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Existence and uniqueness of the solution of fractional damped dynamical systems

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Abstract

In this paper, we consider the existence and uniqueness of the solution of fractional damped dynamical systems. First, we obtain the spreading form of the Gronwall inequality. Furthermore, several sufficient conditions for the existence of the solutions are derived from the application of fixed point theorems and inequalities such as the Hölder and Gronwall inequalities.

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1 Introduction

Recently, fractional differential equations [1–5] have been studied intensively. The motivation for this work arises from both the development of the theory of fractional calculus itself and its wide application to various fields of science, such as physics, chemistry, biological, electromagnetic of complex media, robotics, economics, etc.

Much attention has been paid to the existence and uniqueness of the solutions of fractional dynamic systems [6–12] on account of the fact that existence is the fundamental problem and a necessary condition for considering some other properties for fractional dynamic systems, such as controllability, stability, etc. Besides, one has to take into account the peculiarity of the kernel to obtain explicit results in practice not only to reduce such an equation to an integral equation. Moreover, many authors have considered the multi-term fractional order systems [13–17] due to their successful applications in mechanical system, the dynamics of certain gases, the dynamics of sphere. And the existence and uniqueness of the solutions of this multi-term fractional order systems becomes more complicated than one-term fractional order systems which have been considered before by many authors. In this paper, motivated by the above, we will consider the existence and uniqueness of the solution of the following fractional damped dynamical systems:

$$\begin{cases} {}^cD_t^{\alpha}x(t) - A{}^cD_t^{\beta}x(t) = f(t,x(t)), & t \in J := [0,T], \\ x(0) = x_0, & x'(0) = x'_0, \end{cases} \tag{1}$$

where $0 < \beta \leq 1 < \alpha \leq 2$, $0 < T < \infty$, $x \in R^n$, A is an $R^{n\times n}$ matrix and $f : J \times R^n \to R^n$ is jointly continuous.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and facts on fractional calculus, the generalized Gronwall inequality, and fixed point theorems. In Section 3, we will give some results as regards the spreading form of the Gronwall inequality. In Section 4, we present our main results of this paper.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts. Throughout this paper, let $C(J,R^n)$ be the Banach space of all continuous function from J into R^n with the norm $\|x\|_c = \sup\{|x(t)| : t \in J\}$ for $x \in C(J,R^n)$. Let $|x|_{(\cdot)}$ be any vector norm (e.g., := 1, 2, ∞) and $\|(\cdot)\|$ be the matrix norm induced by this vector. The symbol $\sim$ means that the quotient of both sides converges to 1. Then we recall the following well-known definitions.

Definition 2.1 The fractional integral of order α with the lower zero for a function $f : [0,\infty) \to R$ is defined as

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0, \alpha > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is a gamma function.

Definition 2.2 The Riemann-Liouville derivative of order α with the lower limit zero for a function $f : [0,\infty) \to R$ can be defined as

$${}^L D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}}\, ds, \quad t > 0, n-1 < \alpha < n.$$

Definition 2.3 The Caputo derivative of order α for a function $f : [0,\infty) \to R$ can be defined as

$${}^c D_t^\alpha f(t) = {}^L D_t^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \quad t > 0, n-1 < \alpha < n.$$

Lemma 2.4 ([1]) *From the definition of fractional integrals and Caputo derivatives, we have*

$$I_t^\alpha \left({}^c D_t^\alpha x(t) \right) = x(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0), \quad t > 0, n-1 < \alpha < n.$$

Especially, when $1 < \alpha < 2$, *then we have*

$$I_t^\alpha \left({}^c D_t^\alpha x(t) \right) = x(t) - x(0) - t x'(0).$$

Lemma 2.5 ([18]) *Let* $0 < \beta < 1 < \alpha < 2$, *then we have*

$$I_t^\alpha \left({}^c D_t^\beta x(t) \right) = I_t^{\alpha-\beta} x(t) - \frac{x(0) t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}.$$

Taking the fractional integral of order α on both sides on the system (1), *due to the above two lemmas we can easily obtain the integral equation of the system* (1):

$$x(t) = x_0 + tx_0' - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x_0 + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds.$$

For a measurable function $\delta : J \to R$, define the norm

$$\|\delta\|_{L^p(J)} = \begin{cases} (\int_J |\delta(t)|^p\,dt)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \inf_{\mu(\bar{J})=0}\{\sup_{t\in J-\bar{J}} |\delta(t)|\}, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J,R)$ be the Banach space of all Lebesgue measure functions $\delta : J \to R$ with $\|\delta\|_{L^p(J)} < \infty$.

