(t) - a_{33}\overline{\psi}^2(t) + \frac{a_{32}\overline{\varphi}(t)\overline{\psi}(t)}{1 + a\overline{\varphi}(t) + b\overline{\psi}(t)} \\
&\quad = D_3k_2\lambda^2e^{-\lambda t} + ck_2\lambda e^{-\lambda t} + (a_1 - q_3e_3)\bigl(k_2 + k_2e^{-\lambda t}\bigr) - a_{33}\bigl(k_2 + k_2e^{-\lambda t}\bigr)^2 \\
&\qquad + \frac{a_{32}\overline{\varphi}(t)(k_2 + k_2e^{-\lambda t})}{1 + a\overline{\varphi}(t) + b(k_2 + k_2e^{-\lambda t})} \\
&\quad \le D_3k_2\lambda^2e^{-\lambda t} + ck_2\lambda e^{-\lambda t} + (a_1 - q_3e_3)\bigl(k_2 + k_2e^{-\lambda t}\bigr) - a_{33}\bigl(k_2 + k_2e^{-\lambda t}\bigr)^2 \\
&\qquad + \frac{a_{32}(k_1 + k_1e^{-\lambda t})(k_2 + k_2e^{-\lambda t})}{1 + a(k_1 + k_1e^{-\lambda t}) + b(k_2 + k_2e^{-\lambda t})} \\
&\quad = D_3k_2\lambda^2e^{-\lambda t} + ck_2\lambda e^{-\lambda t} + (a_1 - q_3e_3)k_2 + (a_1 - q_3e_3)k_2e^{-\lambda t} - a_{33}k_2^2 \\
&\qquad - 2a_{33}k_2^2e^{-\lambda t} - a_{33}k_2^2e^{-2\lambda t} + \frac{a_{32}k_1k_2(1 + e^{-\lambda t})^2}{1 + ak_1(1 + e^{-\lambda t}) + bk_2(1 + e^{-\lambda t})} \\
&\quad = k_2e^{-\lambda t}\bigl[\Delta_2(-\lambda, c) - a_{33}k_2\bigr] \\
&\qquad - k_2\left[a_{33}k_2e^{-\lambda t} + a_{33}k_2e^{-2\lambda t} + \frac{a_{32}k_1}{1 + ak_1 + bk_2} - \frac{a_{32}k_1(1 + e^{-\lambda t})^2}{1 + ak_1(1 + e^{-\lambda t}) + bk_2(1 + e^{-\lambda t})}\right].
\end{aligned}$$

Note that $\Delta_2(0, c) - a_{33}k_2 = m_2 - a_{33}k_2 = -\frac{a_{32}k_1}{1+ak_1+bk_2} < 0$, which implies that there exists a λ_2^* such that $\Delta_2(\lambda, c) < 0$ for $\lambda \in (0, \lambda_2^*)$. Moreover, by the assumption $a_{33}k_2 \ge \frac{4a_{32}k_1}{1+ak_1+bk_2}$, we get

$$\begin{aligned}
I_2(\lambda, t) &:= a_{33}k_2e^{-\lambda t} + a_{33}k_2e^{-2\lambda t} + \frac{a_{32}k_1}{1 + ak_1 + bk_2} - \frac{a_{32}k_1(1 + e^{-\lambda t})^2}{1 + ak_1(1 + e^{-\lambda t}) + bk_2(1 + e^{-\lambda t})} \\
&\ge a_{33}k_2e^{-\lambda t} + a_{33}k_2e^{-2\lambda t} + \frac{a_{32}k_1}{1 + ak_1 + bk_2} - \frac{a_{32}k_1(1 + e^{-\lambda t})^2}{1 + ak_1 + bk_2} \\
&= e^{-\lambda t}\left[a_{33}k_2 + a_{33}k_2e^{-\lambda t} - \frac{2a_{32}k_1}{1 + ak_1 + bk_2} - \frac{a_{32}k_1}{1 + ak_1 + bk_2}e^{-\lambda t}\right] \\
&= e^{-\lambda t}\left\{a_{33}k_2 - \frac{2a_{32}k_1}{1 + ak_1 + bk_2} + e^{-\lambda t}\left[a_{33}k_2 - \frac{a_{32}k_1}{1 + ak_1 + bk_2}\right]\right\} > 0.
\end{aligned}$$

Therefore, we have

$$D_3\overline{\psi}''(t) - c\overline{\psi}'(t) + (a_1 - q_3e_3)\overline{\psi}(t) - a_{33}\overline{\psi}^2(t) + \frac{a_{32}\overline{\varphi}(t)\overline{\psi}(t)}{1 + a\overline{\varphi}(t) + b\overline{\psi}(t)} \le 0$$

for $\lambda \in (0, \lambda_2^*)$.

