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Existence of positive solutions for integral boundary value problems of fractional differential equations with p-Laplacian

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Abstract
This paper is concerned with the existence of positive solutions for integral boundary value problems of Caputo fractional differential equations with p-Laplacian operator. By means of the properties of the Green's function, Avery-Peterson fixed point theorems, we establish conditions ensuring the existence of positive solutions for the problem. As an application, an example is given to demonstrate the main result.

Keywords: fractional differential equations; Caputo derivative; p-Laplacian operator; integral boundary conditions; positive solutions

1 Introduction

Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, such as rheology, dynamical processes in self-similar and porous structures, heat conduction, control theory, electroanalytical chemistry, chemical physics, economics, *etc.* Many researchers have shown their interest in fractional differential equations. The motivation for this work stems from both the intensive development of the theory of fractional calculus itself and the applications. Many papers and books have appeared on fractional calculus and fractional differential equations (see [1–10]).

It is well known that the p-Laplacian operator is also used in analyzing mechanics, physics, and dynamic systems, and the related fields of mathematical modeling. However, there are few studies of the existence of positive solution of fractional differential equations with the p-Laplacian operator; see [11–18] and the references therein.

In [11], Liu *et al.* studied the solvability of the Caputo fractional differential equation with boundary value conditions involving the p-Laplacian operator. The existence and uniqueness of the problem is found by the Banach fixed point theorem. The problem is given in the following:

$$\begin{cases} (\varphi_p(D_{0+}^{\alpha}x(t)))' = f(t,x(t)), & \text{for } t \in (0,1), \\ x(0) = r_0 x(1), \\ x'(0) = r_1 x'(1), \\ x^{(j)}(0) = 0, \end{cases}$$

where $i = 2, 3, \ldots, [\alpha] - 1$. Here, φ_p is the p-Laplacian operator and $D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $1 < \alpha \in \mathbb{R}$, and the nonlinear function $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ is given.

In [12], Lu *et al.* studied the existence of non-negative solutions of a nonlinear fractional boundary value problem with the p-Laplacian operator

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) = f(t, u(t)), \quad \text{for } 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \\ D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 3$, and D_{0+}^{α}, D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives. Green's functions, the Guo-Krasnoselskii theorem, and the Leggett-Williams fixed point theorems are used.

Boundary value problems with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems. They include two-point, three-point and multi-point boundary value problems as special cases. For an overview of the literature on integral boundary value problems and symmetric solutions, see [19–26] and the references therein.

In [24], Zhi *et al.* studied the existence of positive solutions for nonlocal boundary value problem of the fractional differential equations with p-Laplacian operator. The problem is given in the following:

$$\begin{cases} (\varphi_p(D_{0+}^{\alpha}u(t)))'' = f(t, u(t), D_{0+}^{\beta}u(t)), \quad \text{for } 0 < t < 1, \\ u(0) = u''(0) = 0, \qquad u(1) = \int_0^1 g(s)u(s)\,ds, \\ (\varphi_p(D_{0+}^{\alpha}u(0)))' = \lambda_1(\varphi_p(D_{0+}^{\alpha}u(\xi_1)))', \\ \varphi_p(D_{0+}^{\alpha}u(1)) = \lambda_2(\varphi_p(D_{0+}^{\alpha}u(\xi_2))), \end{cases}$$

where $0 < \xi_1 \leq \xi_2 < 1$, $2 < \alpha < 3$, $1 < \beta < \alpha - 1 < 2$, $0 \leq \lambda_1, \lambda_2 < 1$. D_{0+}^{α} is the Caputo fractional derivative of order α.

In [26], Mahmudov and Unul studied the existence of the solutions of the fractional differential equation with p-Laplacian operator and integral conditions is discussed. The problem is given in the following:

$$\begin{cases} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}u(t)) = f(t, u(t), D_{0+}^{\gamma}u(t)), \\ u(0) + \mu_1 u(1) = \sigma_1 \int_0^1 g(s, u(s))\,ds, \\ u'(0) + \mu_2 u'(1) = \sigma_2 \int_0^1 h(s, u(s))\,ds, \\ D_{0+}^{\alpha}u(0) = 0, \\ D_{0+}^{\alpha}u(1) = \upsilon D_{0+}^{\alpha}u(\eta), \end{cases}$$

where D_{0+}^{α}, D_{0+}^{β} are the Caputo fractional derivative operators with $1 < \alpha < 2$, $1 < \beta < 2$. μ_1, σ_i $(i = 1, 2)$ are non-negative constants. f, g, h are continuous functions.

Motivated by the above work, we investigate the following integral boundary value problems (for short, BVP) of fractional differential equations with p-Laplacian:

$$\begin{cases} D_{0+}^{\beta}\varphi_p(D_{0+}^{\alpha}x(t)) = f(t, x(t), D_{0+}^{\beta}x(t)), \quad t \in (0,1) \\ (\varphi_p(D_{0+}^{\alpha}x(0)))^{(i)} = \varphi_p(D_{0+}^{\alpha}x(1)) = 0, \quad i = 1, 2, \ldots, m-1, \\ x(1) = \int_0^1 g_1(s)x(s)\,ds, \\ x'(0) = \int_0^1 g_2(s)x(s)\,ds, \\ x^{(j)}(0) = 0, \quad j = 2, 3, \ldots, n-1, \end{cases} \tag{1.1}$$

where φ_p is the p-Laplacian operator, $1 < n-1 < \alpha < n$, $1 < m-1 < \beta < m$, $\alpha - \beta > 1$, and $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Caputo fractional derivatives. $g_k \in C([0,1],[0,+\infty))$, $k = 1,2$, $f \in C([0,1] \times [0,+\infty) \times (-\infty,0],[0,+\infty))$ are given functions.

In this paper, a positive solution $x = x(t)$ of BVP (1.1) means a solution of (1.1) satisfying $x(t) > 0$, $t \in [0,1]$.

Throughout this paper, we always assume that the following condition is satisfied.

(L_0) $g_1(t) > g_2(t) \geq 0$, $0 \leq \int_0^1 g_2(s)\,ds$, $\int_0^1 g_1(s)\,ds < 1$.