Lemma 2.6 (Hölder's inequality [7]) *Assume that $p,q \ge 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $\delta \in L^p(J,R)$, $\lambda \in L^q(J,R)$, then, for $1 \le q \le \infty$, $\delta\lambda \in L^1(J,R)$, and*

$$\|\delta\lambda\|_{L^1(J)} \le \|\delta\|_{L^p(J)}\|\lambda\|_{L^q(J)}.$$

Lemma 2.7 ([19]) *Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that P and Q map $\mathcal{C}$ into X such that*:
- (i) *$Px + Qy \in \mathcal{C}$ where $x, y \in \mathcal{C}$*;
- (ii) *P is a compact and continuous*;
- (iii) *Q is a contraction mapping.*

Then there exists $z \in \mathcal{C}$ such that $z = Pz + Qz$.

Lemma 2.8 ([19]) *Let X be a Banach spaces and $F : X \to X$ be a completely continuous operator, if the set*

$$E(F) = \bigl\{y \in X : y = \lambda Fy \text{ for some } \lambda \in [0,1]\bigr\}$$

is bounded, then F has at least a fixed point.

Lemma 2.9 ([19]) *Let $\mathcal{C}$ be a nonempty convex subset of X. Let U be a nonempty open subset of $\mathcal{C}$ with $0 \in U$ and $F : \overline{U} \to \mathcal{C}$ be a compact and continuous operators. Then either*:
- (i) *F has fixed points, or*
- (ii) *there exist $y \in \partial U$ and $\lambda^* \in [0,1]$ with $y = \lambda^* F(y)$.*

Lemma 2.10 ([20]) *Suppose $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $[0,T)$ and $\widetilde{g}(t)$ is a nonnegative, nondecreasing, continuous function defined on $[0,T)$; $\widetilde{g}(t) \le M$, where M is constant. Suppose $x(t)$ is a nonnegative and locally integrable on $[0,T)$ with*

$$x(t) \le \widetilde{a}(t) + \widetilde{g}(t)\int_0^t (t-s)^{\beta-1}x(s)\,ds, \quad t \in [0,T).$$

Then

$$x(t) \le \widetilde{a}(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\widetilde{g}(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1}\widetilde{a}(t) \right] ds, \quad t \in [0,T).$$

Corollary 2.11 ([20]) *Under the hypothesis of Lemma* 2.10, *let* $a(t)$ *be a nondecreasing function on* $[0,T)$. *Then*

$$x(t) \le \widetilde{a}(t) E_\beta\bigl(\widetilde{g}(t)\Gamma(\beta)t^\beta\bigr),$$

where E_β *is the Mittag-Leffler function defined by*

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}, \quad z \in \mathrm{C}, \mathrm{Re}(\beta) > 0.$$

3 Some results of the Gronwall inequality

In this section, we will give some results as regards the extended form of the Gronwall inequality.

Lemma 3.1 *Let* $p > 0$, $q > 0$, *then*

$$\int_\tau^t (t-s)^{p-1}(s-\tau)^{q-1}\,ds = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}(t-\tau)^{p+q-1}.$$

Proof Let $s = \tau + z(t-\tau)$, then we have

$$\begin{aligned} \int_\tau^t (t-s)^{p-1}(s-\tau)^{q-1}\,ds &= (t-\tau)^{p+q-1}\int_0^1 (1-z)^{p-1}z^{(q-1)}\,dz \\ &= (t-\tau)^{p+q-1}B(p,q) \\ &= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}(t-\tau)^{p+q-1}. \end{aligned}$$

□

Theorem 3.2 *Suppose* $\alpha > 0$, $\beta > 0$, $a(t)$ *is a nonnegative function locally integrable on* $[0,T)$, $\widetilde{g}(t)$, *and* $\overline{g}(t)$ *are nonnegative, nondecreasing, continuous functions defined on* $[0,T)$; $\widetilde{g}(t) \le \widetilde{M}$, $\overline{g}(t) \le \overline{M}$, *where* $\widetilde{M}$ *and* $\overline{M}$ *are constants. Suppose* $x(t)$ *is a nonnegative and locally integrable on* $[0,T)$ *with*

$$x(t) \le a(t) + \widetilde{g}(t)\int_0^t (t-s)^{\alpha-1}x(s)\,ds + \overline{g}(t)\int_0^t (t-s)^{\beta-1}x(s)\,ds, \quad t \in [0,T).$$

Then

$$\begin{aligned} x(t) \le{}& a(t) + \int_0^t \sum_{n=1}^{\infty} \bigl[g(t)\bigr]^n \\ &\times \sum_{k=0}^{n} \frac{C_n^k[\Gamma(\alpha)]^{n-k}[\Gamma(\beta)]^k}{\Gamma[(n-k)\alpha+k\beta]}(t-s)^{(n-k)\alpha+k\beta-1}a(s)\,ds, \quad t \in [0,T), \end{aligned} \tag{2}$$

where $g(t) = \widetilde{g}(t) + \overline{g}(t)$ *and* $C_n^k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