By the above argument, we see that $(\overline{\varphi}(t), \overline{\psi}(t))$ is an upper solution of (3.1). This completes the proof. □

In order to prove $(\underline{\varphi}(t), \underline{\psi}(t))$ is a lower solution of (3.1), we need the following lemma.

Lemma 3.8 *Assume that*

$$a_{22}k_1 \geq \frac{(3+2\sqrt{2})a_{23}M_3}{1+ak_1+bk_2}, \qquad a_{33}k_2 \geq \frac{4a_{32}k_1}{1+ak_1+bk_2}$$

hold. Then there exist $\varepsilon_1 \in (0, (\sqrt{2}-1)k_1)$ *and* $\varepsilon_2 \in (0, \frac{k_2}{2})$ *such that*

$$-a_{22}\varepsilon_1^2 + (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 + \frac{a_{23}k_1k_2}{1+ak_1+bk_2} - \frac{a_{23}(k_1-\varepsilon_1)M_3}{1+a(k_1-\varepsilon_1)+bk_2} > \varepsilon_0, \tag{3.6}$$

$$a_{33}k_2\varepsilon_2 - a_{33}\varepsilon_2^2 - \frac{a_{32}k_1k_2}{1+ak_1+bk_2} > \varepsilon_0, \tag{3.7}$$

where $\varepsilon_0 > 0$ *is a constant.*

Proof Let

$$\begin{aligned}
g_1(\varepsilon_1) &= -a_{22}\varepsilon_1^2 + (2\sqrt{2}-2)a_{22}k_1\varepsilon_1, \\
g_2(\varepsilon_1) &= -\frac{a_{23}k_1k_2}{1+ak_1+bk_2} + \frac{a_{23}(k_1-\varepsilon_1)M_3}{1+a(k_1-\varepsilon_1)+bk_2}, \\
g_3(\varepsilon_2) &= a_{33}k_2\varepsilon_2 - a_{33}\varepsilon_2^2.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&g_1(0) = g_1\big((2\sqrt{2}-2)k_1\big) = 0, \qquad \max\big\{g_1(\varepsilon_1)\big\} = g_1\big((\sqrt{2}-1)k_1\big) = (3-2\sqrt{2})a_{22}k_1^2, \\
&g_2(\varepsilon_1) \leq \frac{a_{23}k_1M_3}{1+ak_1+bk_2}, \qquad \max\big\{g_3(\varepsilon_2)\big\} = g_3\left(\frac{k_2}{2}\right) = \frac{a_{33}k_2^2}{4}.
\end{aligned}$$

If $a_{22}k_1 \geq \frac{(3+2\sqrt{2})a_{23}M_3}{1+ak_1+bk_2}$, $a_{33}k_2 \geq \frac{4a_{32}k_1}{1+ak_1+bk_2}$, then there exist ε_i^*, ε_i^+ $(i = 1, 2)$ such that

$$0 < \varepsilon_1^* < (\sqrt{2}-1)k_1 < \varepsilon_1^+ < (2\sqrt{2}-2)k_1, \qquad 0 < \varepsilon_2^* < \frac{k_2}{2} < \varepsilon_2^+ < k_2$$

and

$$\begin{aligned}
&g_1(\varepsilon_1) \geq g_2(\varepsilon_1) \quad \text{for } \varepsilon_1^* \leq \varepsilon_1 < (\sqrt{2}-1)k_1, \\
&g_3(\varepsilon_2) \geq \frac{a_{32}k_1k_2}{1+ak_1+bk_2} \quad \text{for } \varepsilon_2^* \leq \varepsilon_2 < \frac{k_2}{2}.
\end{aligned}$$

The proof is complete. □

Lemma 3.9 *Assume that* $a_{22}k_1 \geq \frac{(3+2\sqrt{2})a_{23}M_3}{1+ak_1+bk_2}$ *and* $a_{33}k_2 \geq \frac{4a_{32}k_1}{1+ak_1+bk_2}$ *hold. Then* $(\underline{\varphi}(t), \underline{\psi}(t))$ *is a lower solution of* (3.1).

Proof For $\underline{\varphi}(t) \in C(\mathbb{R},\mathbb{R})$, if $t \le t_3$, we have $\underline{\phi}(t) = e^{\lambda_1 t} - qe^{\eta\lambda_1 t}$, $\overline{\psi}(t) \le e^{\lambda_3 t} + qe^{\eta\lambda_3 t}$. If $t - y - c\tau \le t_3$, then $\underline{\phi}(t-y-c\tau) = e^{\lambda_1(t-y-c\tau)} - qe^{\eta\lambda_1(t-y-c\tau)}$; if $t-y-c\tau > t_3$, then $\underline{\phi}(t-y-c\tau) = k_1 - \varepsilon_1 e^{-\lambda(t-y-c\tau)} \ge k_1 - \varepsilon_1 e^{-\lambda t_3} = e^{\lambda_1 t_3} - qe^{\eta\lambda_1 t_3} \ge e^{\lambda_1(t-y-c\tau)} - qe^{\eta\lambda_1(t-y-c\tau)}$. Therefore, we have

