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Asymptotics and oscillation of higher-order functional dynamic equations with Laplacian and deviating arguments

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Abstract

In this paper, we deal with the asymptotics and oscillation of the solutions of higher-order nonlinear dynamic equations with Laplacian and mixed nonlinearities of the form

$$\{r_{n-1}(t)\phi_{\alpha_{n-1}}[(r_{n-2}(t)(\cdots(r_1(t)\phi_{\alpha_1}[x^{\Delta}(t)])^{\Delta}\cdots)^{\Delta})^{\Delta}]\}^{\Delta} + \sum_{\nu=0}^{N} p_\nu(t)\phi_{\gamma_\nu}(x(g_\nu(t))) = 0$$

on an above-unbounded time scale. By using a generalized Riccati transformation and integral averaging technique we study asymptotic behavior and derive some new oscillation criteria for the cases without any restrictions on $g(t)$ and $\sigma(t)$ and when n is even and odd. Our results obtained here extend and improve the results of Chen and Qu (J. Appl. Math. Comput. 44(1-2):357-377, 2014) and Zhang *et al.* (Appl. Math. Comput. 275:324-334, 2016).

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1 Introduction

We are concerned with the asymptotic and oscillatory behavior of the higher-order nonlinear functional dynamic equation

$$\{r_{n-1}(t)\phi_{\alpha_{n-1}}[(r_{n-2}(t)(\cdots(r_1(t)\phi_{\alpha_1}[x^{\Delta}(t)])^{\Delta}\cdots)^{\Delta})^{\Delta}]\}^{\Delta} + \sum_{\nu=0}^{N} p_\nu(t)\phi_{\gamma_\nu}(x(g_\nu(t))) = 0 \tag{1.1}$$

on an above-unbounded time scale $\mathbb{T}$, assuming without loss of generality that $t_0 \in \mathbb{T}$. For $A \subset \mathbb{T}$ and $B \subset \mathbb{R}$, we denote by $C_{\mathrm{rd}}(A,B)$ the space of right-dense continuous functions from A to B and by $C^1_{\mathrm{rd}}(A,B)$ the set of functions in $C_{\mathrm{rd}}(A,B)$ with right-dense continuous

Δ-derivatives. We refer the readers to the books by Bohner and Peterson [3, 4] for an excellent introduction of calculus of time scales. Throughout this paper, we suppose that:

(i) $n, N \in \mathbb{N}$, $n \geq 2$, and $\phi_\beta(u) := |u|^{\beta-1}u$, $\beta > 0$;

(ii) $r_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ for $i = 1, 2, \ldots, n-1$ are such that

$$\int_{t_0}^{\infty} r_i^{-1/\alpha_i}(\tau)\Delta\tau = \infty; \tag{1.2}$$

(iii) $\alpha_i > 0$, $i = 1, 2, \ldots, n-1$, and $\gamma_\nu > 0$, $\nu = 0, 1, \ldots, N$, are constants such that

$$\gamma_\nu > \gamma_0, \quad \nu = 1, 2, \ldots, l \quad \text{and} \quad \gamma_\nu < \gamma_0, \quad \nu = l+1, l+2, \ldots, N; \tag{1.3}$$

(iv) $p_\nu \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, [0,\infty))$, $\nu = 0, 1, \ldots, N$, are such that not all of the $p_\nu(t)$ vanish in a neighborhood of infinity;

(v) $g_\nu : \mathbb{T} \to \mathbb{T}$ are rd-continuous functions such that $\lim_{t\to\infty} g_\nu(t) = \infty$, $\nu = 0, 1, \ldots, N$.

By a solution of equation (1.1) we mean a function $x \in C^1_{\mathrm{rd}}([T_x,\infty)_{\mathbb{T}}, \mathbb{R})$ for some $T_x \geq 0$ such that $x^{[i]} \in C^1_{\mathrm{rd}}([T_x,\infty)_{\mathbb{T}}, \mathbb{R})$, $i = 1, 2, \ldots, n-1$, that satisfies equation (1.1) on $[T_x,\infty)_{\mathbb{T}}$, where

$$x^{[i]} := r_i\phi_{\alpha_i}\bigl[\bigl(x^{[i-1]}\bigr)^{\Delta}\bigr], \quad i = 1, 2, \ldots, n, \text{with } r_n = 1, \alpha_n = 1, \text{and } x^{[0]} = x. \tag{1.4}$$

A solution $x(t)$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is nonoscillatory.

Oscillation criteria for higher-order dynamic equations on time scales have been studied by many authors. For instance, Grace *et al.* [5] obtained sufficient conditions for oscillation for the higher-order nonlinear dynamic equation

$$x^{\Delta^n}(t) + p(t)\bigl(x^\sigma\bigl(g(t)\bigr)\bigr)^\gamma = 0,$$

where γ is the quotient of positive odd integers, and where $g(t) \leq t$. In [5], some comparison criteria have been studied when $g(t) \leq t$, and some oscillation criteria are given when n is even and $g(t) = t$. The results in [5] have been proved when

$$\int_{t_0}^{\infty}\int_{t}^{\infty}\int_{s}^{\infty} p(u)\Delta u\Delta s\Delta t = \infty. \tag{1.5}$$

Wu *et al.* [6] established Kamanev-type oscillation criteria for the higher-order nonlinear dynamic equation

$$\bigl\{r_{n-1}(t)\bigl[\bigl(r_{n-2}(t)\bigl(\cdots\bigl(r_1(t)x^{\Delta}(t)\bigr)^{\Delta}\cdots\bigr)^{\Delta}\bigr)^{\Delta}\bigr]^{\alpha}\bigr\}^{\Delta} + f\bigl(t, x\bigl(g(t)\bigr)\bigr) = 0,$$

where α is the quotient of positive odd integers, $g : \mathbb{T} \to \mathbb{T}$ with $g(t) > t$ and $\lim_{t\to\infty} g(t) = \infty$, and there exists a positive rd-continuous function $p(t)$ such that $\frac{f(t,u)}{u^\alpha} \geq p(t)$ for $u \neq 0$. Sun *et al.* [7] proved some criteria for oscillation and asymptotic behavior of the dynamic equation

$$\bigl\{r_{n-1}(t)\bigl[\bigl(r_{n-2}(t)\bigl(\cdots\bigl(r_1(t)x^{\Delta}(t)\bigr)^{\Delta}\cdots\bigr)^{\Delta}\bigr)^{\Delta}\bigr]^{\alpha}\bigr\}^{\Delta} + f\bigl(t, x\bigl(g(t)\bigr)\bigr) = 0,$$

where $\alpha \geq 1$ is the quotient of positive odd integers, $g : \mathbb{T} \to \mathbb{T}$ is an increasing differentiable function with $g(t) \leq t$, $g \circ \sigma = \sigma \circ g$, and $\lim_{t\to\infty} g(t) = \infty$, and there exists a positive rd-continuous function $p(t)$ such that $\frac{f(t,u)}{u^\beta} \geq p(t)$ for $u \neq 0$ and $\beta \geq 1$ is the quotient of positive odd integers. Sun *et al.* [8] studied quasilinear dynamic equations of the form

$$\left\{r_{n-1}(t)\left[\left(r_{n-2}(t)\left(\cdots\left(r_1(t)x^\Delta(t)\right)^\Delta\cdots\right)^\Delta\right)^\Delta\right]^\alpha\right\}^\Delta + p(t)x^\beta(t) = 0,$$

where α, β are the quotients of positive odd integers. Also, the results obtained in [6–8] are presented when

$$\int_{t_0}^\infty \frac{1}{r_{n-2}(t)}\left\{\int_t^\infty \left[\frac{1}{r_{n-1}(s)}\int_s^\infty p(u)\Delta u\right]^{1/\alpha}\Delta s\right\}\Delta t = \infty. \tag{1.6}$$

Hassan and Kong [9] obtained asymptotics and oscillation criteria for the nth-order half-linear dynamic equation

$$\left(x^{[n-1]}\right)^\Delta(t) + p(t)\phi_{\alpha[1,n-1]}\left(x\left(g(t)\right)\right) = 0,$$

where $\alpha[1,n-1] := \alpha_1 \cdots \alpha_{n-1}$, and Grace and Hassan [10] further studied the asymptotics and oscillation for the higher-order nonlinear dynamic equation

$$\left(x^{[n-1]}\right)^\Delta(t) + p(t)\phi_\gamma\left(x^\sigma\left(g(t)\right)\right) = 0.$$