Proof Let $g(t) = \widetilde{g}(t) + \overline{g}(t)$, then

$$x(t) \le a(t) + g(t) \int_0^t \left[(t-s)^{\alpha-1} + (t-s)^{\beta-1}\right] x(s)\,ds.$$

Define the operator

$$Bx(t) = g(t) \int_0^t \left[(t-s)^{\alpha-1} + (t-s)^{\beta-1}\right] x(s)\,ds,$$

then

$$x(t) \le a(t) + Bx(t),$$

which implies that

$$x(t) \le \sum_{k=0}^{n-1} B^k a(t) + B^n x(t). \tag{3}$$

Now we prove

$$B^n x(t) \le \left[g(t)\right]^n \int_0^t \sum_{k=0}^{n} \frac{C_n^k [\Gamma(\alpha)]^{n-k} [\Gamma(\beta)]^k}{\Gamma[(n-k)\alpha + k\beta]} (t-s)^{(n-k)\alpha + k\beta - 1} x(s)\,ds. \tag{4}$$

When $n = 1$, the inequality (4) obviously holds. We assume that it holds for $n = m$. When $n = m + 1$, we have

$$\begin{aligned} B^{m+1}x(t) &= B\big(B^m x(t)\big) \\ &\le g(t) \int_0^t \left[(t-s)^{\alpha-1} + (t-s)^{\beta-1}\right] \\ &\quad \times \left[\int_0^s \left[g(s)\right]^m \sum_{k=0}^{m} \frac{C_m^k [\Gamma(\alpha)]^{m-k} [\Gamma(\beta)]^k}{\Gamma[(m-k)\alpha + k\beta]} (s-\tau)^{(m-k)\alpha + k\beta - 1} x(\tau)\,d\tau \right] ds. \end{aligned} \tag{5}$$

Since $\widetilde{g}(t)$, $\overline{g}(t)$ is nondecreasing which implies $g(t)$ is also nondecreasing and by Lemma 3.1, the inequality (5) can be rewritten as

$$\begin{aligned} B^{m+1}x(t) \le{} & \left[g(t)\right]^{m+1} \int_0^t \left[(t-s)^{\alpha-1} + (t-s)^{\beta-1}\right] \\ & \times \left[\int_0^s \sum_{k=0}^{m} \frac{C_m^k [\Gamma(\alpha)]^{m-k} [\Gamma(\beta)]^k}{\Gamma[(m-k)\alpha + k\beta]} (s-\tau)^{(m-k)\alpha + k\beta - 1} x(\tau)\,d\tau \right] ds. \end{aligned} \tag{6}$$

By interchanging the order of integration, the inequality (6) can be rewritten

$$\begin{aligned} B^{m+1}x(t) \le{} & \left[g(t)\right]^{m+1} \int_0^t \left[\int_\tau^t \sum_{k=0}^{m} \frac{C_m^k [\Gamma(\alpha)]^{m-k} [\Gamma(\beta)]^k}{\Gamma[(m-k)\alpha + k\beta]} (t-s)^{\alpha-1} (s-\tau)^{(m-k)\alpha + k\beta - 1} \right. \\ & \left. + \sum_{k=0}^{m} \frac{C_m^k [\Gamma(\alpha)]^{m-k} [\Gamma(\beta)]^k}{\Gamma[(m-k)\alpha + k\beta]} (t-s)^{\beta-1} (s-\tau)^{(m-k)\alpha + k\beta - 1}\,ds \right] x(\tau)\,d\tau \end{aligned}$$

$$\begin{aligned}
&= \left[g(t)\right]^{m+1}\int_0^t \left[\sum_{k=0}^{m} \frac{C_m^k[\Gamma(\alpha)]^{m+1-k}[\Gamma(\beta)]^k}{\Gamma[(m+1-k)\alpha+k\beta]}(t-s)^{(m+1-k)\alpha+k\beta-1}\right.\\
&\quad \left.+\sum_{k=0}^{m} \frac{C_m^k[\Gamma(\alpha)]^{m-k}[\Gamma(\beta)]^{k+1}}{\Gamma[(m-k)\alpha+(k+1)\beta]}(t-s)^{(m-k)\alpha+(k+1)\beta-1}\right]x(s)\,ds\\
&= \left[g(t)\right]^{m+1}\int_0^t \left[\sum_{k=0}^{m} \frac{C_m^k[\Gamma(\alpha)]^{m+1-k}[\Gamma(\beta)]^k}{\Gamma[(m+1-k)\alpha+k\beta]}(t-s)^{(m+1-k)\alpha+k\beta-1}\right.\\
&\quad \left.+\sum_{k=1}^{m+1} \frac{C_m^{k-1}[\Gamma(\alpha)]^{m+1-k}[\Gamma(\beta)]^{k}}{\Gamma[(m+1-k)\alpha+k\beta]}(t-s)^{(m+1-k)\alpha+k\beta-1}\right]x(s)\,ds\\
&= \left[g(t)\right]^{m+1}\int_0^t \left[\frac{C_m^0[\Gamma(\alpha)]^{m+1}}{\Gamma[(m+1)\alpha]}(t-s)^{(m+1)\alpha-1}\right.\\
&\quad +\sum_{k=1}^{m} \frac{(C_m^k+C_m^{k-1})[\Gamma(\alpha)]^{m+1-k}[\Gamma(\beta)]^{k}}{\Gamma[(m+1-k)\alpha+k\beta]}(t-s)^{(m+1-k)\alpha+k\beta-1}\\
&\quad \left.+\frac{C_m^m[\Gamma(\beta)]^{m+1}}{\Gamma[(m+1)\beta]}(t-s)^{(m+1)\beta-1}\right]\\
&= \left[g(t)\right]^{m+1}\int_0^t \sum_{k=0}^{m+1} \frac{C_{m+1}^k[\Gamma(\alpha)]^{m+1-k}[\Gamma(\beta)]^k}{\Gamma[(m+1-k)\alpha+k\beta]}(t-s)^{(m+1-k)\alpha+k\beta-1}x(s)\,ds.
\end{aligned}$$