$$\begin{aligned}
&D_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\underline{\varphi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\underline{\varphi}(t)\\
&\quad - a_{22}\underline{\varphi}^2(t) - \frac{a_{23}\underline{\varphi}(t)\overline{\psi}(t)}{1+a\underline{\varphi}(t)+b\overline{\psi}(t)}\\
&\quad\ge D_2\big(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\big)'' - c\big(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\big)' + me^{-d_1\tau}\\
&\qquad\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\big(e^{\lambda_1(t-y-c\tau)} - qe^{\eta\lambda_1(t-y-c\tau)}\big)\,dy - (d_2+q_2e_2)\big(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\big)\\
&\qquad - a_{22}\big(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\big)^2 - \frac{a_{23}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1+a(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})}\\
&\quad= -q\Delta_1(\eta\lambda_1,c)e^{\eta\lambda_1 t} - a_{22}\big(e^{\lambda_1 t} - qe^{\eta\lambda_1 t}\big)^2 - \frac{a_{23}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1+a(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})}\\
&\quad= -q\Delta_1(\eta\lambda_1,c)e^{\eta\lambda_1 t} - a_{22}e^{2\lambda_1 t} + 2a_{22}qe^{\lambda_1 t}e^{\eta\lambda_1 t} - a_{22}q^2e^{2\eta\lambda_1 t}\\
&\qquad - \frac{a_{23}(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})(e^{\lambda_3 t} + qe^{\eta\lambda_3 t})}{1+a(e^{\lambda_1 t} - qe^{\eta\lambda_1 t})}\\
&\quad\ge -q\Delta_1(\eta\lambda_1,c)e^{\eta\lambda_1 t} - a_{22}e^{2\lambda_1 t} - a_{22}q^2e^{2\eta\lambda_1 t} - a_{23}e^{\lambda_1 t}\big(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\big)\\
&\quad\ge -q\Delta_1(\eta\lambda_1,c)e^{\eta\lambda_1 t} - a_{22}e^{2\lambda_1 t} - a_{22}q^2e^{2\eta\lambda_1 t} - a_{23}e^{\eta\lambda_1 t} - a_{23}qe^{(\lambda_1+\eta\lambda_3)t}\\
&\quad= -e^{\eta\lambda_1 t}\big[q\Delta_1(\eta\lambda_1,c) + a_{22}e^{(2\lambda_1-\eta\lambda_1)t} + a_{22}q^2e^{\eta\lambda_1 t} + a_{23} + a_{23}qe^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t}\big]\\
&\quad\le -e^{\eta\lambda_1 t}\big[q\Delta_1(\eta\lambda_1,c) + a_{22}e^{(2\lambda_1-\eta\lambda_1)t_3} + a_{22}q^2e^{\eta\lambda_1 t_3} + a_{23} + a_{23}qe^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t_3}\big]\\
&\quad= -e^{\eta\lambda_1 t}\left\{q\left[\Delta_1(\eta\lambda_1,c) + \frac{a_{22}}{q}e^{(2\lambda_1-\eta\lambda_1)t_3} + a_{22}qe^{\eta\lambda_1 t_3} + \frac{a_{23}}{q} + a_{23}e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t_3}\right]\right\}\\
&\quad= -e^{\eta\lambda_1 t}\left\{q\left[\Delta_1(\eta\lambda_1,c) + \frac{a_{22}}{q}e^{(2\lambda_1-\eta\lambda_1)t_3} + a_{22}q^{-\frac{1}{\eta-1}}\eta^{-\frac{\eta}{\eta-1}} + \frac{a_{23}}{q} + a_{23}e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t_3}\right]\right\}\\
&\quad\ge 0\quad\text{for enough large } q.
\end{aligned}$$

If $t > t_3$, then $\underline{\phi}(t) = k_1 - \varepsilon_1 e^{-\lambda t}$, $\overline{\psi}(t) \le k_2 + k_2e^{-\lambda t}$. If $t-y-c\tau \le t_3$, then $\underline{\phi}(t-y-c\tau) = e^{\lambda_1(t-y-c\tau)} - qe^{\eta\lambda_1(t-y-c\tau)} \ge k_1 - \varepsilon_1 e^{-\lambda(t-y-c\tau)}$; if $t-y-c\tau > t_3$, then $\underline{\phi}(t-y-c\tau) = k_1 - \varepsilon_1 e^{-\lambda(t-y-c\tau)}$. Thus,