However, the establishment of the results in [10] requires the restriction on the time scale $\mathbb{T}$ that $g^* \circ \sigma = \sigma \circ g^*$ with $g^*(t) = \min\{t, g(t)\}$, which is hardly satisfied. Hassan [11] improved the results in [9, 10] and established oscillation criteria for the higher-order quasilinear dynamic equation

$$\left(x^{[n-1]}\right)^\Delta(t) + p(t)\phi_\gamma\left(x\left(g(t)\right)\right) = 0$$

when n is even or odd and when $\alpha > \gamma$, $\alpha = \gamma$, and $\alpha < \gamma$ with $\alpha = \alpha_1 \cdots \alpha_{n-1}$. Chen and Qu [1] considered the even-order advanced type dynamic equation with mixed nonlinearities

$$\left\{r(t)\phi_{\gamma_0}\left(x^{\Delta^{n-1}}(t)\right)\right\}^\Delta + \sum_{\nu=0}^N p_\nu(t)\phi_{\gamma_\nu}\left(x\left(g_\nu(t)\right)\right) = 0, \tag{1.7}$$

where $n \geq 2$ is even, $\gamma_\nu > 0$, $g_\nu(t) \geq t$, and $\gamma_1 > \cdots > \gamma_l > \gamma_0 > \gamma_{l+1} > \cdots > \gamma_N > 0$. Zhang *et al.* [2] studied the dynamic equation (1.7), where $n \geq 2$ is integer and $g_\nu^\Delta(t) > 0$, and obtained some of the results in [2] when $\gamma_0 \geq 1$. Also, the results obtained in [1, 2] are given when

$$\int_{t_0}^\infty \left[\int_\nu^\infty \left(r^{-1}(s)\int_s^\infty \sum_{\nu=0}^N p_\nu(\tau)\Delta\tau\right)^{1/\gamma_0}\Delta s\right]\Delta\nu = \infty. \tag{1.8}$$

Huang [12] extended the work in [1] to the neutral advanced dynamic equation

$$\left\{r(t)\phi_\alpha\left(y^{\Delta^{n-1}}(t)\right)\right\}^\Delta + \sum_{\nu=0}^N p_\nu(t)\phi_{\gamma_\nu}\left(x\left(g_\nu(t)\right)\right) = 0,$$

where $n \geq 2$ is integer, $y(t) := x(t) + p(t)x(g(t))$, $\gamma_\nu > 0$, $g(t) \leq t$, and $g_\nu(t) \geq t$. For more results on dynamic equations, we refer the reader to the papers [13–29].

In this paper, we will discuss the higher-order nonlinear dynamic equation (1.1) with mixed nonlinearities on a general time scale without any restrictions on $g(t)$ and $\sigma(t)$ and also without conditions (1.5), (1.6), and (1.8). The results in this paper improve the results in [1, 2, 5–10] on the oscillation of various dynamic equations.

2 Main results

We introduce the following notations:

$$k_+ := \max\{k, 0\}, \qquad k_- := \max\{-k, 0\} \quad \text{for any } k \in \mathbb{R},$$

and

$$\alpha[h,k] := \begin{cases} \alpha_h \cdots \alpha_k, & h \leq k, \\ 1, & h > k, \end{cases} \tag{2.1}$$

with $\alpha = \gamma_0 = \alpha[1, n-1]$ and $\beta_i = \alpha[1, i]$. For any $t, s \in \mathbb{T}$ and for a fixed $m \in \{0, 1, \ldots, n-1\}$, define the functions $R_{m,j}(t,s)$, $j = 0, 1, \ldots, m$, and $\hat{p}_j(t)$, $j = 0, 1, \ldots, n-1$, by the following recurrence formulas:

$$R_{m,j}(t,s) := \begin{cases} 1, & j = 0, \\ \int_s^t [\frac{R_{m,j-1}(\tau,s)}{r_{m-j+1}(\tau)}]^{1/\alpha_{m-j+1}} \Delta\tau, & j = 1, 2, \ldots, m, \end{cases} \tag{2.2}$$

and

$$\hat{p}_j(t) := \begin{cases} \sum_{\nu=0}^{N} p_\nu(t), & j = 0, \\ [\frac{1}{r_{n-j}(t)} \int_t^\infty \hat{p}_{j-1}(\tau)\Delta\tau]^{1/\alpha_{n-j}}, & j = 1, 2, \ldots, n-1. \end{cases}$$

For a fixed $m \in \{0, \ldots, n-1\}$, define the functions $\bar{p}_{m,j}(t,s)$, $j = 0, 1, 2, \ldots, n-1$, by the recurrence formula

$$\bar{p}_{m,j}(t,s) := \begin{cases} p_m(t,s), & j = 0, \\ [\frac{1}{r_{n-j}(t)} \int_t^\infty \bar{p}_{m,j-1}(\tau,s)\Delta\tau]^{1/\alpha_{n-j}}, & j = 1, 2, \ldots, n-1, \end{cases} \tag{2.3}$$

with

$$\varphi_{m,\nu}(t,t_1) := \begin{cases} 1, & g_\nu(t) \geq \sigma(t), \\ \frac{R_{m,m}(g_\nu(t),t_1)}{R_{m,m}(\sigma(t),t_1)}, & g_\nu(t) \leq \sigma(t), \end{cases}$$

and

$$p_m(t,s) = p_0(t)\phi_\alpha\big(\varphi_{m,0}(t,s)\big) + \prod_{\nu=1}^{N}\left[\frac{p_\nu(t)\phi_{\gamma_\nu}(\varphi_{m,\nu}(t,s))}{\eta_\nu}\right]^{\eta_\nu}$$

such that

$$\sum_{\nu=1}^{N} \gamma_\nu \eta_\nu = \alpha \quad \text{and} \quad \sum_{\nu=1}^{N} \eta_\nu = 1, \tag{2.4}$$

where

$$\delta(t,s) := \begin{cases} \left[\int_t^\infty \bar{p}_{m,n-m-1}(\tau,s)\Delta\tau\right]^{1/\beta_m - 1}, & 0 < \beta_m \leq 1, \\ R_{m,m}^{\beta_m - 1}(t,s), & \beta_m \geq 1, \end{cases}$$

provided that the improper integrals involved are convergent.

In the sequel, we present conditions that guarantee the following conclusions:

(C) (i) every solution of equation (1.1) is oscillatory if n is even;
(ii) every solution of equation (1.1) either is oscillatory or tends to zero eventually if n is odd.

Theorem 2.1 *Let conditions* (i)-(v) *hold. Furthermore, for each* $i \in \{1, 2, \ldots, n-1\}$ *and sufficiently large* $T, T_1 \in [t_0, \infty)_{\mathbb{T}}$, *one of the following conditions is satisfied*:

(a) *either* $\int_T^\infty \bar{p}_{i,n-i-1}(\tau, T_1)\Delta\tau = \infty$, *or* $\int_T^\infty \bar{p}_{i,n-i-1}(\tau, T_1)\Delta\tau < \infty$ *and either*

$$\limsup_{t\to\infty} R_{i,i}^{\beta_i}(t, T_1) \int_t^\infty \bar{p}_{i,n-i-1}(\tau, T_1)\Delta\tau > 1$$

or

$$\limsup_{t\to\infty} R_{i,i}(t, T_1)\left(\int_t^\infty \bar{p}_{i,n-i-1}(\tau, T_1)\Delta\tau\right)^{1/\beta_i} > 1;$$

(b) *there exists* $\rho_i \in C_{\mathrm{rd}}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *such that*

$$\limsup_{t\to\infty} \int_T^t \left[\rho_i(\tau)\bar{p}_{i,n-i-1}(\tau, T_1) - \frac{(\rho_i^\Delta(\tau))_+}{R_{i,i}^{\beta_i}(\sigma(\tau), T_1)}\right]\Delta\tau = \infty; \tag{2.5}$$

(c) *there exists* $\rho_i \in C_{\mathrm{rd}}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *such that*

$$\begin{aligned}\limsup_{t\to\infty} \int_T^t \Bigg[&\rho_i(\tau)\bar{p}_{i,n-i-1}(\tau, T_1) \\ &- \frac{1}{\rho_i^{\beta_i}(\tau)}\left[\frac{(\rho_i^\Delta(\tau))_+}{1+\beta_i}\right]^{1+\beta_i}\left[\frac{r_1(\tau)}{R_{i,i-1}(\tau, T_1)}\right]^{\beta_i/\alpha_1}\Bigg]\Delta\tau = \infty;\end{aligned} \tag{2.6}$$

(d) *there exist* $\rho_i \in C_{\mathrm{rd}}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ *and* $H_i, h_i \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$, *where* $\mathbb{D} \equiv \{(t, \tau) : t \geq \tau \geq t_0\}$, *such that*

$$H_i(t,t) = 0, \quad t \geq t_0, \qquad H_i(t, \tau) > 0, \quad t > \tau \geq t_0, \tag{2.7}$$

and H_i *has a nonpositive continuous* Δ*-partial derivative* $H_i^{\Delta_\tau}(t, \tau)$ *with respect to the second variable and satisfies*

$$H_i^{\Delta_\tau}(t, \tau) + H_i(t, \tau)\frac{\rho_i^\Delta(\tau)}{\rho_i^\sigma(\tau)} = -\frac{h_i(t, \tau)}{\rho_i^\sigma(\tau)} H_i^{\beta_i/(1+\beta_i)}(t, \tau) \tag{2.8}$$

and

$$\limsup_{t\to\infty}\frac{1}{H_i(t,T)}\int_T^t\left[\rho_i(\tau)\bar{p}_{i,n-i-1}(\tau,T_1)H_i(t,\tau)\right.$$
$$\left.-\frac{1}{\rho_i^{\beta_i}(\tau)}\left[\frac{(h_i(t,\tau))_-}{1+\beta_i}\right]^{1+\beta_i}\left[\frac{r_1(\tau)}{R_{i,i-1}(\tau,T_1)}\right]^{\beta_i/\alpha_1}\right]\Delta\tau=\infty; \tag{2.9}$$