The organization of the paper is as follows. In Section 2, we present some necessary definitions and lemmas which will be used to prove our main results. In Section 3, by using the Avery-Peterson fixed point theorems, the results for the existence of multiple positive solutions of BVP (1.1) are established. In Section 4, we give an example to demonstrate the main result.

2 Preliminaries and lemmas

In this section, we give some definitions and basic lemmas that will be used and important to us in the following.

Definition 2.1 (see [1]) The Riemann-Liouville integral of fractional order $\alpha > 0$ of a function g is defined as

$$I_{0^+}^{\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,ds.$$

Definition 2.2 (see [2]) The Caputo derivative of fractional order $\alpha > 0$ of a function g is defined as

$$D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)\,ds,$$

where n is the smallest integer greater than or equal to α.

Definition 2.3 (see [4]) Let E be a real Banach space. A nonempty, closed, and convex set $P \subset E$ is a cone if the following two conditions are satisfied:

(1) if $x \in P$ and $\mu \geq 0$, then $\mu x \in P$;

(2) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces the ordering in E given by $x_1 \leq x_2$ if and only if $x_2 - x_1 \in P$.

Definition 2.4 (see [4]) The map γ is said to be a continuous non-negative convex (concave) function on a cone P of a real Banach space E provided that $\gamma : P \to [0,+\infty)$ is continuous and

$$\gamma\big(tx + (1-t)y\big) \leq (\geq)\, t\gamma(x) + (1-t)\gamma(y), \quad x,y \in P, t \in [0,1].$$

Lemma 2.1 (see [8]) *Let $\alpha > 0$, assume that $u, D_{0^+}^{\alpha} u \in C(0,1) \cap L^1(0,1)$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}$$

holds for some $C_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2.2 (see [26]) *The Caputo fractional derivative of order $n-1 < \alpha < n$ for t^β is given by*

$$D_{0^+}^{\alpha} t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \beta > n-1, \\ 0, & \beta \in \{0, 1, \ldots, n-1\}. \end{cases}$$

Lemma 2.3 *Let $h \in C[0,1]$ and $1 < m-1 < \beta < m$. Then the BVP*

$$\begin{cases} D_{0^+}^{\beta} u(t) = h(t), & 0 < t < 1, \\ u(1) = u^{(i)}(0) = 0, & i = 1, 2, \ldots, m-1, \end{cases} \tag{2.1}$$

has an unique solution

$$u(t) = -\int_0^1 H(t,s) h(s)\, ds, \tag{2.2}$$

where

$$H(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} (1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \le s \le t \le 1, \\ (1-s)^{\beta-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Proof From (2.1) and Lemma 2.1, we have

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)\, ds + C_0 + C_1 t + C_2 t^2 + \cdots + C_{m-1} t^{m-1}. \tag{2.4}$$

Since $u^{(i)}(0) = 0$ ($i = 1, 2, \ldots, m-1$), we have $C_1 = C_2 = \cdots = C_{m-1} = 0$ and

$$u(1) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\, ds + C_0. \tag{2.5}$$

Combined with $u(1) = 0$, we know

$$C_0 = -\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\, ds.$$

Thus

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)\, ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\, ds \\ &= \frac{1}{\Gamma(\beta)} \int_0^t \big((t-s)^{\beta-1} - (1-s)^{\beta-1} \big) h(s)\, ds \\ &\quad - \frac{1}{\Gamma(\beta)} \int_t^1 (1-s)^{\beta-1} h(s)\, ds \\ &= -\int_0^1 H(t,s) h(s)\, ds, \end{aligned}$$

where $H(t,s)$ is given by (2.3).

The proof is completed. □

Lemma 2.4 *Assume* (L_0) *holds, let* $y \in C[0,1]$ *and* $1 < n-1 < \alpha < n, 1 < m-1 < \beta < m$. *Then the following boundary value problems*:

$$\begin{cases} D_{0+}^{\alpha} x(t) = y(t), \quad 0 < t < 1, \\ x^{(j)}(0) = 0, \quad j = 2,3,\dots,n-1, \\ x(1) = \int_0^1 g_1(s)x(s)\,ds, \\ x'(0) = \int_0^1 g_2(s)x(s)\,ds, \end{cases} \tag{2.6}$$

has an unique solution

$$x(t) = -\int_0^1 G(t,s)y(s)\,ds \tag{2.7}$$

and

$$D_{0+}^{\beta} x(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)\,ds, \tag{2.8}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.9}$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ (1-s)^{\alpha-1}, & 0 \le t \le s \le 1, \end{cases} \tag{2.10}$$

$$G_2(t,s) = \delta\left(P(t) \int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + Q(t) \int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau \right), \tag{2.11}$$

here

$$\delta^{-1} = (1-M_1)(1-N_2) + N_1(1-M_2), \tag{2.12}$$

$$P(t) = 1 - N_2 + N_1 t, \qquad Q(t) = M_2 - 1 + (1-M_1)t, \tag{2.13}$$

$$M_1 = \int_0^1 g_1(s)\,ds, \qquad M_2 = \int_0^1 s g_1(s)\,ds, \tag{2.14}$$

$$N_1 = \int_0^1 g_2(s)\,ds, \qquad N_2 = \int_0^1 s g_2(s)\,ds. \tag{2.15}$$

Proof From Lemma 2.1, considering BVP (2.6), we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1}.$$

Because $x^{(j)}(0) = 0$ $(j = 1,2,\dots,n-1)$, we have $C_2 = C_3 = \cdots = C_{n-1} = 0$, and

$$x(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\,ds + C_0 + C_1.$$

From the second condition of BVP (2.6), we have

$$\int_0^1 g_1(s)x(s)\,ds = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds + C_0 + C_1. \tag{2.16}$$

From the last condition of BVP (2.6), we have

$$C_1 = x'(0) = \int_0^1 g_2(s)x(s)\,ds. \tag{2.17}$$

By (2.16) and (2.17), we obtain

$$C_0 = \int_0^1 g_1(s)x(s)\,ds - \int_0^1 g_2(s)x(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds. \tag{2.18}$$