$$\begin{aligned}
&D_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\underline{\varphi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\underline{\varphi}(t)\\
&\quad - a_{22}\underline{\varphi}^2(t) - \frac{a_{23}\underline{\varphi}(t)\overline{\psi}(t)}{1+a\underline{\varphi}(t)+b\overline{\psi}(t)}\\
&\quad\ge D_2\big(k_1 - \varepsilon_1 e^{-\lambda t}\big)'' - c\big(k_1 - \varepsilon_1 e^{-\lambda t}\big)'\\
&\qquad + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\big(k_1 - \varepsilon_1 e^{-\lambda(t-y-c\tau)}\big)\,dy\\
&\qquad - (d_2+q_2e_2)\big(k_1 - \varepsilon_1 e^{-\lambda t}\big) - a_{22}\big(k_1 - \varepsilon_1 e^{-\lambda t}\big)^2 - \frac{a_{23}(k_1 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1+a(k_1 - \varepsilon_1 e^{-\lambda t}) + b\overline{\psi}(t)}
\end{aligned}$$

$$
\begin{aligned}
&= -D_2\varepsilon_1\lambda^2 e^{-\lambda t} - c\varepsilon_1\lambda e^{-\lambda t} + me^{-d_1\tau}\left[k_1 - \varepsilon_1 e^{-\lambda(t-c\tau)+D_1\lambda^2\tau}\right] - (d_2+q_2e_2)\left(k_1 - \varepsilon_1 e^{-\lambda t}\right) \\
&\quad - a_{22}\left(k_1 - \varepsilon_1 e^{-\lambda t}\right)^2 - \frac{a_{23}(k_1 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1 + a(k_1 - \varepsilon_1 e^{-\lambda t}) + b\overline{\psi}(t)} \\
&= -e^{-\lambda t}\varepsilon_1\Delta_1(-\lambda, c) + me^{-d_1\tau}k_1 - (d_2+q_2e_2)k_1 - a_{22}k_1^2 + 2a_{22}k_1\varepsilon_1 e^{-\lambda t} \\
&\quad - a_{22}\varepsilon_1^2 e^{-2\lambda t} - \frac{a_{23}(k_1 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1 + a(k_1 - \varepsilon_1 e^{-\lambda t}) + b\overline{\psi}(t)} \\
&= -e^{-\lambda t}\varepsilon_1\Delta_1(-\lambda, c) + \frac{a_{23}k_1k_2}{1 + ak_1 + bk_2} + 2a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} \\
&\quad - \frac{a_{23}(k_1 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1 + a(k_1 - \varepsilon_1 e^{-\lambda t}) + b\overline{\psi}(t)} \\
&= \varepsilon_1 e^{-\lambda t}\left[-\Delta_1(-\lambda, c) + (4 - 2\sqrt{2})a_{22}k_1\right] + (2\sqrt{2} - 2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} \\
&\quad + \frac{a_{23}k_1k_2}{1 + ak_1 + bk_2} - \frac{a_{23}(k_1 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1 + a(k_1 - \varepsilon_1 e^{-\lambda t}) + b\overline{\psi}(t)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
-\Delta_1(0, c) + (4 - 2\sqrt{2})a_{22}k_1 &= -m_1 + (4 - 2\sqrt{2})a_{22}k_1 \\
&= -m_1 + (4 - 2\sqrt{2})\left(m_1 - \frac{a_{23}k_2}{1 + ak_1 + bk_2}\right) \\
&= (3 - 2\sqrt{2})m_1 - (4 - 2\sqrt{2})\frac{a_{23}k_2}{1 + ak_1 + bk_2} \\
&> (3 - 2\sqrt{2})m_1 - (4 - 2\sqrt{2})\frac{a_{23}M_3}{1 + ak_1 + bk_2} \\
&> (3 - 2\sqrt{2})m_1 - (4 - 2\sqrt{2})\frac{m_1}{4 + 2\sqrt{2}} = 0,
\end{aligned}
$$

it follows that there exists a $\lambda_3^* > 0$ such that $-\Delta_1(-\lambda, c) + (4 - 2\sqrt{2})a_{22}k_1 > 0$ for $\lambda \in (0, \lambda_3^*)$. Let

$$
I_3(\lambda, t) := (2\sqrt{2} - 2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} + \frac{a_{23}k_1k_2}{1 + ak_1 + bk_2} - \frac{a_{23}(k_1 - \varepsilon_1 e^{-\lambda t})\overline{\psi}(t)}{1 + a(k_1 - \varepsilon_1 e^{-\lambda t}) + b\overline{\psi}(t)}.
$$

By Lemma 3.2, we can choose $\delta_1 > 0$ such that $a_{22}(4\varepsilon_1\delta_1 + 2\delta_1^2) < \varepsilon_0$ and

$$
(2\sqrt{2} - 2)a_{22}k_1\delta - a_{22}\delta^2 + \frac{a_{23}k_1k_2}{1 + ak_1 + bk_2} - \frac{a_{23}(k_1 - \delta)M_3}{1 + a(k_1 - \delta) + bk_2} > \frac{\varepsilon_0}{2} > 0
$$

for $\varepsilon_1 \in (\varepsilon_1^*, (\sqrt{2} - 1)k_1)$, $\delta \in [\varepsilon_1, \varepsilon_1 + \delta_1]$. Let $\delta^* = \varepsilon_1 + \delta_1$.