(e) *there exists* $\rho_i\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t\left[\rho_i(\tau)\bar{p}_{i,n-i-1}(\tau,T_1)\right.$$
$$\left.-\frac{(\rho_i^{\Delta}(\tau))^2}{4\beta_i\rho_i(\tau)\delta^{\sigma}(\tau,T_1)}\left[\frac{r_1(\tau)}{R_{i,i-1}(\tau,T_1)}\right]^{1/\alpha_1}\right]\Delta\tau=\infty; \tag{2.10}$$

(f) *there exist* $\rho_i\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *and* $H_i,h_i\in C_{\mathrm{rd}}(\mathbb{D},\mathbb{R})$, *where* $\mathbb{D}\equiv\{(t,\tau):t\geq\tau\geq t_0\}$, *such that* (2.7) *holds and* H_i *has a nonpositive continuous* Δ*-partial derivative* $H_i^{\Delta_\tau}(t,\tau)$ *with respect to the second variable and satisfies*

$$H_i^{\Delta_\tau}(t,\tau)+H_i(t,\tau)\frac{\rho_i^{\Delta}(\tau)}{\rho_i^{\sigma}(\tau)}=-\frac{h_i(t,\tau)}{\rho_i^{\sigma}(\tau)}\sqrt{H_i(t,\tau)} \tag{2.11}$$

and

$$\limsup_{t\to\infty}\frac{1}{H_i(t,T)}\int_T^t\left[\rho_i(\tau)\bar{p}_{i,n-i-1}(\tau,T_1)H_i(t,\tau)\right.$$
$$\left.-\frac{[(h_i(t,\tau))_-]^2}{4\beta_i\rho_i(\tau)\delta^{\sigma}(\tau,T_1)}\left[\frac{r_1(\tau)}{R_{i,i-1}(\tau,T_1)}\right]^{1/\alpha_1}\right]\Delta\tau=\infty. \tag{2.12}$$

Moreover, for the case where n is odd, assume that, for an integer $j\in\{0,1,\dots,n-1\}$,

$$\int_T^\infty\hat{p}_j(\tau)\Delta\tau=\infty. \tag{2.13}$$

Then conclusions (C) *hold.*

Example 2.1 Consider the higher-order nonlinear dynamic equation (1.1), where $\beta_i=\alpha[1,i]\leq 1$ and $r_1(t):=\frac{t^\xi}{\beta_1}$ with

$$\xi=\begin{cases}>0 & \text{if } n \text{ is even},\\ \leq 0 & \text{if } n \text{ is odd},\end{cases}$$

and where

$$r_i(t):=\frac{t^{\alpha_i}}{\beta_i},\quad i=2,\dots,n-1\quad\text{and}\quad p_0(t):=\frac{\zeta}{t^{\alpha+1}\phi_\alpha(\varphi_{i,0}(t,t_0))}\quad\text{with } \zeta>0.$$

Choose an n-tuple $(\eta_1,\eta_2,\dots,\eta_n)$ with $0<\eta_j<1$ satisfying (2.4). It is clear that conditions (1.2) hold since

$$\int_{t_0}^\infty r_1^{-1/\alpha_1}(\tau)\Delta\tau=\beta_1^{1/\beta_1}\int_{t_0}^\infty\frac{\Delta\tau}{\tau^{\xi/\alpha_1}}=\infty\quad\text{and}\quad\int_{t_0}^\infty r_i^{-1/\alpha_i}(\tau)\Delta\tau=\beta_i^{1/\alpha_i}\int_{t_0}^\infty\frac{\Delta\tau}{\tau}=\infty$$

by [3], Example 5.60. By the Pötzsche chain rule we get

$$\begin{aligned}
\hat{p}_1(t) &= \left[\frac{1}{r_{n-1}(t)}\int_t^\infty \hat{p}_0(\tau)\Delta\tau\right]^{1/\alpha_{n-1}} \\
&\geq \zeta^{1/\alpha_{n-1}}\left[\frac{\beta_{n-1}}{t^{\alpha_{n-1}}}\int_t^\infty \frac{1}{\tau^{\beta_{n-1}+1}}\Delta\tau\right]^{1/\alpha_{n-1}} \\
&\geq \zeta^{1/\alpha_{n-1}}\left[\frac{1}{t^{\alpha_{n-1}}}\int_t^\infty \left(\frac{-1}{\tau^{\beta_{n-1}}}\right)^{\Delta}\Delta\tau\right]^{1/\alpha_{n-1}} \\
&= \frac{\zeta^{1/\alpha_{n-1}}}{t^{\beta_{n-2}+1}} = \frac{\zeta^{1/\alpha[n-1,n-1]}}{t^{\beta_{n-2}+1}}.
\end{aligned}$$

Also, since (1.2) implies $\lim_{t\to\infty}\frac{\varphi_{i,v}(t,T_1)}{\varphi_{i,v}(t,t_0)} = 1$, we obtain

$$\begin{aligned}
\bar{p}_{i,1}(t,T_1) &= \left[\frac{1}{r_{n-1}(t)}\int_t^\infty \bar{p}_{i,0}(\tau,T_1)\Delta\tau\right]^{1/\alpha_{n-1}} \\
&\geq \zeta^{1/\alpha_{n-1}}\left[\frac{\beta_{n-1}}{t^{\alpha_{n-1}}}\int_t^\infty \frac{1}{\tau^{\beta_{n-1}+1}}\Delta\tau\right]^{1/\alpha_{n-1}} \\
&\geq \frac{\zeta^{1/\alpha[n-1,n-1]}}{t^{\beta_{n-2}+1}}.
\end{aligned}$$

It is easy to see that

$$\hat{p}_j(t), \bar{p}_{i,j}(t,T_1) \geq \frac{\zeta^{1/\alpha[n-j,n-1]}}{t^{\beta_{n-j-1}+1}}, \quad j = 0,1,\ldots,n-2.$$

Therefore, we can find $T_* \geq T \geq T_1$ such that $R_{i,i-1}(t,T_1) \geq 1$ for $t \geq T_*$. Let us take $\rho_i(t) = t^{\beta_i}$. Then, by the Pötzsche chain rule,

$$\rho_i^\Delta(t) = \left(t^{\beta_i}\right)^\Delta = \beta_i\int_0^1 \left(t + h\mu(t)\right)^{\beta_i-1}dh \leq \beta_i t^{\beta_i-1}.$$

Hence,

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t \Bigg[\rho_i(\tau)\bar{p}_{i,n-i-1}(\tau,T_1) \\
&\qquad - \frac{1}{\rho_i^{\beta_i}(\tau)}\left[\frac{(\rho_i^\Delta(\tau))_+}{1+\beta_i}\right]^{1+\beta_i}\left[\frac{r_1(\tau)}{R_{i,i-1}(\tau,T_1)}\right]^{\beta_i/\alpha_1}\Bigg]\Delta\tau \\
&\quad\geq \left[\zeta^{1/\alpha[i+1,n-1]} - \left[\frac{1}{\alpha_1}\right]^{\beta_i/\alpha_1}\left[\frac{\beta_i}{1+\beta_i}\right]^{1+\beta_i}\right]\limsup_{t\to\infty}\int_{T^*}^t \frac{1}{\tau}\Delta\tau \\
&\quad= \infty
\end{aligned}$$

if

$$\zeta^{1/\alpha[i+1,n-1]} > \left[\frac{1}{\alpha_1}\right]^{\beta_i/\alpha_1}\left[\frac{\beta_i}{1+\beta_i}\right]^{1+\beta_i},$$

and hence (2.6) holds. Also,

$$\begin{aligned}\hat{p}_{n-1}(t) &= \left[\frac{1}{r_1(t)}\int_t^\infty \hat{p}_{n-2}(\tau)\Delta\tau\right]^{1/\alpha_1} \\ &\geq \zeta^{1/\alpha}\left[\frac{\alpha_1}{t^\xi}\int_t^\infty \frac{1}{\tau^{\alpha_1+1}}\Delta\tau\right]^{1/\alpha_1} \\ &\geq \zeta^{1/\alpha}\left[\frac{1}{t^\xi}\int_t^\infty \left(\frac{-1}{\tau^{\alpha_1}}\right)^\Delta \Delta\tau\right]^{1/\alpha_1} = \frac{\zeta^{1/\alpha}}{t^{1+\xi/\alpha_1}}.\end{aligned}$$

If n is odd, then

$$\int_T^\infty \hat{p}_{n-1}(\tau)\Delta\tau = \zeta^{1/\alpha}\int_T^\infty \frac{\Delta\tau}{\tau^{1+\xi/\alpha_1}} = \infty,$$

so that condition (2.13) holds. Then, by Theorem 2.1(c) conclusions (C) hold if

$$\zeta^{1/\alpha[i+1,n-1]} > \left[\frac{1}{\alpha_1}\right]^{\beta_i/\alpha_1}\left[\frac{\beta_i}{1+\beta_i}\right]^{1+\beta_i}.$$

3 Lemmas

In order to prove the main results, we need the following lemmas. The first two lemmas are extensions of Lemmas 1 and 2 in [9] to the nonlinear equation (1.1) with exactly the same proof.