So

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + C_0 + C_1 t \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + \int_0^1 g_1(s)x(s)\,ds - \int_0^1 g_2(s)x(s)\,ds \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds + t\int_0^1 g_2(s)x(s)\,ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \big((t-s)^{\alpha-1} - (1-s)^{\alpha-1}\big)y(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}y(s)\,ds \\
&\quad + \int_0^1 g_1(s)x(s)\,ds - \int_0^1 g_2(s)x(s)\,ds + t\int_0^1 g_2(s)x(s)\,ds \\
&= -\int_0^1 G_1(t,s)y(s)\,ds + A_1 - A_2 + tA_2,
\end{aligned} \tag{2.19}$$

where $G_1(t,s)$ is given in (2.10), and

$$\begin{aligned}
A_1 &= \int_0^1 g_1(s)x(s)\,ds, \\
A_2 &= \int_0^1 g_2(s)x(s)\,ds.
\end{aligned}$$

In view of (2.19), we get

$$g_1(t)x(t) = -g_1(t)\int_0^1 G_1(t,s)y(s)\,ds + A_1 g_1(t) - A_2 g_1(t) + tA_2 g_1(t). \tag{2.20}$$

Integrating (2.20) from 0 to 1, we obtain

$$\begin{aligned}
A_1 &= \int_0^1 g_1(s)x(s)\,ds \\
&= -\int_0^1 g_1(s)\left(\int_0^1 G_1(s,\tau)y(\tau)\,d\tau\right) ds
\end{aligned}$$

$$\begin{aligned} &+ A_1 \int_0^1 g_1(s)\,ds - A_2 \int_0^1 g_1(s)\,ds + A_2 \int_0^1 s g_1(s)\,ds \\ &= I_1 + A_1 M_1 - A_2 M_1 + A_2 M_2, \end{aligned} \tag{2.21}$$

where M_1 and M_2 are given in (2.14), and

$$I_1 = -\int_0^1 g_1(s)\left(\int_0^1 G_1(s,\tau)y(\tau)\,d\tau\right) ds = -\int_0^1 \left(\int_0^1 g_1(s)G_1(s,\tau)\,ds\right) y(\tau)\,d\tau.$$

Similarly, we obtain

$$\begin{aligned} A_2 &= \int_0^1 g_2(s)x(s)\,ds \\ &= -\int_0^1 g_2(s)\left(\int_0^1 G_1(s,\tau)y(\tau)\,d\tau\right) ds \\ &\quad + A_1 \int_0^1 g_2(s)\,ds - A_2 \int_0^1 g_2(s)\,ds + A_2 \int_0^1 s g_2(s)\,ds \\ &= I_2 + A_1 N_1 - A_2 N_1 + A_2 N_2, \end{aligned} \tag{2.22}$$

where N_1 and N_2 are given in (2.15), and

$$I_2 = -\int_0^1 g_2(s)\left(\int_0^1 G_1(s,\tau)y(\tau)\,d\tau\right) ds = -\int_0^1 \left(\int_0^1 g_2(s)G_1(s,\tau)\,ds\right) y(\tau)\,d\tau.$$

From (2.21) and (2.22), we get

$$\begin{aligned} A_1 &= \bigl(I_2(M_2 - M_1) + I_1(1 + N_1 - N_2)\bigr)\delta, \\ A_2 &= \bigl(I_2(1 - M_1) + I_1 N_1\bigr)\delta, \end{aligned}$$

where δ^{-1} is given in (2.12). Hence,

$$\begin{aligned} x(t) &= -\int_0^1 G_1(t,s)y(s)\,ds + A_1 - A_2 + tA_2 \\ &= -\int_0^1 G_1(t,s)y(s)\,ds + \delta\bigl(I_2(M_2 - 1 + t - M_1 t) + I_1(1 - N_2 + N_1 t)\bigr) \\ &= -\int_0^1 G_1(t,s)y(s)\,ds + \delta I_1 P(t) + \delta I_2 Q(t) \\ &= -\int_0^1 G_1(t,s)y(s)\,ds - \int_0^1 \delta P(t)\left(\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau\right) y(s)\,ds \\ &\quad - \int_0^1 \delta Q(t)\left(\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right) y(s)\,ds \\ &= -\int_0^1 G_1(t,s)y(s)\,ds - \int_0^1 \delta\left(P(t)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau \right. \\ &\quad \left. + Q(t)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right) y(s)\,ds \end{aligned}$$

$$= -\int_0^1 G_1(t,s)y(s)\,ds - \int_0^1 G_2(t,s)y(s)\,ds$$
$$= -\int_0^1 G(t,s)y(s)\,ds,$$

where $G_2(t,s)$ is given in (2.11) and $P(t)$, $Q(t)$ are given in (2.13).

On the other hand, in view of (2.19), because $m-1<\beta<\alpha-1<n-1$, by Lemma 2.2, we have

$$\begin{aligned}
D_{0+}^{\beta}x(t) &= D_{0+}^{\beta}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + C_0 + C_1 t\right)\\
&= D_{0+}^{\beta}\bigl(I_{0+}^{\alpha}y(t) + C_0 + C_1 t\bigr)\\
&= D_{0+}^{\beta}I_{0+}^{\alpha}y(t) + D_{0+}^{\beta}(C_0) + D_{0+}^{\beta}(C_1 t)\\
&= D_{0+}^{\beta}I_{0+}^{\alpha}y(t)\\
&= I_{0+}^{\alpha-\beta}y(t)\\
&= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}y(s)\,ds.
\end{aligned} \tag{2.23}$$

The proof is completed. □

Lemma 2.5 *BVP* (1.1) *equivalent to the following integral equation*:

$$x(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds \tag{2.24}$$

and

$$D_{0+}^{\beta}x(t) = -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds, \tag{2.25}$$

where $H(t,s)$ *and* $G(t,s)$ *are given in* (2.3) *and* (2.9).

Proof From Lemma 2.4 and Lemma 2.5, let $y(t) = \varphi_q(u(t))$, $h(t) = f(t,x(t),D_{0+}^{\beta}x(t))$, we have

$$\begin{aligned}
y(t) = \varphi_q\bigl(u(t)\bigr) &= \varphi_q\left(-\int_0^1 H(t,s)f\bigl(s,x(s),D_{0+}^{\beta}x(s)\bigr)\,ds\right)\\
&= -\varphi_q\left(\int_0^1 H(t,s)f\bigl(s,x(s),D_{0+}^{\beta}x(s)\bigr)\,ds\right).
\end{aligned}$$

Immediately we obtain

$$x(t) = -\int_0^1 G(t,s)y(s)\,ds = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds.$$

From (2.23), we have (2.25).