If $t \in (t_3, 0]$, note that $\varepsilon_1 e^{-\lambda t}$ is decreasing on $(t_3, 0]$, we can choose $\lambda_4^* > 0$ small enough, such that $\varepsilon_1 e^{-\lambda t_3} = \varepsilon_1 + \delta_1 = \delta^*$ for given $\lambda \in (0, \lambda_4^*)$. Thus, we have $\varepsilon_1 \le \varepsilon_1 e^{-\lambda t} < \delta^*$ for $t \in (t_3, 0]$. Therefore,

$$
\begin{aligned}
I_3(\lambda, t) \ge{}& (2\sqrt{2} - 2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} + \frac{a_{23}k_1k_2}{1 + ak_1 + bk_2} \\
& - \frac{a_{23}(k_1 - \varepsilon_1 e^{-\lambda t})(k_2 + k_2 e^{-\lambda t})}{1 + a(k_1 - \varepsilon_1 e^{-\lambda t}) + b(k_2 + k_2 e^{-\lambda t})}
\end{aligned}
$$

$$\begin{aligned}
&\geq (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} + \frac{a_{23}k_1k_2}{1+ak_1+bk_2} - \frac{a_{23}(k_1-\varepsilon_1 e^{-\lambda t})M_3}{1+a(k_1-\varepsilon_1 e^{-\lambda t})+bk_2} \\
&\geq (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 - a_{22}\delta^{*2} + \frac{a_{23}k_1k_2}{1+ak_1+bk_2} - \frac{a_{23}(k_1-\varepsilon_1)M_3}{1+a(k_1-\varepsilon_1)+bk_2} \\
&= (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 - a_{22}\varepsilon_1^2 + \frac{a_{23}k_1k_2}{1+ak_1+bk_2} - \frac{a_{23}(k_1-\varepsilon_1)M_3}{1+a(k_1-\varepsilon_1)+bk_2} \\
&\quad - a_{22}\big(2\varepsilon_1\delta_1 + \delta_1^2\big) \\
&> \frac{\varepsilon_0}{2} - a_{22}\big(2\varepsilon_1\delta_1 + \delta_1^2\big) > 0.
\end{aligned} \tag{3.8}$$

If $t > 0$, then we have

$$\begin{aligned}
I_3(\lambda,t) &\geq (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} + \frac{a_{23}k_1k_2}{1+ak_1+bk_2} \\
&\quad - \frac{a_{23}(k_1-\varepsilon_1 e^{-\lambda t})(k_2+k_2e^{-\lambda t})}{1+a(k_1-\varepsilon_1 e^{-\lambda t})+b(k_2+k_2e^{-\lambda t})} \\
&\geq (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} + \frac{a_{23}k_1k_2}{1+ak_1+bk_2} - \frac{a_{23}k_1(k_2+k_2e^{-\lambda t})}{1+ak_1+bk_2} \\
&= (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-2\lambda t} - \frac{a_{23}k_1k_2e^{-\lambda t}}{1+ak_1+bk_2} \\
&\geq (2\sqrt{2}-2)a_{22}k_1\varepsilon_1 e^{-\lambda t} - a_{22}\varepsilon_1^2 e^{-\lambda t} - \frac{a_{23}k_1k_2e^{-\lambda t}}{1+ak_1+bk_2} \\
&= e^{-\lambda t}\left[(2\sqrt{2}-2)a_{22}k_1\varepsilon_1 - a_{22}\varepsilon_1^2 - \frac{a_{23}k_1k_2}{1+ak_1+bk_2}\right].
\end{aligned}$$

Since $\max\{(2\sqrt{2}-2)a_{22}k_1\varepsilon_1 - a_{22}\varepsilon_1^2\} = (3-2\sqrt{2})a_{22}k_1^2$ and $a_{22}k_1 \geq \frac{(3+2\sqrt{2})a_{23}M_3}{1+ak_1+bk_2}$, we know that there exists ε_1^{**} ($0 < \varepsilon_1^{**} < (\sqrt{2}-1)k_1$) such that $(2\sqrt{2}-2)a_{22}k_1\varepsilon_1 - a_{22}\varepsilon_1^2 - \frac{a_{23}k_1k_2}{1+ak_1+bk_2} > 0$ for $\varepsilon_1 \in (\varepsilon_1^{**}, (\sqrt{2}-1)k_1)$. Taking $\varepsilon_1' = \max\{\varepsilon_1^*, \varepsilon_1^{**}\}$, we obtain $I_3(\lambda,t) \geq 0$ for $\varepsilon_1 \in (\varepsilon_1', (\sqrt{2}-1)k_1)$.