Lemma 3.1 *Let* $x(t) \in C^n_{\mathrm{rd}}(\mathbb{T},[0,\infty))$. *Assume that* $(x^{[n-1]})^\Delta(t)$ *is of eventually one sign and not identically zero. Then there exists an integer* $m \in \{0,1,\dots,n-1\}$ *with* $m+n$ *odd for* $(x^{[n-1]})^\Delta(t) \leq 0$ *or with* $m+n$ *even for* $(x^{[n-1]})^\Delta(t) \geq 0$ *such that*

$$x^{[k]}(t) > 0 \quad \text{for } k = 0,1,\dots,m \tag{3.1}$$

and

$$(-1)^{m+k}x^{[k]}(t) > 0 \quad \text{for } k = m,m+1,\dots,n-1 \tag{3.2}$$

eventually.

Lemma 3.2 *Assume that equation* (1.1) *has an eventually positive solution* $x(t)$ *and* $m \in \{0,1,\dots,n-1\}$ *is given in Lemma* 3.1 *such that* (3.1) *and* (3.2) *hold for* $t \in [t_1,\infty)_\mathbb{T}$ *for some* $t_1 \in [t_0,\infty)_\mathbb{T}$. *Then the following hold for* $t \in (t_1,\infty)_\mathbb{T}$:

(a) *for* $i = 0,1,\dots,m$,

$$\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)} \quad \text{is strictly decreasing;} \tag{3.3}$$

(b) *for* $i \in \{0,1,\dots,m\}$ *and* $j = 0,1,\dots,m-i$,

$$x^{[j]}(t) \geq \phi^{-1}_{\alpha[j+1,m-i]}\left[\frac{x^{[m-i]}(t)}{R_{m,i}(t,t_1)}\right]R_{m,m-j}(t,t_1). \tag{3.4}$$

Lemma 3.3 *Assume that equation* (1.1) *has an eventually positive solution* $x(t)$ *and* m *is given in Lemma* 3.1 *such that* $m \in \{1,2,\ldots,n-1\}$ *and* (3.1) *and* (3.2) *hold for* $t \geq t_1 \in [t_0,\infty)_{\mathbb{T}}$. *Then, for* $t \in [t_2,\infty)_{\mathbb{T}}$, *where* $g_\nu(t) > t_1$ *for* $t \geq t_2$, *and for* $j = m, m+1, \ldots, n-1$,

$$\int_t^\infty \bar{p}_{m,n-j-1}(\tau,t_1)\Delta\tau < \infty$$

and

$$(-1)^{m+j}x^{[j]}(t) \geq \phi_{\alpha[1,j]}\bigl(x^\sigma(t)\bigr)\int_t^\infty \bar{p}_{m,n-j-1}(\tau,t_1)\Delta\tau. \tag{3.5}$$

Proof We show it by a backward induction. By Lemma 3.1 with $m \geq 1$ we see that $x(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. As a result, (3.1) and (3.2) hold for $t \in [t_1,\infty)_{\mathbb{T}}$. Let $t \in [t_1,\infty)_{\mathbb{T}}$ be fixed. Then, for $\nu = 0,1,\ldots,N$, if $g_\nu(t) \geq \sigma(t)$, then $x(g_\nu(t)) \geq x(t)$ by the fact that $x(t)$ is strictly increasing. Now consider the case where $g_\nu(t) \leq \sigma(t)$. In view of Lemma 3.2(a), we see that for $i = m$, $\frac{x(t)}{R_{m,m}(t,t_1)}$ is decreasing on $(t_1,\infty)_{\mathbb{T}}$ and that there exists $t_2 \geq t_1$ such that $g_\nu(t) > t_1$ for $t \geq t_2$, so that

$$x\bigl(g_\nu(t)\bigr) \geq \frac{R_{m,m}(g_\nu(t),t_1)}{R_{m,m}(\sigma(t),t_1)}x^\sigma(t) \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

In both cases, we have

$$x\bigl(g_\nu(t)\bigr) \geq \varphi_{m,\nu}(t,t_1)x^\sigma(t) \quad \text{for } t \in [t_2,\infty)_{\mathbb{T}}.$$

Therefore,

$$\begin{aligned}\sum_{\nu=0}^N p_\nu(t)\phi_{\gamma_\nu}\bigl(x\bigl(g_\nu(t)\bigr)\bigr) &\geq \sum_{\nu=0}^N p_\nu(t)\phi_{\gamma_\nu}\bigl(\varphi_{m,\nu}(t,t_1)\bigr)\bigl[x^\sigma(t)\bigr]^{\gamma_\nu} \\ &= \phi_\alpha\bigl(x^\sigma(t)\bigr)\sum_{\nu=0}^N p_\nu(t)\phi_{\gamma_\nu}\bigl(\varphi_{m,\nu}(t,t_1)\bigr)\bigl[x^\sigma(t)\bigr]^{\gamma_\nu-\alpha}.\end{aligned}$$

Using the arithmetic-geometric mean inequality (see [30], p.17), we have

$$\sum_{\nu=1}^N \eta_\nu v_\nu \geq \prod_{\nu=1}^N v_\nu^{\eta_\nu} \quad \text{for any } v_\nu \geq 0, \nu = 1,\ldots,N.$$

Then, for $t \geq T_1$,

$$\begin{aligned}&\sum_{\nu=0}^N p_\nu(t)\phi_{\gamma_\nu}\bigl(\varphi_{m,\nu}(t,t_1)\bigr)\bigl[x^\sigma(t)\bigr]^{\gamma_\nu-\alpha} \\ &\quad = p_0(t)\phi_\alpha\bigl(\varphi_{m,0}(t,t_1)\bigr) + \sum_{\nu=1}^N \eta_\nu \frac{p_\nu(t)\phi_{\gamma_\nu}(\varphi_{m,\nu}(t,t_1))}{\eta_\nu}\bigl[x^\sigma(t)\bigr]^{\gamma_\nu-\alpha} \\ &\quad \geq p_0(t)\phi_\alpha\bigl(\varphi_{m,0}(t,t_1)\bigr) + \prod_{\nu=1}^N \biggl[\frac{p_\nu(t)\phi_{\gamma_\nu}(\varphi_{m,\nu}(t,t_1))}{\eta_\nu}\biggr]^{\eta_\nu}\bigl[x^\sigma(t)\bigr]^{\eta_\nu(\gamma_\nu-\alpha)}.\end{aligned}$$

In view of (2.4), we have

$$\sum_{\nu=1}^{N}\gamma_\nu\eta_\nu-\alpha\sum_{\nu=1}^{N}\eta_\nu=0.$$

Hence,

$$\begin{aligned}&\sum_{\nu=0}^{N}p_\nu(t)\phi_{\gamma_\nu}\big(\varphi_{m,\nu}(t,t_1)\big)\big[x^\sigma(t)\big]^{\gamma_\nu-\alpha}\\&\quad\geq p_0(t)\phi_\alpha\big(\varphi_{m,0}(t,t_1)\big)+\prod_{\nu=1}^{N}\left[\frac{p_\nu(t)\phi_{\gamma_\nu}(\varphi_{m,\nu}(t,t_1))}{\eta_\nu}\right]^{\eta_\nu}=p(t,t_1).\end{aligned}$$

This, together with (1.1), shows that, for $t\in[t_2,\infty)_{\mathbb{T}}$,

$$-\big(x^{[n-1]}(t)\big)^\Delta\geq p(t,t_1)\phi_\alpha\big(x^\sigma(t)\big)=\bar{p}_{m,0}(t,t_1)\phi_\alpha\big(x^\sigma(t)\big). \tag{3.6}$$