The inequality (4) is proved. Since $\widetilde{g}(t) \leq \widetilde{M}, \overline{g}(t) \leq \overline{M}, t \in [0,T)$, which means that $g(t) \leq \widetilde{M}+\overline{M}$, therefore it can be seen that

$$B^n x(t) \leq \int_0^t \sum_{k=0}^{n} \frac{(\widetilde{M}+\overline{M})^n C_n^k[\Gamma(\alpha)]^{n-k}[\Gamma(\beta)]^k}{\Gamma[(n-k)\alpha+k\beta]}(t-s)^{(n-k)\alpha+k\beta-1}x(s)\,ds,$$

which implies that $B^n x(t) \to 0$ as $n \to \infty$ for $t \in [0,T)$. Indeed, applying mean value theorems and using Stirling's formula of the gamma function, $\Gamma(z+1) \sim \sqrt{2\pi z}(\frac{z}{e})^z$, there exists a constant $\xi \in [0,T)$ such that

$$\begin{aligned}
&\lim_{n\to\infty} B^n x(t)\\
&\quad \leq \lim_{n\to\infty} x(\xi)\sum_{k=0}^{n} \frac{(\widetilde{M}+\overline{M})^n C_n^k[\Gamma(\alpha)]^{n-k}[\Gamma(\beta)]^k}{\Gamma[(n-k)\alpha+k\beta]}\int_0^t (t-s)^{(n-k)\alpha+k\beta-1}\,ds\\
&\quad = \lim_{n\to\infty} x(\xi)\sum_{k=0}^{n} \frac{(\widetilde{M}+\overline{M})^n C_n^k[\Gamma(\alpha)T^\alpha]^{n-k}[\Gamma(\beta)T^\beta]^k}{\Gamma[(n-k)\alpha+k\beta+1]}\\
&\quad = \lim_{n\to\infty} x(\xi)\sum_{k=0}^{n} \frac{(\widetilde{M}+\overline{M})^n C_n^k}{\sqrt{2\pi[(n-k)\alpha+k\beta]}}\left(\frac{\Gamma(\alpha)T^\alpha}{(\frac{(n-k)\alpha+k\beta}{e})^\alpha}\right)^{n-k}\left(\frac{\Gamma(\beta)T^\beta}{(\frac{(n-k)\alpha+k\beta}{e})^\beta}\right)^{k}\\
&\quad \leq \lim_{n\to\infty} x(\xi)\frac{[(\widetilde{M}+\overline{M})(C_1+C_2)]^n}{\sqrt{2n\pi\gamma}},
\end{aligned}$$

where $\gamma = \min\{\alpha,\beta\}$, $C_1 = \frac{\Gamma(\alpha)T^\alpha}{(\frac{(n-k)\alpha+k\beta}{e})^\alpha}$, $C_2 = \frac{\Gamma(\beta)T^\beta}{(\frac{(n-k)\alpha+k\beta}{e})^\beta}$. When n is so large that $(\widetilde{M}+\overline{M})(C_1+C_2) < 1$, which implies $[(\widetilde{M}+\overline{M})(C_1+C_2)]^n \to 0$ as $n \to \infty$, we can say that $B^n x(t) \to 0$ as

$n \to \infty$ with $\frac{x(\xi)}{\sqrt{2n\pi\gamma}} \to 0$ as $n \to \infty$. Therefore, by equation (3), we can easily obtain the conclusion. The proof is completed. □

Corollary 3.3 *Under the hypothesis of Theorem* 3.2, *let* $a(t)$ *be a nondecreasing function on* $[0, T)$. *Then*

$$x(t) \le a(t)E_\gamma\big[g(t)\big(\Gamma(\alpha)t^\alpha + \Gamma(\beta)t^\beta\big)\big],$$

where $\gamma = \min\{\alpha, \beta\}$.