Summarizing the above discussion, we have

$$\begin{aligned}
&D_2\underline{\varphi}''(t) - c\underline{\varphi}'(t) + me^{-d_1\tau}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\underline{\varphi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\underline{\varphi}(t) \\
&\quad - a_{22}\underline{\varphi}^2(t) - \frac{a_{23}\underline{\varphi}(t)\overline{\psi}(t)}{1+a\underline{\varphi}(t)+b\overline{\psi}(t)} \geq 0
\end{aligned}$$

for any $\lambda \in (0, \min\{\lambda_3^*, \lambda_4^*\})$.

For $\underline{\psi}(t)$, if $t \leq t_4$, then $\underline{\psi}(t) = e^{\lambda_3 t} - qe^{\eta\lambda_3 t}$, $\underline{\varphi}(t) \geq e^{\lambda_1 t} - qe^{\eta\lambda_1 t}$. Therefore, we obtain

$$\begin{aligned}
&D_3\underline{\psi}''(t) - c\underline{\psi}'(t) + (a_1 - q_3e_3)\underline{\psi}(t) - a_{33}\underline{\psi}^2(t) + \frac{a_{32}\underline{\varphi}(t)\underline{\psi}(t)}{1+a\underline{\varphi}(t)+b\underline{\psi}(t)} \\
&\geq D_3\big(e^{\lambda_3 t} - qe^{\eta\lambda_3 t}\big)'' - c\big(e^{\lambda_3 t} - qe^{\eta\lambda_3 t}\big)' \\
&\quad + (a_1 - q_3e_3)\big(e^{\lambda_3 t} - qe^{\eta\lambda_3 t}\big) - a_{33}\big(e^{\lambda_3 t} - qe^{\eta\lambda_3 t}\big)^2 \\
&\geq -qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) - a_{33}e^{2\lambda_3 t} - a_{33}q^2e^{2\eta\lambda_3 t} \\
&= -e^{\eta\lambda_3 t}\big[q\Delta_2(\eta\lambda_3, c) + a_{33}e^{(2\lambda_3 - \eta\lambda_3)t} + a_{33}q^2e^{\eta\lambda_3 t}\big]
\end{aligned}$$

$$\begin{aligned}
&\geq -e^{\eta\lambda_3 t}\big[q\Delta_2(\eta\lambda_3,c)+a_{33}e^{(2\lambda_3-\eta\lambda_3)t_4}+a_{33}q^2e^{\eta\lambda_3 t_4}\big]\\
&= -e^{\eta\lambda_3 t}\big[q\Delta_2(\eta\lambda_3,c)+a_{33}e^{(2\lambda_3-\eta\lambda_3)t_4}+a_{33}q^{\frac{\eta-2}{\eta-1}}\eta^{-\frac{\eta}{\eta-1}}\big]\\
&= -e^{\eta\lambda_3 t}\left\{q\left[\Delta_2(\eta\lambda_3,c)+\frac{a_{33}}{q}e^{(2\lambda_3-\eta\lambda_3)t_4}+a_{33}q^{-\frac{1}{\eta-1}}\eta^{-\frac{\eta}{\eta-1}}\right]\right\}\\
&\geq 0 \quad \text{for enough large } q.
\end{aligned}$$

If $t>0$, then $\underline{\psi}(t)=k_2-\varepsilon_2e^{-\lambda t}$, $\underline{\varphi}(t)=k_1-\varepsilon_1e^{-\lambda t}$, we have