Replacing t by τ in (3.6), integrating from $t\in[t_2,\infty)_{\mathbb{T}}$ to $\nu\in[t,\infty)_{\mathbb{T}}$, and using (3.2), we have

$$\begin{aligned}x^{[n-1]}(t)&>-x^{[n-1]}(\nu)+x^{[n-1]}(t)\geq\int_t^\nu\bar{p}_{m,0}(\tau,t_1)\phi_\alpha\big(x^\sigma(\tau)\big)\Delta\tau\\&\geq\phi_\alpha\big(x^\sigma(t)\big)\int_t^\nu\bar{p}_{m,0}(\tau,t_1)\Delta\tau.\end{aligned}$$

Hence, by taking limits as $\nu\to\infty$ we obtain that

$$x^{[n-1]}(t)\geq\phi_\alpha\big(x^\sigma(t)\big)\int_t^\infty\bar{p}_{m,0}(\tau,t_1)\Delta\tau.$$

This shows that $\int_t^\infty\bar{p}_{m,0}(\tau,t_1)\Delta\tau<\infty$ and (3.5) holds for $j=n-1$. Assume that $\int_t^\infty\bar{p}_{m,n-j-1}(\tau,t_1)\Delta\tau<\infty$ and (3.5) holds for some $j\in\{m+1,m+2,\ldots,n-1\}$. Then, for (3.5),

$$\begin{aligned}(-1)^{m+j}\big[x^{[j-1]}(t)\big]^\Delta&=(-1)^{m+j}\phi_{\alpha_j}^{-1}\left[\frac{x^{[j]}(t)}{r_j(t)}\right]\\&\geq\phi_{\alpha_j}^{-1}\big\{\phi_{\alpha[1,j]}\big(x^\sigma(t)\big)\big\}\left[\frac{1}{r_j(t)}\int_t^\infty\bar{p}_{m,n-j-1}(\tau,t_1)\Delta\tau\right]^{1/\alpha_j}\\&=\phi_{\alpha[1,j-1]}\big(x^\sigma(t)\big)\bar{p}_{m,n-j}(t,t_1).\end{aligned}$$

Replacing t by τ and then integrating it from $t\in[t_2,\infty)_{\mathbb{T}}$ to $\nu\in[t,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}(-1)^{m+j-1}x^{[j-1]}(t)&>(-1)^{m+j}\big(x^{[j-1]}(\nu)-x^{[j-1]}(t)\big)\\&\geq\int_t^\nu\phi_{\alpha[1,j-1]}\big(x^\sigma(\tau)\big)\bar{p}_{m,n-j}(\tau,t_1)\Delta\tau\\&\geq\phi_{\alpha[1,j-1]}\big(x^\sigma(t)\big)\int_t^\nu\bar{p}_{m,n-j}(\tau,t_1)\Delta\tau.\end{aligned}$$

Taking limits as $\nu \to \infty$, we obtain that

$$(-1)^{m+j-1}x^{[j-1]}(t) \geq \phi_{\alpha[1,j-1]}\bigl(x^{\sigma}(t)\bigr)\int_t^{\infty}\bar{p}_{m,n-j}(\tau,t_1)\Delta\tau.$$

This shows that $\int_t^{\infty}\bar{p}_{m,n-j}(\tau,t_1)\Delta\tau < \infty$ and (3.5) holds for $j-1$. Therefore, the conclusion holds. □

The following lemma improves [31], Lemma 1; also see [32–34].

Lemma 3.4 *Let* (1.3) *hold. Then, there exists an N-tuple* $(\eta_1,\eta_2,\ldots,\eta_N)$ *with* $\eta_\nu > 0$ *satisfying* (2.4).

Lemma 3.5 (see [35]) *Let* $\omega(u) = au - bu^{1+1/\beta}$, *where* $a,u \geq 0$ *and* $b,\beta > 0$. *Then*

$$\omega(u) \leq \left(\frac{\beta}{b}\right)^{\beta}\left(\frac{a}{1+\beta}\right)^{1+\beta}.$$

4 Proofs of main results

Proof of Theorem 2.1 Assume that equation (1.1) has a nonoscillatory solution $x(t)$. Then, without loss of generality, assume that $x(t) > 0$ and $x(g_\nu(t)) > 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. It follows from Lemma 3.1 that there exists an integer $m \in \{0,1,\ldots,n-1\}$ with $m+n$ odd such that (3.1) and (3.2) hold for $t \in [t_1,\infty)_{\mathbb{T}}$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Let $t_2 \geq t_1$ be such that $g_\nu(t) > t_1$ for $t \in [t_2,\infty)_{\mathbb{T}}$.

(i) Assume that $m \geq 1$.

Part I: Assume that (a) holds. By Lemma 3.3 we have that, for $j = m$,

$$\int_t^{\infty}\bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau < \infty,$$

which contradicts $\int_t^{\infty}\bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau = \infty$. If $\int_t^{\infty}\bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau < \infty$, then by Lemma 3.3 we have that, for $j = m$,

$$\begin{aligned} x^{[m]}(t) &\geq \phi_{\alpha[1,m]}\bigl(x^{\sigma}(t)\bigr)\int_t^{\infty}\bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau \\ &\geq \phi_{\beta_m}\bigl(x(t)\bigr)\int_t^{\infty}\bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau. \end{aligned} \tag{4.1}$$

By Lemma 3.2(b) with $i = 0$ and $j = 0$ we get

$$\begin{aligned} x(t) &\geq \phi^{-1}_{\alpha[1,m]}\bigl(x^{[m]}(t)\bigr)R_{m,m}(t,t_1) \\ &= \phi^{-1}_{\beta_m}\bigl(x^{[m]}(t)\bigr)R_{m,m}(t,t_1). \end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1), we obtain that

$$1 \geq R^{\beta_m}_{m,m}(t,t_1)\int_t^{\infty}\bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau,$$

which contradicts $\limsup_{t\to\infty} R^{\beta_m}_{m,m}(t,t_1)\int_t^\infty \bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau > 1$. Substituting (4.1) into (4.2), we obtain that

$$1 \geq R_{m,m}(t,t_1)\left(\int_t^\infty \bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau\right)^{1/\beta_m},$$

which contradicts $\limsup_{t\to\infty} R_{m,m}(t,t_1)(\int_t^\infty \bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau)^{1/\beta_m} > 1$.

Part II: Assume that (b) holds. Define

$$w_m(t) := \rho_m(t)\frac{x^{[m]}(t)}{x^{\beta_m}(t)}. \tag{4.3}$$

By the product rule and the quotient rule we have

$$\begin{aligned}
w_m^\Delta(t) &= \rho_m(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^\Delta + \rho_m^\Delta(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^\sigma \\
&= \rho_m(t)\left(\frac{x^{\beta_m}(t)(x^{[m]}(t))^\Delta - (x^{\beta_m}(t))^\Delta x^{[m]}(t)}{(x^{\beta_m}(t))^\sigma x^{\beta_m}(t)}\right) + \rho_m^\Delta(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^\sigma \\
&= \rho_m(t)\frac{(x^{[m]}(t))^\Delta}{(x^{\beta_m}(t))^\sigma} - \rho_m(t)\frac{(x^{\beta_m}(t))^\Delta}{(x^{\beta_m}(t))^\sigma}\frac{x^{[m]}(t)}{x^{\beta_m}(t)} + \rho_m^\Delta(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^\sigma.
\end{aligned} \tag{4.4}$$

From Lemma 3.3 with $j = m+1$ we have

$$-x^{[m+1]}(t) \geq \phi_{\alpha[1,m+1]}\big(x^\sigma(t)\big)\int_t^\infty \bar{p}_{m,n-m-2}(\tau,t_1)\,\Delta\tau, \tag{4.5}$$

which, together with (2.3), implies that, for $t \in [t_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
-\big(x^{[m]}(t)\big)^\Delta &\geq \phi_{\alpha[1,m]}\big(x^\sigma(t)\big)\left[\frac{1}{r_{m+1}(t)}\int_t^\infty \bar{p}_{m,n-m-2}(\tau,t_1)\,\Delta\tau\right]^{1/\alpha_{m+1}} \\
&= \phi_{\beta_m}\big(x^\sigma(t)\big)\,\bar{p}_{m,n-m-1}(t,t_1).
\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.4), we obtain

$$w_m^\Delta(t) \leq -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^\Delta(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^\sigma - \rho_m(t)\frac{(x^{\beta_m}(t))^\Delta}{(x^{\beta_m}(t))^\sigma}\frac{x^{[m]}(t)}{x^{\beta_m}(t)}.$$

When $0 < \beta_m \leq 1$, since $x(t)$ is strictly increasing, by Pötzsche chain rule ([3], Thm. 1.90) we obtain

$$\begin{aligned}
\big(x^{\beta_m}(t)\big)^\Delta &= \beta_m\int_0^1 \big[x(t) + h\,\mu(t)x^\Delta(t)\big]^{\beta_m-1}\,dh\,x^\Delta(t) \\
&= \beta_m\int_0^1 \big[(1-h)x(t) + h\,x^\sigma(t)\big]^{\beta_m-1}\,dh\,x^\Delta(t) \\
&\geq \beta_m\,\big[x^\sigma(t)\big]^{\beta_m-1}x^\Delta(t).
\end{aligned}$$