The proof is completed. □

Lemma 2.6 *Assume* (L_0) *holds, then the function* $H(t,s)$ *defined by* (2.3) *and the function* $G(t,s)$ *defined by* (2.9) *satisfies*

(1) *$H(t,s) \geq 0$ is continuous for all $t,s \in [0,1]$;*
(2) *$H(t,s) \leq H(s,s)$ for all $t,s \in [0,1]$;*
(3) *$G(t,s) \geq 0$ is continuous for all $t,s \in [0,1]$.*

Proof (1) From (2.3), it is easy to show that $H(t,s)$ is continuous on $[0,1] \times [0,1]$ and obviously, $H(t,s) \geq 0$, for $s \geq t$.

For $0 \leq s \leq t \leq 1$, we have

$$(1-s)^{\beta-1} - (t-s)^{\beta-1} = (1-s)^{\beta-1}\left(1 - \left(\frac{t-s}{1-s}\right)^{\beta-1}\right) \geq 0.$$

By (2.3), we know $H(t,s) \geq 0$, $t,s \in [0,1]$, and $H(t,s) > 0$, $t,s \in (0,1)$.

(2) For $0 \leq s \leq t \leq 1$, we have

$$H(t,s) = (1-s)^{\beta-1} - (t-s)^{\beta-1} \leq (1-s)^{\beta-1} = H(s,s)$$

and for $s \geq t$, $H(t,s) = H(s,s)$, so that $H(t,s) \leq H(s,s)$, for all $t,s \in [0,1]$.

(3) From (2.9), we know $G(t,s) = G_1(t,s) + G_2(t,s)$. Firstly, from (2.10), similarly, we can obtain $G_1(t,s) \geq 0$, $t,s \in [0,1]$, and $G_1(t,s) > 0$, $t,s \in (0,1)$. On the other hand, from (L_0), we know, for $t \in (0,1)$,

$$g_1(t) > tg_1(t) > tg_2(t)$$

and

$$\begin{aligned} 1 &> \int_0^1 g_1(s)\,ds > \int_0^1 sg_1(s)\,ds > \int_0^1 sg_2(s)\,ds > 0, \\ 1 &> \int_0^1 g_1(s)\,ds > \int_0^1 g_2(s)\,ds > \int_0^1 sg_2(s)\,ds > 0. \end{aligned}$$

That implies

$$1 > M_1 > M_2 > N_2 > 0, \qquad 1 > M_1 > N_1 > N_2 > 0. \tag{2.26}$$

Therefore

$$\delta^{-1} = (1 - M_1)(1 - N_2) + N_1(1 - M_2) > 0.$$

From (L_0) and $G_1(t,s) \geq 0$, we know

$$\frac{\partial G_2(t,s)}{\partial t} = \delta\left(N_1 \int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (1 - M_1)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right) > 0, \tag{2.27}$$

which implies that $G_2(t,s)$ is a monotone increasing function with respect to $t \in [0,1]$.

So, from (2.11), we have

$$\begin{aligned} G_2(t,s) \geq G_2(0,s) &= \delta\left(P(0)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + Q(0)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right) \\ &= \delta\left((1 - N_2)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (M_2 - 1)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right) \end{aligned}$$

$$\begin{aligned}
&= \delta\left(\int_0^1 \big((1-N_2)g_1(\tau)+(M_2-1)g_2(\tau)\big)G_1(\tau,s)\,d\tau\right)\\
&\geq \delta\left(\int_0^1 \big((1-N_2)g_2(\tau)+(M_2-1)g_2(\tau)\big)G_1(\tau,s)\,d\tau\right)\\
&= \delta(M_2-N_2)\left(\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right)\\
&\geq 0.
\end{aligned}$$

So $G_2(t,s)\geq 0$. Hence, $G(t,s)\geq 0$.

The proof is completed. □

Lemma 2.7 *Let $\eta\in(0,\frac{1}{2})$, denote $I_\eta=[0,\eta]$ and $\rho_1=1-(\frac{\eta}{1-\eta})^{\alpha-1}$, then*

$$\min_{t\in I_\eta} G_1(t,s)\geq \rho_1 G_1(s,s)=\rho_1\max_{t\in[0,1]} G_1(t,s). \tag{2.28}$$

Proof For $0\leq s<t\leq 1$ and $t\in I_\eta$,

$$G_1(t,s)=(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\leq (1-s)^{\alpha-1}=G_1(s,s)$$

and

$$\begin{aligned}
\frac{G_1(t,s)}{G_1(s,s)} &= \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{(1-s)^{\alpha-1}}\\
&= 1-\left(\frac{t-s}{1-s}\right)^{\alpha-1}\\
&\geq 1-\left(\frac{\eta}{1-s}\right)^{\alpha-1}\\
&\geq 1-\left(\frac{\eta}{1-\eta}\right)^{\alpha-1}\\
&= \rho_1>0.
\end{aligned} \tag{2.29}$$

For $s\geq t$ and $t\in I_\eta$,

$$G_1(t,s)=G_1(s,s)>\left(1-\left(\frac{\eta}{1-\eta}\right)^{\alpha-1}\right)G_1(s,s)=\rho_1 G_1(s,s). \tag{2.30}$$

Therefore,

$$\min_{t\in I_\eta} G_1(t,s)\geq \rho_1 G_1(s,s).$$

The proof is completed. □

Lemma 2.8 *Assume* (L_0) *holds, then the function $G_2(t,s)$ satisfies the following properties*:

(1) $G_2(t,s)\leq G_2(1,s)=\max_{t\in[0,1]} G_2(t,s)$;

(2) $\min_{t\in I_\eta} G_2(t,s)\geq \rho_2\max_{t\in[0,1]} G_2(t,s)$, *where* $0<\rho_2=\frac{M_2-N_2}{1-N_2+N_1}<1$.