$$\begin{aligned}
&D_3\underline{\psi}''(t)-c\underline{\psi}'(t)+(a_1-q_3e_3)\underline{\psi}(t)-a_{33}\underline{\psi}^2(t)+\frac{a_{32}\underline{\varphi}(t)\underline{\psi}(t)}{1+a\underline{\varphi}(t)+b\underline{\psi}(t)}\\
&= D_3\big(k_2-\varepsilon_2e^{-\lambda t}\big)''-c\big(k_2-\varepsilon_2e^{-\lambda t}\big)'+(a_1-q_3e_3)\big(k_2-\varepsilon_2e^{-\lambda t}\big)\\
&\quad -a_{33}\big(k_2-\varepsilon_2e^{-\lambda t}\big)^2+\frac{a_{32}(k_1-\varepsilon_1e^{-\lambda t})(k_2-\varepsilon_2e^{-\lambda t})}{1+a(k_1-\varepsilon_1e^{-\lambda t})+b(k_2-\varepsilon_2e^{-\lambda t})}\\
&= -D_3\varepsilon_2\lambda^2e^{-\lambda t}-c\varepsilon_2\lambda e^{-\lambda t}+(a_1-q_3e_3)\big(k_2-\varepsilon_2e^{-\lambda t}\big)-a_{33}k_2^2+2a_{33}k_2\varepsilon_2e^{-\lambda t}\\
&\quad -a_{33}\varepsilon_2^2e^{-2\lambda t}+\frac{a_{32}(k_1-\varepsilon_1e^{-\lambda t})(k_2-\varepsilon_2e^{-\lambda t})}{1+a(k_1-\varepsilon_1e^{-\lambda t})+b(k_2-\varepsilon_2e^{-\lambda t})}\\
&= -\varepsilon_2\Delta_2(-\lambda,c)e^{-\lambda t}+2a_{33}k_2\varepsilon_2e^{-\lambda t}-a_{33}\varepsilon_2^2e^{-2\lambda t}\\
&\quad -\frac{a_{32}k_1k_2}{1+ak_1+bk_2}+\frac{a_{32}(k_1-\varepsilon_1e^{-\lambda t})(k_2-\varepsilon_2e^{-\lambda t})}{1+a(k_1-\varepsilon_1e^{-\lambda t})+b(k_2-\varepsilon_2e^{-\lambda t})}\\
&\geq \varepsilon_2e^{-\lambda t}\big[-\Delta_2(-\lambda,c)+a_{33}k_2\big]+a_{33}k_2\varepsilon_2e^{-\lambda t}-a_{33}\varepsilon_2^2e^{-\lambda t}\\
&\quad -\frac{a_{32}k_1\varepsilon_2e^{-\lambda t}}{1+ak_1+bk_2}-\frac{a_{32}k_2\varepsilon_1e^{-\lambda t}}{1+ak_1+bk_2}.
\end{aligned}$$

Since $-\Delta_2(0,c)+a_{33}k_2=\frac{a_{32}k_1}{1+ak_1+bk_2}>0$, we can choose $\lambda_5^*>0$ such that $-\Delta_2(-\lambda,c)+a_{33}k_2>0$ for $\lambda\in(0,\lambda_5^*)$. Using $\varepsilon_1\in(0,(\sqrt{2}-1)k_1)$ and $\varepsilon_2\in(0,\frac{k_2}{2})$, we get

$$\begin{aligned}
I_4(\lambda,t) &:= a_{33}k_2\varepsilon_2e^{-\lambda t}-a_{33}\varepsilon_2^2e^{-\lambda t}-\frac{a_{32}k_1\varepsilon_2}{1+ak_1+bk_2}e^{-\lambda t}-\frac{a_{32}k_2\varepsilon_1}{1+ak_1+bk_2}e^{-\lambda t}\\
&\geq a_{33}k_2\varepsilon_2e^{-\lambda t}-\frac{1}{2}a_{33}k_2\varepsilon_2e^{-\lambda t}-\frac{a_{32}k_1\varepsilon_2}{1+ak_1+bk_2}e^{-\lambda t}-\frac{a_{32}(\sqrt{2}-1)k_1k_2}{1+ak_1+bk_2}e^{-\lambda t}\\
&= e^{-\lambda t}\left[\frac{1}{2}a_{33}k_2\varepsilon_2-\frac{a_{32}k_1\varepsilon_2}{1+ak_1+bk_2}-\frac{a_{32}(\sqrt{2}-1)k_1k_2}{1+ak_1+bk_2}\right]\\
&\geq e^{-\lambda t}\left[\frac{a_{32}k_1\varepsilon_2}{1+ak_1+bk_2}-\frac{a_{32}(\sqrt{2}-1)k_1k_2}{1+ak_1+bk_2}\right].
\end{aligned}$$

The last expression is due to $a_{33}k_2\geq\frac{4a_{32}k_1}{1+ak_1+bk_2}$. On the other hand, if $\varepsilon_2=\frac{k_2}{2}$, then $\frac{a_{32}k_1\frac{k_2}{2}}{1+ak_1+bk_2}-\frac{a_{32}(\sqrt{2}-1)k_1k_2}{1+ak_1+bk_2}>0$. Thus, there exists ε_2' such that $\frac{a_{32}k_1\varepsilon_2}{1+ak_1+bk_2}-\frac{a_{32}(\sqrt{2}-1)k_1k_2}{1+ak_1+bk_2}>0$ for $\varepsilon_2'<\varepsilon_2<\frac{k_2}{2}$.