Hence,

$$\begin{aligned} w_m^{\Delta}(t) &\le -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma} - \beta_m\rho_m(t)\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma} \\ &\le -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma}. \end{aligned} \tag{4.7}$$

When $\beta_m \ge 1$, since $x(t)$ is strictly increasing, again by Pötzsche chain rule we obtain

$$\begin{aligned} \left(x^{\beta_m}(t)\right)^{\Delta} &= \beta_m \int_0^1 \left[x(t) + h\,\mu(t)x^{\Delta}(t)\right]^{\beta_m-1} dh\, x^{\Delta}(t) \\ &= \beta_m \int_0^1 \left[(1-h)x(t) + h\,x^{\sigma}(t)\right]^{\beta_m-1} dh\, x^{\Delta}(t) \\ &\ge \beta_m \left[x(t)\right]^{\beta_m-1} x^{\Delta}(t). \end{aligned}$$

Therefore,

$$\begin{aligned} w_m^{\Delta}(t) &\le -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma} - \beta_m\rho_m(t)\frac{x^{\Delta}(t)}{x(t)}\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma} \\ &\le -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma}. \end{aligned} \tag{4.8}$$

Then, for $\beta_m > 0$,

$$w_m^{\Delta}(t) \le -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma}. \tag{4.9}$$

By using Lemma 3.2 (b) with $i = 0$ and $j = 0$ we see that

$$x(t) \ge \phi_{\alpha[1,m]}^{-1}\left(x^{[m]}(t)\right)R_{m,m}(t,t_1),$$

which implies

$$\frac{x^{[m]}(t)}{x^{\beta_m}(t)} \le \frac{1}{R_{m,m}^{\beta_m}(t,t_1)}. \tag{4.10}$$

Substituting (4.10) into (4.9), we get

$$\begin{aligned} w_m^{\Delta}(t) &\le -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \frac{\rho_m^{\Delta}(t)}{R_{m,m}^{\beta_m}(\sigma(t),t_1)} \\ &\le -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \frac{(\rho_m^{\Delta}(t))_+}{R_{m,m}^{\beta_m}(\sigma(t),t_1)} \qquad \text{for } t \in [t_2,\infty)_{\mathbb{T}}. \end{aligned}$$

Integrating both sides from t_2 to t we get

$$\int_{t_2}^{t}\left[\rho_m(\tau)\,\bar{p}_{m,n-m-1}(\tau,t_1) - \frac{(\rho_m^{\Delta}(\tau))_+}{R_{m,m}^{\beta_m}(\sigma(\tau),t_1)}\right]\Delta\tau \le w_m(t_2) - w_m(t) \le w_m(t_2),$$

which contradicts (2.5).

Part III: Assume that (c) holds. When $0 < \beta_m \leq 1$, by the definition of $w_m(t)$, since $x(t)$ is strictly increasing, (4.7) can be written as

$$w_m^{\Delta}(t) \leq -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma} - \beta_m\rho_m(t)\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}. \tag{4.11}$$

By using Lemma 3.2 (b) with $i = 0$ and $j = 1$ we see that

$$x^{[1]}(t) \geq \phi_{\alpha[2,m]}^{-1}\big(x^{[m]}(t)\big)R_{m,m-1}(t,t_1), \tag{4.12}$$

which implies

$$\begin{aligned} \frac{x^{\Delta}(t)}{x^{\sigma}(t)} &\geq \frac{\phi_{\alpha[1,m]}^{-1}(x^{[m]}(t))}{x^{\sigma}(t)}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} \\ &\geq \frac{\phi_{\alpha[1,m]}^{-1}(x^{[m]}(t))}{x^{\sigma}(t)}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} \\ &\geq \left[\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma}\right]^{1/\beta_m}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} \\ &= \left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}\right]^{1/\beta_m}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}. \end{aligned} \tag{4.13}$$

Substituting (4.13) into (4.11), we get, for $0 < \beta_m \leq 1$,

$$\begin{aligned} w_m^{\Delta}(t) &\leq -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma} \\ &\quad - \beta_m\rho_m(t)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}\right]^{1+1/\beta_m}. \end{aligned}$$

When $\beta_m \geq 1$, by the definition of $w_m(t)$, (4.8) can be written as

$$w_m^{\Delta}(t) \leq -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma} - \beta_m\rho_m(t)\frac{x^{\Delta}(t)}{x(t)}\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}. \tag{4.14}$$

By using Lemma 3.2 (b) with $i = 0$ and $j = 1$ we see that

$$x^{[1]}(t) \geq \phi_{\alpha[2,m]}^{-1}\big(x^{[m]}(t)\big)R_{m,m-1}(t,t_1),$$

which implies

$$\begin{aligned} \frac{x^{\Delta}(t)}{x(t)} &= \frac{x^{\Delta}(t)}{x(t)} \geq \frac{\phi_{\alpha[1,m]}^{-1}(x^{[m]}(t))}{x(t)}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} \\ &\geq \frac{\phi_{\alpha[1,m]}^{-1}(x^{[m]}(t))}{x(t)}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} \\ &= \left[\left(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\right)^{\sigma}\right]^{1/\beta_m}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1} \\ &= \left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}\right]^{1/\beta_m}\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}. \end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.14), we get, for $\beta_m \geq 1$,

$$w_m^{\Delta}(t) \leq -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma} - \beta_m \rho_m(t)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}\right]^{1+1/\beta_m}.$$

Hence, for $\beta_m > 0$ and $t \in [t_2,\infty)_{\mathbb{T}}$,

$$\begin{aligned} w_m^{\Delta}(t) &\leq -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma} \\ &\quad - \beta_m \rho_m(t)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}\right]^{1+1/\beta_m} \qquad (4.16)\\ &\leq -\rho_m(t)\,\bar{p}_{m,n-m-1}(t,t_1) + \left(\rho_m^{\Delta}(t)\right)_+\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma} \\ &\quad - \beta_m \rho_m(t)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}\right]^{1+1/\beta_m}. \qquad (4.17) \end{aligned}$$

Using Lemma 3.5 with

$$a := \left(\rho_m^{\Delta}(t)\right)_+, \qquad b := \beta_m \rho_m(t)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}, \qquad \beta := \beta_m \quad \text{and} \quad u := \left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma},$$

we obtain

$$\begin{aligned} &\left(\rho_m^{\Delta}(t)\right)_+\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma} - \beta_m \rho_m(t)\left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma}\right]^{1+1/\beta_m} \\ &\leq \left(\frac{\beta_m}{\beta_m \rho_m(t)}\left[\frac{r_1(t)}{R_{m,m-1}(t,t_1)}\right]^{1/\alpha_1}\right)^{\beta_m}\left[\frac{(\rho_m^{\Delta}(t))_+}{1+\beta_m}\right]^{1+\beta_m} \\ &= \frac{1}{\rho_m^{\beta_m}(t)}\left[\frac{(\rho_m^{\Delta}(t))_+}{1+\beta_m}\right]^{1+\beta_m}\left[\frac{r_1(t)}{R_{m,m-1}(t,t_1)}\right]^{\beta_m/\alpha_1}. \end{aligned}$$

From this and from (4.17) we have

$$w_m^{\Delta}(t) \leq -\rho_m(t)\bar{p}_{m,n-m-1}(t,t_1) + \frac{1}{\rho_m^{\beta_m}(t)}\left[\frac{(\rho_m^{\Delta}(t))_+}{1+\beta_m}\right]^{1+\beta_m}\left[\frac{r_1(t)}{R_{m,m-1}(t,t_1)}\right]^{\beta_m/\alpha_1}.$$

Integrating both sides from t_2 to t, we get

$$\begin{aligned} &\int_{t_2}^{t}\Bigg[\rho_m(\tau)\bar{p}_{m,n-m-1}(\tau,t_1) \\ &\quad - \frac{1}{\rho_m^{\beta_m}(\tau)}\left[\frac{(\rho_m^{\Delta}(\tau))_+}{1+\beta_m}\right]^{1+\beta_m}\left[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\right]^{\beta_m/\alpha_1}\Bigg]\Delta\tau \leq w_m(t_2) - w_m(t) \leq w_m(t_2), \end{aligned}$$

which contradicts (2.6).