Proof From Lemma 2.7 and (2.27), we obtain $G_2(t,s) > 0$ and

$$\begin{aligned}\max_{t\in[0,1]} G_2(t,s) &= G_2(1,s)\\ &= \delta\left((1-N_2+N_1)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (M_2-M_1)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right)\end{aligned}$$

and

$$\begin{aligned}\min_{t\in I_\eta} G_2(t,s) &= G_2(0,s)\\ &= \delta\left((1-N_2)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (M_2-1)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau\right)\\ &\geq 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{G_2(0,s)}{G_2(1,s)} &= \frac{(1-N_2)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (M_2-1)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau}{(1-N_2+N_1)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (M_2-M_1)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau}\\ &> \frac{(1-N_2)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (M_2-1)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau}{(1-N_2+N_1)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau + (M_2-M_1)\int_0^1 g_2(\tau)G_1(\tau,s)\,d\tau}\\ &> \frac{(1-N_2+M_2-1)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau}{(1-N_2+N_1)\int_0^1 g_1(\tau)G_1(\tau,s)\,d\tau}\\ &= \frac{M_2-N_2}{1-N_2+N_1}\\ &= \rho_2\end{aligned}$$

and

$$0 < \rho_2 = \frac{M_2-N_2}{1-N_2+N_1} < \frac{M_2-N_2}{1-N_2} < 1.$$

Hence,

$$\min_{t\in I_\eta} G_2(t,s) \geq \rho_2 G_2(1,s).$$

The proof is completed. □

From Lemma 2.7 and Lemma 2.8, we can easily show that the following result holds.

Lemma 2.9

$$\min_{t\in I_\eta} G(t,s) \geq \rho\bigl(G_1(s,s)+G_2(1,s)\bigr) \quad \textit{and} \quad G(t,s) \leq G_1(s,s)+G_2(1,s),$$

where $\rho = \min\{\rho_1,\rho_2\}$.

3 Main results

In this section we deduce the existence of positive solutions to BVP (1.1) by using the well-known Avery-Peterson fixed point theorem; see [4].

Let γ and θ be non-negative continuous convex functionals on P, φ be a non-negative continuous concave functional on P, and ψ be a non-negative continuous functional on P. For $a, b, c, d > 0$, we define the following convex set:

$$P(\gamma; d) = \{x \in P : \gamma(x) < d\},$$
$$P(\gamma, \varphi; b, d) = \{x \in P : b \le \varphi(x), \gamma(x) \le d\},$$
$$P(\gamma, \theta, \varphi; b, c, d) = \{x \in P : b \le \varphi(x), \theta(x) \le c, \gamma(x) \le d\},$$

and a closed set

$$P(\gamma, \psi; a, d) = \{x \in P : a \le \psi(x), \gamma(x) \le d\}.$$

Lemma 3.1 (see [4]) *Let P be a cone in a real Banach space E. Let γ and θ be non-negative continuous convex functionals on P, φ be a non-negative continuous concave functional on P, and ψ be a non-negative continuous functional on P satisfying $\psi(\lambda x) \le \lambda\psi(x)$ for $0 \le \lambda \le 1$, such that, for some positive numbers M and d,*

$$\varphi(x) \le \psi(x), \qquad \|x\| \le M\gamma(x) \tag{3.1}$$

for all $x \in \overline{P(\gamma; d)}$. Suppose

$$T : \overline{P(\gamma; d)} \to \overline{P(\gamma; d)}$$

is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(H1) *$\{x \in P(\gamma, \theta, \varphi; b, c, d) : \varphi(x) > b\} \neq \emptyset$, and $\varphi(x) > b$ for $x \in P(\gamma, \theta, \varphi; b, c, d)$;*

(H2) *$\varphi(Tx) > b$ for $x \in P(\gamma, \varphi; b, d)$ with $\theta(Tx) > c$;*

(H3) *$0 \notin P(\gamma, \psi; a, d)$ and $\psi(Tx) < a$ for $x \in P(\gamma, \psi; a, d)$ with $\psi(x) = a$.*

Then T has at least three fixed point $x_1, x_2, x_3 \in \overline{P(\gamma; d)}$ such that

$$\gamma(x_i) \le d, \quad i = 1, 2, 3; \qquad \varphi(x_1) > b, \qquad a < \varphi(x_2), \qquad \psi(x_2) < b; \qquad \psi(x_3) < a.$$

Let $E = \{x \in C[0,1] : D_{0+}^{\beta} x \in C[0,1], x'(0) = \int_0^1 g_2(s)x(s)\,ds, x^{(j)}(0) = 0, j = 2, 3, \ldots, m-1\}$ be endowed with the norm

$$\|x\| = \max\Bigl\{\max_{t\in[0,1]} |x(t)|, \max_{t\in[0,1]} |D_{0+}^{\beta} x(t)|\Bigr\},$$

then E is a Banach space.

We define a set $P \subset E$ by

$$P = \Bigl\{x \in E : x(t) \ge 0, D_{0+}^{\beta} x(t) \le 0, \min_{t\in I_\eta} x(t) \ge \rho \max_{t\in[0,1]} x(t)\Bigr\}.$$

For $x, y \in P$ and $k_1, k_2 \geq 0$, it is easy to obtain

$$k_1 x(t) + k_2 y(t) \geq 0, \qquad D_{0+}^{\beta}\big(k_1 x(t) + k_2 y(t)\big) = k_1 D_{0+}^{\beta} x(t) + k_2 D_{0+}^{\beta} y(t) \leq 0$$

and

$$\begin{aligned}
\min_{t\in I_\eta}\big\{k_1 x(t) + k_2 y(t)\big\} &\geq \min_{t\in I_\eta}\big\{k_1 x(t)\big\} + \min_{t\in I_\eta}\big\{k_2 y(t)\big\} \\
&\geq \rho \max_{t\in[0,1]}\big\{k_1 x(t)\big\} + \rho \max_{t\in[0,1]}\big\{k_2 y(t)\big\} \\
&= \rho\Big(\max_{t\in[0,1]}\big\{k_1 x(t)\big\} + \max_{t\in[0,1]}\big\{k_2 y(t)\big\}\Big) \\
&\geq \rho \max_{t\in[0,1]}\big\{k_1 x(t) + k_2 y(t)\big\}.
\end{aligned}$$

Thus, for $x, y \in P$ and $k_1, k_2 \geq 0$, $k_1 x(t) + k_2 y(t) \in P$. And if $x \in P$, $x \neq 0$, it is easy to prove that $-x \notin P$. Therefore, P is a cone in E.