Proof When $a(t)$ is a nondecreasing function on $[0, T)$, the inequality (2) can be rewritten as

$$\begin{aligned}
x(t) &\le a(t)\left[1 + \int_0^t \sum_{n=1}^{\infty}\big[g(t)\big]^n \sum_{k=0}^{n} \frac{C_n^k[\Gamma(\alpha)]^{n-k}[\Gamma(\beta)]^k}{\Gamma[(n-k)\alpha + k\beta]}(t-s)^{(n-k)\alpha+k\beta-1}\,ds\right] \\
&= a(t)\left[1 + \sum_{n=1}^{\infty}\big[g(t)\big]^n \sum_{k=0}^{n} \frac{C_n^k[\Gamma(\alpha)]^{n-k}[\Gamma(\beta)]^k}{\Gamma[(n-k)\alpha + k\beta + 1]}(t-s)^{(n-k)\alpha+k\beta}\right] \\
&\le a(t)\left[1 + \sum_{n=1}^{\infty} \frac{[g(t)]^n \sum_{k=0}^{n} C_n^k[\Gamma(\alpha)t^\alpha]^{n-k}[\Gamma(\beta)t^\beta]^k}{\Gamma[n\gamma + 1]}\right] \\
&= a(t)\left[1 + \sum_{n=1}^{\infty} \frac{[g(t)]^n[\Gamma(\alpha)t^\alpha + \Gamma(\beta)t^\beta]^n}{\Gamma[n\gamma + 1]}\right] \\
&= a(t)E_\gamma\big[g(t)\big(\Gamma(\alpha)t^\alpha + \Gamma(\beta)t^\beta\big)\big].
\end{aligned}$$

The proof is completed. □

4 Main results

In this section, we will introduce the existence and uniqueness of the solutions of the system (1). Before stating and proving the main results, we need to give the following hypotheses.

(H1) $f : J \times R^n \to R^n$ is jointly continuous.

(H2) For all $t \in J$ and $x \in R^n$, there exists $q_1 \in (0, 1)$ and a real function $m \in L^{\frac{1}{q_1}}(J, R^n)$ such that $|f(t, x)| \le m(t)$.

(H3) For all $t \in J$ and $x_1, x_2 \in R^n$, there exist $q_2 \in (0, 1)$ and a real function $h \in L^{\frac{1}{q_2}}(J, R^n)$ such that $|f(t, x_1) - f(t, x_2)| \le h(t)|x_1 - x_2|$.

For brevity, let us denote

$$a = \frac{\alpha - 1}{1 - q_1}, \qquad b = \frac{\alpha - 1}{1 - q_2}.$$

Theorem 4.1 *Assume that* (H1)-(H3) *hold, if*

$$\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha - \beta + 1)} + \frac{\|h\|_{L^{\frac{1}{q_2}}(J)}T^{(1+b)(1-q_2)}}{\Gamma(\alpha)(1+b)^{(1-q_2)}} < 1, \tag{7}$$

then the system of (1) *has a unique solution on* J.

Proof First, we construct a space

$$C_r = \{x \in C(J, R^n) : \|x\|_c \le r\},$$

where r satisfies

$$|x_0| + T|x_0'| + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{r\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\|m\|_{L^{\frac{1}{q_1}}(J)} T^{(1+a)(1-q_1)}}{\Gamma(\alpha)(1+a)^{1-q_1}} \le r.$$

It is obvious that the space C_r is the Banach space. Then we define an operator $F : C_r \to C(J, R^n)$ by

$$\begin{aligned}(Fx)(t) &= x_0 + tx_0' - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x_0 + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, x(s))\,ds. \end{aligned} \tag{8}$$

Obviously, F is well defined due to (H1).

On the one hand, for every $x \in C_r(J, R^n)$, we have

$$\begin{aligned}|(Fx)(t)| &\le |x_0| + T|x_0'| + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x(s)|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s, x(s))|\,ds \\ &\le |x_0| + T|x_0'| + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{r\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\ &\quad + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q_1}}\,ds\right)^{1-q_1}\left(\int_0^t m(s)^{\frac{1}{q_1}}\,ds\right)^{q_1} \\ &\le |x_0| + T|x_0'| + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{r\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\|m\|_{L^{\frac{1}{q_1}}(J)} T^{(1+a)(1-q_1)}}{\Gamma(\alpha)(1+a)^{1-q_1}} \\ &\le r, \end{aligned}$$

which implies that $\|Fx\|_c \le r$.