Part IV: Assume that (d) holds. Multiplying both sides of (4.16), with t replaced by τ, by $H_m(t,\tau)$ and integrating with respect to τ from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_{t_2}^{t} \rho_m(\tau)\bar{p}_{m,n-m-1}(\tau,t_1)H_m(t,\tau)\Delta\tau \\
&\quad \le -\int_{t_2}^{t} H_m(t,\tau)w_m^{\Delta}(\tau)\Delta\tau \\
&\qquad + \int_{t_2}^{t} H_m(t,\tau)\rho_m^{\Delta}(\tau)\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma}\Delta\tau \\
&\qquad - \beta_m\int_{t_2}^{t} \rho_m(\tau)H_m(t,\tau)\left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma}\right]^{1+1/\beta_m}\Delta\tau.
\end{aligned}$$

Integrating by parts and using (2.7) and (2.8), we obtain

$$\begin{aligned}
&\int_{t_2}^{t} \rho_m(\tau)\bar{p}_{m,n-m-1}(\tau,t_1)H_m(t,\tau)\Delta\tau \\
&\quad \le H_m(t,t_2)w_m(t_2) + \int_{t_2}^{t} H_m^{\Delta_\tau}(t,\tau)w_m^{\sigma}(\tau)\Delta\tau \\
&\qquad + \int_{t_2}^{t} H_m(t,\tau)\rho_m^{\Delta}(\tau)\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma}\Delta\tau \\
&\qquad - \beta_m\int_{t_2}^{t} \rho_m(\tau)H_m(t,\tau)\left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma}\right]^{1+1/\beta_m}\Delta\tau \\
&\quad \le H_m(t,t_2)w(t_2) + \int_{t_2}^{t}\Bigg[\big(h_m(t,\tau)\big)_-\big(H_m(t,\tau)\big)^{\frac{\beta_m}{1+\beta_m}}\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma} \\
&\qquad - \beta_m\rho_m(\tau)H_m(t,\tau)\left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma}\right]^{1+1/\beta_m}\Bigg]\Delta\tau. \qquad (4.18)
\end{aligned}$$

Using Lemma 3.5 with

$$a := \big(h_m(t,\tau)\big)_-\big(H_m(t,\tau)\big)^{\frac{\beta_m}{1+\beta_m}}, \qquad b := \beta_m\rho_m(\tau)H_m(t,\tau)\left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}\right]^{1/\alpha_1},$$

and

$$\beta := \beta_m, \qquad u := \left(\frac{w_m(t)}{\rho_m(t)}\right)^{\sigma},$$

we get

$$\begin{aligned}
&\big(h_m(t,\tau)\big)_-\big(H_m(t,\tau)\big)^{\frac{\beta_m}{1+\beta_m}}\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma} \\
&\qquad - \beta_m\rho_m(\tau)H_m(t,\tau)\left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^{\sigma}\right]^{1+1/\beta_m} \\
&\quad \le \frac{1}{(1+\beta_m)^{1+\beta_m}}\frac{[(h_m(t,\tau))_-]^{1+\beta_m}}{\rho_m^{\beta_m}(\tau)}\left[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\right]^{\beta_m/\alpha_1} \\
&\quad = \frac{1}{\rho_m^{\beta_m}(\tau)}\left[\frac{(h_m(t,\tau))_-}{1+\beta_m}\right]^{1+\beta_m}\left[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\right]^{\beta_m/\alpha_1}.
\end{aligned}$$

From this last inequality and from (4.18) we have

$$\int_{t_2}^{t}\Bigg[\rho_m(\tau)\bar{p}_{m,n-m-1}(\tau,t_1)H_m(t,\tau) - \frac{1}{\rho_m^{\beta_m}(\tau)}\bigg[\frac{(h_m(t,\tau))_-}{1+\beta_m}\bigg]^{1+\beta_m}\bigg[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\bigg]^{\beta_m/\alpha_1}\Bigg]\Delta\tau \le H_m(t,t_2)w_m(t_2),$$

which implies that

$$\frac{1}{H_m(t,t_2)}\int_{t_2}^{t}\Bigg[\rho_m(\tau)\bar{p}_{m,n-m-1}(\tau,t_1)H_m(t,\tau) - \frac{1}{\rho_m^{\beta_m}(\tau)}\bigg[\frac{(h_m(t,\tau))_-}{1+\beta_m}\bigg]^{1+\beta_m}\bigg[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\bigg]^{\beta_m/\alpha_1}\Bigg]\Delta\tau \le w_m(t_2),$$

contradicting assumption (2.9).

Part V: Assume that (e) holds. From (4.16) we have

$$\begin{aligned} w_m^{\Delta}(t) &\le -\rho_m(t)\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma} \\ &\quad - \beta_m\rho_m(t)\bigg[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\bigg]^{1/\alpha_1}\bigg[\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma}\bigg]^{1+1/\beta_m} \\ &\le -\rho_m(t)\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma} \\ &\quad - \beta_m\rho_m(t)\bigg[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\bigg]^{1/\alpha_1}\bigg[\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma}\bigg]^{1/\beta_m-1}\bigg[\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma}\bigg]^{2}. \end{aligned} \tag{4.19}$$

When $0 < \beta_m \le 1$, in view of the definition of w and (4.1), we get

$$\bigg[\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma}\bigg]^{1/\beta_m-1} = \bigg[\bigg(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\bigg)^{\sigma}\bigg]^{1/\beta_m-1} \ge \bigg[\int_{\sigma(t)}^{\infty}\bar{p}_{m,n-m-1}(\tau,t_1)\Delta\tau\bigg]^{1/\beta_m-1}. \tag{4.20}$$

When $\beta_m \ge 1$, in view of the definition of w and (4.2), we get

$$\bigg[\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma}\bigg]^{1/\beta_m-1} = \bigg[\bigg(\frac{x^{[m]}(t)}{x^{\beta_m}(t)}\bigg)^{\sigma}\bigg]^{1/\beta_m-1} \ge \big[R_{m,m}^{\sigma}(t,t_1)\big]^{\beta_m-1}. \tag{4.21}$$

Thus, by (4.20), (4.21), and the definition of $\delta(t,t_1)$, (4.19) becomes

$$\begin{aligned} w_m^{\Delta}(t) &\le -\rho_m(t)\bar{p}_{m,n-m-1}(t,t_1) + \rho_m^{\Delta}(t)\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma} \\ &\quad - \beta_m\rho_m(t)\delta^{\sigma}(t,t_1)\bigg[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\bigg]^{1/\alpha_1}\bigg[\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma}\bigg]^{2}. \end{aligned} \tag{4.22}$$

Now,

$$\begin{aligned} &\rho_m^{\Delta}(t)\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma} - \beta_m\rho_m(t)\delta^{\sigma}(t,t_1)\bigg[\frac{R_{m,m-1}(t,t_1)}{r_1(t)}\bigg]^{1/\alpha_1}\bigg[\bigg(\frac{w_m(t)}{\rho_m(t)}\bigg)^{\sigma}\bigg]^{2} \\ &\quad = \frac{(\rho_m^{\Delta}(t))^2}{4\beta_m\rho_m(t)\delta^{\sigma}(t,t_1)}\bigg[\frac{r_1(t)}{R_{m,m-1}(t,t_1)}\bigg]^{1/\alpha_1} \end{aligned}$$

$$
\begin{aligned}
&- \left[\sqrt{\beta_m \rho_m(t) \delta^\sigma(t,t_1) \left[\frac{R_{m,m-1}(t,t_1)}{r_1(t)} \right]^{1/\alpha_1}} \left[\left(\frac{w_m(t)}{\rho_m(t)} \right)^\sigma \right] \right. \\
&\left. - \frac{\rho_m^\Delta(t)}{2\sqrt{\beta_m \rho_m(t) \delta^\sigma(t,t_1) [\frac{R_{m,m-1}(t,t_1)}{r_1(t)}]^{1/\alpha_1}}} \right]^2 \\
\le{} & \frac{(\rho_m^\Delta(t))^2}{4\beta_m \rho_m(t) \delta^\sigma(t,t_1)} \left[\frac{r_1(t)}{R_{m,m-1}(t,t_1)} \right]^{1/\alpha_1}.
\end{aligned}
$$

Therefore,

$$
w_m^\Delta(t) \le -\rho_m(t) \bar{p}_{m,n-m-1}(t,t_1) + \frac{(\rho_m^\Delta(t))^2}{4\beta_m \rho_m(t) \delta^\sigma(t,t_1)} \left[\frac{r_1(t)}{R_{m,m-1}(t,t_1)} \right]^{1/\alpha_1}.
$$

Integrating both sides from t_2 to t, we get

$$
\begin{aligned}
&\int_{t_2}^t \left[\rho_m(\tau) \bar{p}_{m,n-m-1}(\tau,t_1) \right. \\
&\left. - \frac{(\rho_m^\Delta(\tau))^2}{4\beta_m \rho_m(\tau) \delta^\sigma(\tau,t_1)} \left[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)} \right]^{1/\alpha_1} \right] \Delta\tau \le w_m(t_2) - w_m(t) \le w_m(t_2),
\end{aligned}
$$

which contradicts (2.10).