If $t_4<t\leq 0$, then $\underline{\psi}(t)=k_2-\varepsilon_2e^{-\lambda t}$. Thus,

$$\begin{aligned}
&D_3\underline{\psi}''(t)-c\underline{\psi}'(t)+(a_1-q_3e_3)\underline{\psi}(t)-a_{33}\underline{\psi}^2(t)+\frac{a_{32}\underline{\varphi}(t)\underline{\psi}(t)}{1+a\underline{\varphi}(t)+b\underline{\psi}(t)}\\
&= -\varepsilon_2\Delta_2(-\lambda,c)e^{-\lambda t}+2a_{33}k_2\varepsilon_2e^{-\lambda t}-a_{33}\varepsilon_2^2e^{-2\lambda t}-\frac{a_{32}k_1k_2}{1+ak_1+bk_2}+\frac{a_{32}\underline{\varphi}(t)\underline{\psi}(t)}{1+a\underline{\varphi}(t)+b\underline{\psi}(t)}
\end{aligned}$$

$$\begin{aligned}
&\geq -\varepsilon_2\Delta_2(-\lambda,c)e^{-\lambda t} + 2a_{33}k_2\varepsilon_2 e^{-\lambda t} - a_{33}\varepsilon_2^2 e^{-2\lambda t} - \frac{a_{32}k_1k_2}{1+ak_1+bk_2} \\
&= \varepsilon_2 e^{-\lambda t}\bigl[-\Delta_2(-\lambda,c) + a_{33}k_2\bigr] + a_{33}k_2\varepsilon_2 e^{-\lambda t} - a_{33}\varepsilon_2^2 e^{-2\lambda t} - \frac{a_{32}k_1k_2}{1+ak_1+bk_2}.
\end{aligned}$$

Similar to the proof in (3.8), we can also deduce that $a_{33}k_2\varepsilon_2 e^{-\lambda t} - a_{33}\varepsilon_2^2 e^{-2\lambda t} - \frac{a_{32}k_1k_2}{1+ak_1+bk_2} \geq 0$. Hence, $\varepsilon_2 e^{-\lambda t}[-\Delta_2(-\lambda,c) + a_{33}k_2] + a_{33}k_2\varepsilon_2 e^{-\lambda t} - a_{33}\varepsilon_2^2 e^{-2\lambda t} - \frac{a_{32}k_1k_2}{1+ak_1+bk_2} \geq 0$ for $\lambda \in (0,\lambda_5^*)$. Therefore, $\underline{\psi}(t)$ satisfies

$$D_3\underline{\psi}''(t) - c\underline{\psi}'(t) + (a_1 - q_3e_3)\underline{\psi}(t) - a_{33}\underline{\psi}^2(t) + \frac{a_{32}\underline{\varphi}(t)\underline{\psi}(t)}{1+a\underline{\varphi}(t)+b\underline{\psi}(t)} \geq 0$$

for $\lambda \in (0,\lambda_5^*)$. Thus, we know that $(\underline{\varphi}(t),\underline{\psi}(t))$ is a lower solution of (3.1). This completes the proof. □

In summary, we have the following results.

Theorem 3.2 *Assume that* $a_{22}k_1 \geq \frac{(3+2\sqrt{2})a_{23}M_3}{1+ak_1+bk_2}$ *and* $a_{33}k_2 \geq \frac{4a_{32}k_1}{1+ak_1+bk_2}$ *hold. Then, for every* $c > c^* = \max\{c_1,c_2\}$, *system* (1.6) *has a traveling wave solution* $(\varphi(t),\psi(t))$ *with wave speed* c, *which connects* $(0,0)$ *and* (k_1,k_2). *Furthermore,*

$$\lim_{t\to-\infty}\varphi(t)e^{-\lambda_1 t} = 1, \qquad \lim_{t\to-\infty}\psi(t)e^{-\lambda_3 t} = 1.$$

4 Discussion

In this paper, we investigate the stability and traveling waves of a stage-structured predator-prey reaction-diffusion systems of Beddington-DeAngelis functional response with both nonlocal delays and harvesting. The predator's functional response is Holling type II in [40], the Beddington-DeAngelis functional response is similar to the Holling type II functional response, but it has an extra term bv in the denominator providing a better description of mutual interference by predators. If $b = 0$, our systems can reduce to the systems in [40]. We establish the stability of the equilibria and reduce the existence of traveling waves to the existence of a pair of upper-lower solutions by using the cross iteration method and the Schauder's fixed point theorem. An extra term bv in our model would not affect the local stability of equilibria E_1 and E_2 but affects the local stability of equilibria E_3 and E_4 according to Theorems 2.1-2.4. On the other hand, an extra term bv does not affect the wave speed according to Lemma 3.6.

It is well known that the construction of upper and lower solutions is very important but difficult. In this paper, we follow the idea of Hong and Weng [40], but the construction of lower solution is different from that in [40]. We define t_3 and t_4 such that the functions $l_1(t)$ and $l_2(t)$ have global maximum at t_3 and t_4, respectively. That is, the lower solution $(\underline{\varphi}(t),\underline{\psi}(t))$ is nondecreasing for $t \in \mathbb{R}$, which is different from [40]. So, the proofs of Lemma 3.7 and Lemma 3.9 are also different.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equal contributions. All authors read and approved the final manuscript

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