On the other hand, for arbitrary $x_1, x_2 \in C_r$, using (H3) and the Hölder inequality, we get

$$\begin{aligned}&|(Fx_1)(t) - (Fx_2)(t)| \\ &\quad\le \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x_1(s) - x_2(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)|x_1(s) - x_2(s)|\,ds \\ &\quad\le \|x_1 - x_2\|_c \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\,ds \\ &\qquad + \|x_1 - x_2\|_c \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q_2}}\,ds\right)^{1-q_2}\left(\int_0^t h(s)^{\frac{1}{q_2}}\,ds\right)^{q_2} \\ &\quad\le \|x_1 - x_2\|_c \left(\frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\|h\|_{L^{\frac{1}{q_2}}(J)} T^{(1+b)(1-q_2)}}{\Gamma(\alpha)(1+b)^{1-q_2}}\right). \end{aligned}$$

Thus, F is a contraction mapping on C_r due to the condition (7). Due to the well-known Banach contraction mapping principle we know that the operator F has a unique fixed point on C_r. Therefore, the system (1) has a unique solution. The proof is completed. □

Theorem 4.2 *Assume that* (H1)-(H3) *hold*, *if*

$$\frac{\|h\|_{L^{\frac{1}{q_2}}(J)} T^{(1+b)(1-q_2)}}{\Gamma(\alpha)(1+b)^{(1-q_2)}} < 1, \tag{9}$$

then the system (1) *has at least one solution on* J.

Proof Consider the C_r defined in Theorem 4.1. We define the operators F_1 and F_2 on C_r as follows:

$$\begin{aligned} (F_1x)(t) &= x_0 + tx_0' - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}x_0 + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,ds, \\ (F_2x)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds. \end{aligned}$$

For any $x, y \in C_r$ and $t \in J$, using (H2) and the Hölder inequality, we have

$$\begin{aligned} \|F_1x + F_2y\|_c &\le |x_0| + T\bigl|x_0'\bigr| + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{r\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\|m\|_{L^{\frac{1}{q_1}}(J)} T^{(1+a)(1-q_1)}}{\Gamma(\alpha)(1+a)^{1-q_1}} \\ &\le r. \end{aligned}$$

So we know that $F_1x + F_2y \in C_r$. In order to use Lemma 2.7, we will verify that F_2 is a contraction mapping, while F_1 is compact and continuous.

Now we prove the operator F_2 is a contraction operator. Let $x_1, x_2 \in C_r$ and $t \in J$, then

$$\begin{aligned} &\bigl|(F_2x_1)(t) - (F_2x_2)(t)\bigr| \\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|f\bigl(s,x_1(s)\bigr) - f\bigl(s,x_2(s)\bigr)\bigr|\,ds \\ &\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\bigl|x_1(s) - x_2(s)\bigr|\,ds \\ &\quad\le \frac{1}{\Gamma(\alpha)}\biggl(\int_0^t (t-s)^{\frac{\alpha-1}{1-q_2}}\,ds\biggr)^{1-q_2}\biggl(\int_0^t h(s)^{\frac{1}{q_2}}\,ds\biggr)^{q_2}\|x_1 - x_2\|_c \\ &\quad\le \frac{\|h\|_{L^{\frac{1}{q_2}}(J)} T^{(1+b)(1-q_2)}}{\Gamma(\alpha)(1+b)^{1-q_2}}\|x_1 - x_2\|_c. \end{aligned}$$

So we see that F_2 is a contraction operator due to the condition (9).

Then we prove the operator F_1 is compact and continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in the $C(J, R^n)$. Then, for each $t \in J$, we have

$$\begin{aligned} \bigl|(F_1x_n)(t) - (F_1x)(t)\bigr| &\le \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\bigl|x_n(s) - x(s)\bigr|\,ds \\ &\le \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\bigl\|x_n(\cdot) - x(\cdot)\bigr\|_c. \end{aligned}$$

Thus, $\|F_1x_n - F_1x\| \to 0$ as $n \to \infty$, which implies that the operator F_1 is continuous on C_r.

For each $t \in J$, we obtain

$$\begin{aligned}
&\left|(F_1x)(t)\right| \\
&\quad\leq |x_0| + T\left|x_0'\right| - \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|x(s)\right|ds \\
&\quad\leq |x_0| + T\left|x_0'\right| - \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{r\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} := \widetilde{r}.
\end{aligned}$$

It shows that, for any $r > 0$, there exists a $\widetilde{r}$ such that, for each $x \in C_r$, we have $\|F_1x\| \leq \widetilde{r}$. That is to say, F_1 maps bounded sets into bounded sets in C_r.