Part VI: Assume that (f) holds. Multiplying both sides of (4.22), with t replaced by τ, by $H_m(t,\tau)$ and integrating with respect to τ from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$, we have

$$
\begin{aligned}
&\int_{t_2}^t \rho_m(\tau) \bar{p}_{m,n-m-1}(\tau,t_1) H_m(t,\tau) \Delta\tau \\
&\le -\int_{t_2}^t H_m(t,\tau) w_m^\Delta(\tau) \Delta\tau + \int_{t_2}^t H_m(t,\tau) \rho_m^\Delta(\tau) \left(\frac{w_m(\tau)}{\rho_m(\tau)} \right)^\sigma \Delta\tau \\
&\quad - \beta_m \int_{t_2}^t \rho_m(\tau) H_m(t,\tau) \delta^\sigma(\tau,t_1) \left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)} \right]^{1/\alpha_1} \left[\left(\frac{w_m(\tau)}{\rho_m(\tau)} \right)^\sigma \right]^2 \Delta\tau.
\end{aligned}
$$

Integrating by parts and using (2.7) and (2.11), we obtain

$$
\begin{aligned}
&\int_{t_2}^t \rho_m(\tau) \bar{p}_{m,n-m-1}(\tau,t_1) H_m(t,\tau) \Delta\tau \\
&\le H_m(t,t_2) w_m(t_2) + \int_{t_2}^t H_m^{\Delta_\tau}(t,\tau) w_m^\sigma(\tau) \Delta\tau + \int_{t_2}^t H_m(t,\tau) \rho_m^\Delta(\tau) \left(\frac{w_m(\tau)}{\rho_m(\tau)} \right)^\sigma \Delta\tau \\
&\quad - \beta_m \int_{t_2}^t \rho_m(\tau) H_m(t,\tau) \delta^\sigma(\tau,t_1) \left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)} \right]^{1/\alpha_1} \left[\left(\frac{w_m(\tau)}{\rho_m(\tau)} \right)^\sigma \right]^2 \Delta\tau \\
&\le H_m(t,t_2) w(t_2) \\
&\quad - \int_{t_2}^t \left[\beta_m \rho_m(\tau) H_m(t,\tau) \delta^\sigma(\tau,t_1) \left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)} \right]^{1/\alpha_1} \left[\left(\frac{w_m(\tau)}{\rho_m(\tau)} \right)^\sigma \right]^2 \right. \\
&\quad \left. - \big(h_m(t,\tau) \big)_- \sqrt{H_m(t,\tau)} \left(\frac{w_m(\tau)}{\rho_m(\tau)} \right)^\sigma \right] \Delta\tau.
\end{aligned}
$$

Now,

$$\begin{aligned}
&\beta_m\rho_m(\tau)H_m(t,\tau)\delta^\sigma(\tau,t_1)\left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}\right]^{1/\alpha_1}\left[\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^\sigma\right]^2\\
&\quad-\left(h_m(t,\tau)\right)_-\sqrt{H_m(t,\tau)}\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^\sigma\\
&=\left[\sqrt{\beta_m\rho_m(\tau)H_m(t,\tau)\delta^\sigma(\tau,t_1)\left[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}\right]^{1/\alpha_1}}\left(\frac{w_m(\tau)}{\rho_m(\tau)}\right)^\sigma\right.\\
&\quad\left.-\frac{(h_m(t,\tau))_-}{2\sqrt{\beta_m\rho_m(\tau)\delta^\sigma(\tau,t_1)[\frac{R_{m,m-1}(\tau,t_1)}{r_1(\tau)}]^{1/\alpha_1}}}\right]^2\\
&\quad-\frac{[(h_m(t,\tau))_-]^2}{4\beta_m\rho_m(\tau)\delta^\sigma(\tau,t_1)}\left[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\right]^{1/\alpha_1}\\
&\geq-\frac{[(h_m(t,\tau))_-]^2}{4\beta_m\rho_m(\tau)\delta^\sigma(\tau,t_1)}\left[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\right]^{1/\alpha_1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{1}{H_m(t,t_2)}\int_{t_2}^t\left[\rho_m(\tau)\bar{p}_{m,n-m-1}(\tau,t_1)H_m(t,\tau)\right.\\
&\quad\left.-\frac{[(h_m(t,\tau))_-]^2}{4\beta_m\rho_m(\tau)\delta^\sigma(\tau,t_1)}\left[\frac{r_1(\tau)}{R_{m,m-1}(\tau,t_1)}\right]^{1/\alpha_1}\right]\Delta\tau\leq w_m(t_2),
\end{aligned}$$

which contradicts assumption (2.12).

(ii) We show that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. In fact, from Lemma 3.1 we see that it is only possible when n is odd. In this case,

$$\begin{aligned}
&(-1)^k x^{[k]}(t) > 0 \quad \text{and}\\
&\left((-1)^k x^{[k]}(t)\right)^\Delta < 0 \quad \text{for } t\in[t_1,\infty)_{\mathbb{T}} \text{ and } k = 0,1,\ldots,n-1.
\end{aligned}\tag{4.23}$$

Hence,

$$\lim_{t\to\infty}(-1)^k x^{[k]}(t) = l_k \geq 0 \quad \text{for } k = 0,1,\ldots,n-1.$$

We claim that $\lim_{t\to\infty} x(t) = l_0 = 0$. Assume that $l_0 > 0$. Then, for sufficiently large $t_2 \in [t_1,\infty)_{\mathbb{T}}$, we have $x(g_\nu(t)) \geq l_0$ for $t \geq t_2$. It follows that

$$\phi_{\gamma_\nu}\left(x\left(g_\nu(t)\right)\right) \geq l_0^{\gamma_\nu} \geq L \quad \text{for } t\in[t_2,\infty)_{\mathbb{T}},$$

where $L := \min_{\nu=0}^N\{l_0^{\gamma_\nu}\} > 0$. Then from (1.1) we obtain

$$-\left(x^{[n-1]}(t)\right)^\Delta \geq L\sum_{\nu=0}^N p_\nu(t) = L\hat{p}_0(t).$$

Integrating this from t to $\nu \in [t,\infty)_{\mathbb{T}}$, we get

$$-x^{[n-1]}(\nu) + x^{[n-1]}(t) \geq L \int_t^{\nu} \hat{p}_0(\tau)\Delta\tau,$$

and by (4.23) we see that $x^{[n-1]}(\nu) > 0$. Hence, by taking limits as $\nu \to \infty$ we have

$$x^{[n-1]}(t) \geq L \int_t^{\infty} \hat{p}_0(\tau)\Delta\tau.$$

If $\int_t^{\infty} \hat{p}_0(\tau)\Delta\tau = \infty$, then we have reached a contradiction. Otherwise,

$$\left(x^{[n-2]}(t)\right)^{\Delta} \geq L^{1/\alpha_{n-1}} \left[\frac{1}{r_{n-1}(t)} \int_t^{\infty} \hat{p}_0(\tau)\Delta\tau\right]^{1/\alpha_{n-1}} = L^{1/\alpha_{n-1}} \hat{p}_1(t).$$

Integrating this from t to $\nu \in [t,\infty)_{\mathbb{T}}$ and letting $\nu \to \infty$, by (4.23) we get

$$-x^{[n-2]}(t) \geq L^{1/\alpha_{n-1}} \int_t^{\infty} \hat{p}_1(\tau)\Delta\tau.$$

If $\int_t^{\infty} \hat{p}_1(\tau)\Delta\tau = \infty$, then we have reached a contradiction. Otherwise,

$$-\left(x^{[n-3]}(t)\right)^{\Delta} \geq L^{1/\alpha[n-2,n-1]} \left[\frac{1}{r_{n-2}(t)} \int_t^{\infty} \hat{p}_1(\tau)\Delta\tau\right]^{1/\alpha_{n-2}} = L^{1/\alpha[n-2,n-1]} \hat{p}_2(t).$$

Continuing this process, we get

$$-x^{[1]}(t) \geq L^{1/\alpha[2,n-1]} \int_t^{\infty} \hat{p}_{n-2}(\tau)\Delta\tau.$$

If $\int_t^{\infty} \hat{p}_{n-2}(\tau)\Delta\tau = \infty$, then we have reached a contradiction. Otherwise,

$$-x^{\Delta}(t) \geq L^{1/\alpha[1,n-1]} \left[\frac{1}{r_1(t)} \int_t^{\infty} \hat{p}_{n-2}(\tau)\Delta\tau\right]^{1/\alpha_1} = L^{1/\alpha} \hat{p}_{n-1}(t).$$

Again, integrating from t_2 to $t \in [t_2,\infty)_{\mathbb{T}}$, we get

$$-x(t) + x(t_2) \geq L^{1/\alpha} \int_{t_2}^{t} \hat{p}_{n-1}(\tau)\Delta\tau.$$

If $\int_t^{\infty} \hat{p}_{n-1}(\tau)\Delta\tau = \infty$, then we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the assumption that $x(t) > 0$ eventually. This shows that if $m = 0$, then $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Competing interests
The author declares that he has no competing interests.

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