Let $T : P \to E$ be the operator defined by

$$Tx(t) := \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, x(\tau), D_{0+}^{\beta} x(\tau)\big)\, d\tau\right) ds.$$

Then we have the following conclusion.

Lemma 3.2 *Assume* (L_0) *holds, then* $T : P \to P$ *is a completely continuous operator.*

Proof For $x \in P$, from the non-negativity and continuity of $G(t,s), H(t,s), f(t, x(t), D_{0+}^{\beta} x(t))$, we know that T is a continuous operator and $Tx(t) \geq 0$. By (2.25), we have

$$D_{0+}^{\beta} Tx(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, x(\tau), D_{0+}^{\beta} x(\tau)\big)\, d\tau\right) ds \leq 0.$$

Furthermore,

$$\begin{aligned}
\min_{t\in I_\eta} Tx(t) &= \min_{t\in I_\eta} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, x(\tau), D_{0+}^{\beta} x(\tau)\big)\, d\tau\right) ds \\
&= \int_0^1 \min_{t\in I_\eta} G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, x(\tau), D_{0+}^{\beta} x(\tau)\big)\, d\tau\right) ds \\
&\geq \int_0^1 \rho \max_{t\in[0,1]} G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, x(\tau), D_{0+}^{\beta} x(\tau)\big)\, d\tau\right) ds = \rho \max_{t\in[0,1]} Tx(t).
\end{aligned}$$

Thus, $T(P) \subseteq P$.

Next, we show that T is uniformly bounded.

Let $D \subset P$ be bounded, *i.e.*, there exists a positive constant r such that $\|x\| \leq r$, for all $x \in D$. Let $M_0 = \max_{t\in[0,1], x\in D} |f(t, x(t), D_{0+}^{\beta} x(t))| + 1 > 0$, for $x \in D$, we have

$$\begin{aligned}
\big|Tx(t)\big| &= \left|\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, x(\tau), D_{0+}^{\beta} x(\tau)\big)\, d\tau\right) ds\right| \\
&\leq \int_0^1 \big|G(t,s)\big|\varphi_q\left(\int_0^1 \big|H(s,\tau)\big|\big|f\big(\tau, x(\tau), D_{0+}^{\beta} x(\tau)\big)\big|\, d\tau\right) ds
\end{aligned}$$

$$\begin{aligned}
&\le \varphi_q(M_0)\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)\right)ds \\
&\le \varphi_q(M_0)\int_0^1 \bigl(G_1(s,s)+G_2(1,s)\bigr)\varphi_q\left(\int_0^1 H(s,\tau)\right)ds := M_{01}.
\end{aligned}$$

Furthermore, for any $x \in D$ and $t \in [0,1]$, we have

$$\begin{aligned}
\bigl|D_{0+}^{\beta}Tx(t)\bigr| &= \left|-\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds\right| \\
&\le \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds \\
&\le \varphi_q(M_0)\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)\,d\tau\right)ds := M_{02}.
\end{aligned}$$

So $\|Tx\| \le \max\{M_{01}, M_{02}\}$, which implies T is uniformly bounded.

In the following of the proof, we will prove that T is equicontinuous. Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. Thus, for any $\epsilon > 0$, there exists a constant $\delta_1 > 0$, such that

$$\bigl|G(t_1,s)-G(t_2,s)\bigr| < \frac{\epsilon}{\varphi_q(M_0)\int_0^1 \varphi_q(\int_0^1 H(s,\tau)\,d\tau)\,ds}$$

for $t_1, t_2 \in [0,1]$ with $|t_1, t_2| < \delta_1$. Therefore,

$$\begin{aligned}
\bigl|Tx(t_1)-Tx(t_2)\bigr| &= \left|\int_0^1 G(t_1,s)\varphi_q\left[\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right]ds\right. \\
&\quad \left.-\int_0^1 G(t_2,s)\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds\right| \\
&\le \int_0^1 \bigl|G(t_1,s)-G(t_2,s)\bigr|\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds \\
&< \frac{\epsilon}{\varphi_q(M_0)\int_0^1 \varphi_q(\int_0^1 H(s,\tau)\,d\tau)\,ds}\varphi_q(M_0)\int_0^1 \varphi_q\left(\int_0^1 H(s,\tau)\,d\tau\right)ds \\
&= \epsilon.
\end{aligned}$$

On the other hand, for $1 < m-1 < \beta < \alpha-1 < n-1$, $t^{\alpha-\beta}$ is uniformly continuous on $[0,1]$. We denote $M_{03} = \frac{\varphi_q(M_0)}{(\alpha-\beta)\Gamma(\alpha-\beta)}$. Then there exists a constant $0 < \delta_2 < (\frac{\epsilon}{4M_{03}})^{(\alpha-\beta)^{-1}}$, such that, for any $0 < t_1 < t_2 < 1$ and $|t_2 - t_1| < \delta_2$, we have

$$\bigl|t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\bigr| < \frac{\epsilon}{2M_{03}}.$$

Thus, from Lemma 2.7, we have

$$\begin{aligned}
&\bigl|D_{0+}^{\beta}Tx(t_2) - D_{0+}^{\beta}Tx(t_1)\bigr| \\
&\quad= \left|\frac{1}{\Gamma(\alpha-\beta)}\int_0^{t_2}(t_2-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds\right. \\
&\qquad\left.-\frac{1}{\Gamma(\alpha-\beta)}\int_0^{t_1}(t_1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\right)ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha-\beta)}\left(\left|\int_0^{t_1}\left((t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right)\right.\right.\\
&\quad\left.\times\varphi_q\left(\int_0^1 H(s,\tau)f\left(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\right)d\tau\right)ds\right|\\
&\quad\left.+\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)f\left(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\right)d\tau\right)ds\right|\right)\\
&\leq \frac{\varphi_q(\int_0^1 M_0H(\tau,\tau)\,d\tau)}{\Gamma(\alpha-\beta)}\left(\left|\int_0^{t_1}\left((t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1}\right)ds\right|\right.\\
&\quad\left.+\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\,ds\right|\right)\\
&= M_{03}\left(\left|\int_0^{t_1}d\left[(t_1-s)^{\alpha-\beta}-(t_2-s)^{\alpha-\beta}\right]\right|+\left|\int_{t_1}^{t_2}d(t_2-s)^{\alpha-\beta}\right|\right)\\
&= M_{03}\left(\left|0-(t_2-t_1)^{\alpha-\beta}-t_1^{\alpha-\beta}+t_2^{\alpha-\beta}\right|+\left|(t_2-t_1)^{\alpha-\beta}\right|\right)\\
&\leq M_{03}\left(\left|2(t_2-t_1)^{\alpha-\beta}\right|+\left|t_2^{\alpha-\beta}-t_1^{\alpha-\beta}\right|\right)\\
&< M_{03}\left(2\left(\left(\frac{\epsilon}{4M_{03}}\right)^{(\alpha-\beta)^{-1}}\right)^{\alpha-\beta}+\frac{\epsilon}{2M_{03}}\right)\\
&= \epsilon.
\end{aligned}
$$

Hence, T is equicontinuous.