And also for $t_1, t_2 \in J$, $0 \leq t_1 < t_2 \leq T$, $x \in C_r$, we have

$$\begin{aligned}
&\left|(F_1x)(t_2) - (F_1x)(t_1)\right| \\
&\quad\leq \left|x_0'\right|(t_2-t_1) + \frac{|x_0|\|A\|}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\right) + \frac{r\|A\|}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\beta-1}\,ds \\
&\qquad + \frac{r\|A\|}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\right]ds \\
&\quad= \left|x_0'\right|(t_2-t_1) + \frac{|x_0|\|A\|}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\right) + \frac{r\|A\|}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\right) \\
&\quad\leq \left|x_0'\right|(t_2-t_1) + \frac{2r\|A\|}{\Gamma(\alpha-\beta+1)}\left(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\right).
\end{aligned}$$

As $t_2 \to t_1$, $|(F_1x)(t_2) - (F_1x)(t_1)| \to 0$, which implies that F_1 is equicontinuous. By applying the well-known Ascoli theorem, we come to the conclusion that the operator F_1 is compact.

Hence from all the above results together with Lemma 2.7, we can see that the operator F has at least one fix point. Therefore, the system (1) has at least one solution on J. The proof is completed. □

Now, we replace (H2) by the following linear growth condition:

(H2′) For each $t \in J$ and all $x \in R^n$, there exists a constant $L > 0$ such that

$$\left|f(t,x)\right| \leq L\left(1 + |x|\right).$$

Theorem 4.3 *Assume that* (H1) *and* (H2′) *hold, then the system* (1) *has at least one solution on J.*

Proof Consider the operator $F : C_r \to C(J, R^n)$ defined as (8) in Theorem 4.1. And we define the set

$$E(F) = \left\{x \in C_r : x = \lambda Fx \text{ for some } \lambda \in [0,1]\right\}.$$

In order to use Lemma 2.8, we need to verify that F is completely continuous and the set $E(F)$ is bounded.

Now we prove the F is completely continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in the $C(J, R^n)$. Then, for each $t \in J$, we have

$$\begin{aligned}
&\bigl|(Fx_n)(t) - (Fx)(t)\bigr| \\
&\quad\le \frac{A}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \bigl|x_n(s) - x(s)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl|f\bigl(s, x_n(s)\bigr) - f\bigl(s, x(s)\bigr)\bigr|\,ds \\
&\quad\le \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \bigl\|x_n(\cdot) - x(\cdot)\bigr\|_c \\
&\qquad + \frac{T^\alpha}{\Gamma(\alpha+1)} \bigl\|f\bigl((\cdot), x_n(\cdot)\bigr) - f\bigl((\cdot), x(\cdot)\bigr)\bigr\|_c.
\end{aligned}$$

Thus, $\|Fx_n - Fx\| \to 0$ as $n \to \infty$, which implies that the operator F is continuous on C_r.

Since for each $t \in J$, we get

$$\begin{aligned}
&\bigl|(Fx)(t)\bigr| \\
&\quad\le |x_0| + T\bigl|x_0'\bigr| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{r\|A\|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl|f\bigl(s, x(s)\bigr)\bigr|\,ds \\
&\quad\le |x_0| + T\bigl|x_0'\bigr| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{r\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{L(1+r)T^\alpha}{\Gamma(\alpha+1)} := \widehat{r}.
\end{aligned}$$

It shows that, for any $r > 0$, there exists a $\widehat{r}$ such that, for each $x \in C_r$, we have $\|Fx\| \le \widehat{r}$. That is to say, F maps bounded sets into bounded sets in C_r.

And also due to (H2′), for $t_1, t_2 \in J$, $0 \le t_1 < t_2 \le T$, $x \in C_r$, we have

$$\begin{aligned}
&\bigl|(Fx)(t_2) - (Fx)(t_1)\bigr| \\
&\quad\le \bigl|x_0'\bigr|(t_2 - t_1) + \frac{|x_0|\|A\|}{\Gamma(\alpha-\beta+1)} \bigl(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\bigr) + \frac{r\|A\|}{\Gamma(\alpha-\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-\beta-1}\,ds \\
&\qquad + \frac{r\|A\|}{\Gamma(\alpha-\beta)} \int_0^{t_1} \bigl[(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\bigr]\,ds + \frac{L(1+r)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds \\
&\qquad + \frac{L(1+r)}{\Gamma(\alpha)} \int_0^{t_1} \bigl[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr]\,ds \\
&\quad\le \bigl|x_0'\bigr|(t_2 - t_1) + \frac{2r\|A\|}{\Gamma(\alpha-\beta+1)} \bigl(t_2^{\alpha-\beta} - t_1^{\alpha-\beta}\bigr) + \frac{L(1+r)}{\Gamma(\alpha+1)} \bigl(t_2^\alpha - t_1^\alpha\bigr).
\end{aligned}$$

As $t_2 \to t_1$, $|(Fx)(t_2) - (Fx)(t_1)| \to 0$, which implies that F is equicontinuous. By applying the well-known Ascoli theorem, we see that the operator F is completely continuous.