According to the Arzelà-Ascoli theorem, T is a completely continuous operator.

The proof is completed. □

Denote the positive constants

$$
J_1 = \int_0^1 \left(G_1(s,s)+G_2(1,s)\right)\varphi_q\left(\int_0^1 H(s,\tau)\,d\tau\right)ds,
$$

$$
J_2 = \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)\,d\tau\right)ds,
$$

and

$$
J_3 = \int_0^1 \left(G_1(s,s)+G_2(1,s)\right)\varphi_q\left(\int_0^{\eta} H(s,\tau)\,d\tau\right)ds.
$$

Define the functionals as follows:

$$
\gamma(x)=\|x\|,\qquad \theta(x)=\psi(x)=\max_{t\in[0,1]}\left|x(t)\right|,\qquad \varphi(x)=\min_{t\in I_\eta}\left|x(t)\right|,
$$

then γ and θ are continuous non-negative convex functionals, φ is a continuous non-negative concave functional, ψ is a continuous non-negative functional, and

$$
\rho\theta(x)\leq\varphi(x)\leq\theta(x)=\psi(x),\qquad \|x\|\leq M\gamma(x),
$$

where $M = 1$. Therefore, condition (3.1) in Lemma 3.1 is satisfied.

Theorem 3.1 *Suppose* (L_0) *holds, and there exist constants* $0 < a, b, d$ *with* $a < b < \rho d \min\{\frac{J_3}{J_1}, \frac{J_3}{J_2}\}$ *and* $c = \frac{b}{\rho}$, *such that*

(L_1) $f(t,x,y) \le \min\{\varphi_p(\frac{d}{J_1}), \varphi_p(\frac{d}{J_2})\}$, $(t,x,y) \in [0,1] \times [0,d] \times [-d,0]$;
(L_2) $f(t,x,y) > \varphi_p(\frac{b}{\rho J_3})$, $(t,x,y) \in [0,\eta] \times [b,\frac{b}{\rho}] \times [-d,0]$;
(L_3) $f(t,x,y) < \varphi_p(\frac{a}{J_1})$, $(t,x,y) \in [0,1] \times [0,a] \times [-d,0]$.

Then BVP (1.1) *has at least three positive solutions* x_1, x_2, x_3, *satisfying*

$$\|x_i\| \le d \quad (i = 1,2,3), \tag{3.2}$$

$$\min_{t\in I_\eta}\bigl|x_1(t)\bigr| > b, \qquad a < \min_{t\in I_\eta}\bigl|x_2(t)\bigr|, \qquad \max_{t\in[0,1]}\bigl|x_2(t)\bigr| < b, \qquad \max_{t\in[0,1]}\bigl|x_3(t)\bigr| < a. \tag{3.3}$$

Proof It is clear that the fixed points of operator T are equivalent the solutions of BVP (1.1). For $x \in \overline{P(\gamma;d)}$, we have

$$\gamma(x) = \|x\| \le d,$$

this implies

$$\max_{t\in[0,1]}\bigl|x(t)\bigr| \le d, \qquad \max_{t\in[0,1]}\bigl|D_{0+}^{\beta}x(t)\bigr| \le d,$$

then

$$0 \le x(t) \le d, \qquad -d \le D_{0+}^{\beta}x(t) \le 0.$$

By (L_1), we have

$$\begin{aligned}
\max_{t\in[0,1]}\bigl|Tx(t)\bigr| &= \max_{t\in[0,1]}\int_0^1 G(t,s)\varphi_q\biggl(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\le \int_0^1 \bigl(G_1(s,s)+G_2(1,s)\bigr)\varphi_q\biggl(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\biggr)\,ds \\
&\le \int_0^1 \bigl(G_1(s,s)+G_2(1,s)\bigr)\varphi_q\biggl(\varphi_p\biggl(\frac{d}{J_1}\biggr)\int_0^1 H(s,\tau)\,d\tau\biggr)\,ds \\
&= \frac{d}{J_1}\int_0^1 \bigl(G_1(s,s)+G_2(1,s)\bigr)\varphi_q\biggl(\int_0^1 H(s,\tau)\,d\tau\biggr)\,ds \\
&= d
\end{aligned}$$

and

$$\begin{aligned}
\max_{t\in[0,1]}\bigl|D_{0+}^{\beta}Tx(t)\bigr| &= \max_{t\in[0,1]}\biggl|\frac{-1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} \\
&\quad \times \varphi_q\biggl(\int_0^1 H(s,\tau)f\bigl(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\bigr)\,d\tau\biggr)\,ds\biggr| \\
&\le \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\biggl(\varphi_p\biggl(\frac{d}{J_2}\biggr)\int_0^1 H(s,\tau)\,d\tau\biggr)\,ds
\end{aligned}$$

$$\begin{aligned}&= \frac{d}{J_2}\frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 H(s,\tau)\,d\tau\right)ds\\&= d,\end{aligned}$$

so

$$\gamma(Tx) = \|Tx\| = \max\Big\{\max_{t\in[0,1]}\big|Tx(t)\big|, \max_{t\in[0,1]}\big|D_{0+}^{\beta}Tx(t)\big|\Big\} \le d.$$

Therefore $T:\overline{P(\gamma;d)}\to\overline{P(\gamma;d)}$.