On the other hand, let $x \in E(F)$, then $x = \lambda Fx$ for some $\lambda \in [0,1]$. For $t \in J$, we have

$$\begin{aligned}
\bigl|x(t)\bigr| &\le |x_0| + T\bigl|x_0'\bigr| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} |x_0| + \frac{\|A\|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \bigl|x(s)\bigr|\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bigl|f\bigl(s, x(s)\bigr)\bigr|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le |x_0| + T|x_0'| + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{LT^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|x(s)|\,ds \\
&\quad + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|x(s)|\,ds.
\end{aligned}
$$

For brevity, let us denote

$$
\begin{aligned}
&l = |x_0| + T|x_0'| + \frac{\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| + \frac{LT^\alpha}{\Gamma(\alpha+1)}, \\
&m = \frac{\|A\|}{\Gamma(\alpha-\beta)}, \qquad n = \frac{L}{\Gamma(\alpha)}.
\end{aligned}
$$

Then by Corollary 3.3, we can obtain

$$
\begin{aligned}
|x(t)| &\le l + m\int_0^t (t-s)^{\alpha-\beta-1}|x(s)|\,ds + n\int_0^t (t-s)^{\alpha-1}|x(s)|\,ds \\
&\le lE_{\alpha-\beta}\big[(m+n)\big(\Gamma(\alpha)t^\alpha + \Gamma(\alpha-\beta)t^{\alpha-\beta}\big)\big] \\
&:= M^*,
\end{aligned}
$$

which implies that there exists a constant $M^* > 0$ such that $|x(t)| \le M^*$. Then we can say that the set $E(F)$ is bounded.

Hence from all the above results together with Lemma 2.8, we deduce that F has a fixed point. Therefore the system (1) has at least one solution. The proof is completed. □

In order to weaken the condition (H2′), we will use Lemma 2.9 to prove the following theorem.

(H2″) There exists a constant $q_3 \in (0,1)$ such that a real valued function $\phi_f(t) \in L^{\frac{1}{q_3}}(J,R^n)$ and there exists a L^1-integrable and nondecreasing $\psi : [0,+\infty) \to [0,+\infty)$ such that $|f(t,x)| \le \phi_f(t)\psi(|x|)$ for each $t \in J$ and $x \in R^n$.

(H4) The inequality

$$
\frac{\eta}{\Lambda + \frac{\psi(\eta)T^{(1+\gamma)(1-q_3)}\vartheta}{\Gamma(\alpha)(1+\gamma)^{1-q_3}}} > 1 \tag{10}
$$

has at least a positive solution, where $\Lambda = |x_0| + T|x_0'| + \frac{2\|x\|_c\|A\|T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$, $\vartheta = \|\phi_f\|_{L^{\frac{1}{q_3}}}(J)$, $\gamma = \frac{\alpha-1}{1-q_3}$.

Theorem 4.4 *Assume that* (H1), (H2″), *and* (H4) *hold, then the system* (1) *has at least one solution on* J.

Proof Proof by contradiction. By the hypothesis (H4), there exists a $N > 0$ such that $\|x\|_c \ne N$. Let $C_N = \{x \in C(J,R^n) : \|x\|_c < N\}$. Consider the operator F defined in Theorem 4.1, we can easily shown that $F : \overline{C}_N \to C(J,R^n)$ is continuous and completely continuous. Assume that there exist $x \in \partial C_N$ such that $x = \lambda^* F(x)$, $\lambda^* \in [0,1]$, then we have

$$\begin{aligned}
\left|x(t)\right| \le \left|(Fx)(t)\right| &\le |x_0| + T\left|x_0'\right| + \frac{\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}|x_0| \\
&\quad + \frac{\|A\|}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1}\left|x(s)\right| ds \\
&\quad + \frac{\psi(\|x\|_c)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left|f\left(s,x(s)\right)\right| ds \\
&\le |x_0| + T\left|x_0'\right| + \frac{2\|x\|_c\|A\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\quad + \frac{\psi(\|x\|_c)}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q_3}}\, ds\right)^{1-q_3} \left(\int_0^t \phi_f(s)^{\frac{1}{q_3}}\, ds\right)^{q_3} \\
&\le \Lambda + \frac{\psi(\|x\|_c) T^{(1+\gamma)(1-q_3)}\vartheta}{\Gamma(\alpha)(1+\gamma)^{1-q_3}}.
\end{aligned}$$

Thus

$$\frac{\eta}{\Lambda + \frac{\psi(\eta)T^{(1+\gamma)(1-q_3)}\vartheta}{\Gamma(\alpha)(1+\gamma)^{1-q_3}}} \le 1,$$

which is a contradiction with the inequality (10) under the hypothesis (H4).

Then we can say that there is no $x \in \partial C_N$ such that $x = \lambda^* F(x)$, $\lambda^* \in [0,1]$, from the selection of C_N. By the Lemma 2.9, we deduce that F has a fixed point $x \in \overline{C}_N$. Therefore, the system (1) has at least one solution. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the manuscript. Both of them read and approved the final manuscript.

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