Let $x(t)=\frac{b}{\rho}$, $x(t)\in P(\gamma,\theta,\varphi;b,c,d)$ and $\varphi(\frac{b}{\rho})>b$, which implies that

$$\big\{x\in P(\gamma,\theta,\varphi;b,c,d):\varphi(x)>b\big\}\neq\varnothing.$$

For $x\in P(\gamma,\theta,\varphi;b,c,d)$, we know that $b\le x(t)\le c=\frac{b}{\rho}$ for $t\in I_\eta$ and $-d\le D_{0+}^{\beta}x(t)\le 0$. In view of ($L_2$),

$$\begin{aligned}\varphi(Tx) &= \min_{t\in I_\eta}\big|Tx(t)\big|\\&= \min_{t\in I_\eta}\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f\big(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\big)\,d\tau\right)ds\\&> \int_0^1 \rho\big(G_1(s,s)+G_2(1,s)\big)\varphi_q\left(\int_0^\eta H(s,\tau)\varphi_p\left(\frac{b}{\rho J_3}\right)d\tau\right)ds\\&= \rho\frac{b}{\rho J_3}\int_0^1\big(G_1(s,s)+G_2(1,s)\big)\varphi_q\left(\int_0^\eta H(s,\tau)\,d\tau\right)ds\\&= b.\end{aligned}\tag{3.4}$$

So $\varphi(Tx)>b$ for all $x\in P(\gamma,\theta,\varphi;b,c,d)$. Hence, the condition (H1) of Lemma 3.1 is satisfied.

By (3.2), for all $x\in P(\gamma,\varphi;b,d)$ with $\theta(Tx)>c=\frac{b}{\rho}$, we have

$$\varphi(Tx)\ge\rho\theta(Tx)>\rho c=\rho\frac{b}{\rho}=b.$$

Thus, the condition (H2) of Lemma 3.1 holds.

Because of $\psi(0)=0<a$, then $0\notin P(\gamma,\psi;a,d)$. For $x\in P(\gamma,\psi;a,d)$ with $\psi(x)=a$, we know $\gamma(x)\le d$. It means that $\max_{t\in[0,1]}x(t)=a$ and $-d\le D_{0+}^{\beta}x(t)\le 0$.

From (L_3), we can obtain

$$\begin{aligned}\psi(Tx) &= \max_{t\in[0,1]}\big|Tx(t)\big|\\&= \max_{t\in[0,1]}\int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau)f\big(\tau,x(\tau),D_{0+}^{\beta}x(\tau)\big)\,d\tau\right)ds\\&< \int_0^1\big(G_1(s,s)+G_2(1,s)\big)\varphi_q\left(\int_0^1 H(s,\tau)\varphi_p\left(\frac{a}{J_1}\right)d\tau\right)ds\\&= \frac{a}{J_1}\int_0^1\big(G_1(s,s)+G_2(1,s)\big)\varphi_q\left(\int_0^1 H(s,\tau)\,d\tau\right)ds\\&= a.\end{aligned}$$

Therefore, the condition (H3) of Lemma 3.1 holds.

To sum up, the conditions of Lemma 3.1 are all verified. Hence, BVP (1.1) has at least three positive solutions x_1, x_2, x_3 satisfying (3.2) and (3.3).

The proof is completed. □

4 Example

In this section, we present an example to illustrate the main result. Consider the following boundary value problems:

$$\begin{cases} D_{0+}^{\frac{5}{4}}\varphi_{\frac{3}{2}}(D_{0+}^{\frac{5}{2}}x(t)) = f(t,x(t),D_{0+}^{\frac{5}{4}}x(t)), & t \in (0,1), \\ (\varphi_{\frac{3}{2}}(D_{0+}^{\frac{5}{2}}x(0)))' = \varphi_{\frac{3}{2}}(D_{0+}^{\frac{5}{2}}x(1)) = 0, \\ x(1) = \int_0^1 sx(s)\,ds, \\ x'(0) = \int_0^1 s^2x(s)\,ds, \\ x''(0) = 0, \end{cases} \tag{4.1}$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{5}{4}$, $p = \frac{3}{2}$, $g_1(t) = t$, $g_2(t) = t^2$, and

$$f(t,x,y) = \begin{cases} \tan(0.6t) + 0.01e^{x/5}x^2 + \sin(y), & 0 \le x \le 18, \\ \tan(0.6t) + \sin(y) + 92, & 18 < x \le 5{,}000. \end{cases} \tag{4.2}$$

Choose $a = 8$, $b = 18$, $d = 5{,}000$, $\eta = \frac{1}{3}$. By a simple computation, we have

$$\begin{aligned} &\rho_1 = \frac{1}{2}, \qquad \rho_2 = \frac{1}{13}, \qquad \rho = \frac{1}{13}, \qquad r = 1, \\ &M_1 = \frac{1}{2}, \qquad M_2 = \frac{1}{3}, \qquad N_1 = \frac{1}{3}, \qquad N_2 = \frac{1}{4}, \qquad \delta = \frac{72}{43}, \\ &J_1 = 0.26227, \qquad J_2 = 0.312484, \qquad J_3 = 0.0286711. \end{aligned}$$

We can check that the nonlinear term $f(t,x,y)$ satisfies

(L_1) $f(t,x,y) \le \min\{\varphi_p(\frac{d}{J_1}), \varphi_p(\frac{d}{J_2})\} \approx 126.494$, $(t,x,y) \in [0,1] \times [0,5{,}000] \times [-5{,}000,0]$;
(L_2) $f(t,x,y) > \varphi_p(\frac{b}{\rho J_3}) \approx 90.3412$, $(t,x,y) \in [0,\frac{1}{3}] \times [18,234] \times [-5{,}000,0]$;
(L_3) $f(t,x,y) < \varphi_p(\frac{a}{J_1}) \approx 5.52294$, $(t,x,y) \in [0,1] \times [0,8] \times [-5{,}000,0]$.

Then all assumptions of Theorem 3.1 are satisfied. Thus, BVP (4.1) has at least three positive solutions x_1, x_2, x_3, satisfying

$$\|x_i\| \le 5{,}000 \quad (i = 1,2,3), \tag{4.3}$$

$$\min_{t\in I_\eta}|x_1| > 18, \qquad 8 < \min_{t\in I_\eta}|x_2|, \qquad \max_{t\in[0,1]}|x_2| < 18, \qquad \max_{t\in[0,1]}|x_3| < 8. \tag{4.4}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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