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S-asymptotically T -periodic solutions for delay fractional differential equations with almost sectorial operator

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Abstract

We address the existence and uniqueness of S-asymptotically T -periodic solution of delay fractional differential equations with almost sectorial operator in infinite dimensional Banach spaces. Under the weak assumptions, we obtain the existence and uniqueness result. An example is presented.

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1 Introduction

Recently, fractional differential systems have played an important role in physics, chemistry, engineering, biology, finance etc., due to the memory character of fractional derivative, which is a generalization of integer-order derivative and can describe many phenomena that an integer derivative cannot characterize (see [1–7] and the references therein).

In this paper, we study the existence of S-asymptotically T -periodic solutions for the following fractional differential equation on a Banach space X :

$$\begin{cases} {}^c D_t^q u(t) = Au(t) + f(t, u_t), & t > 0, \\ u(t) = \phi(t), & t \in [-\delta, 0], \end{cases} \quad (1.1)$$

where $q \in (0, 1)$ and $\delta > 0$. The fractional derivative is understood here in the Caputo sense. A is an almost sectorial operator to be introduced later. $u_t : [-\delta, 0] \rightarrow X$ is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-\delta, 0]$. f is a function to be specified later. $\phi \in C([-\delta, 0], X)$.

The literature concerning S-asymptotically T -periodic functions with values in Banach spaces is very new. There are some papers dealing with the existence of S-asymptotically T -periodic solutions of differential equations and fractional differential equations with a sectorial operator in finite as well as infinite dimensional spaces (cf. [4–6, 8–14]). However, von Wahl first introduced examples of almost sectorial operators which are not sectorial [15]. To the best of the authors' knowledge, there are few papers on the existence of S-asymptotically T -periodic (mild) solutions for fractional differential equation with almost sectorial operator of type (1.1).

We will now present a summary of this work. In Section 2, we recall some fundamental properties of S-asymptotically T -periodic functions and preliminary facts. The existence and uniqueness of S-asymptotically T -periodic mild solution of problem (1.1) are discussed in Section 3, and an example is given to illustrate our result.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space and $L(X)$ be the space of all bounded linear operators from X to X with the usual operator norm $\|x\|_{L(X)}$. $C_b(\mathbb{R}_+, X)$ denotes the space of the continuous bounded functions from $[0, +\infty)$ to X , endowed with the norm $\|x\|_\infty = \sup_{t \geq 0} \|x(t)\|$. $C([-\delta, 0], X)$ denotes the space of the continuous functions from $[-\delta, 0]$ to X with the norm $\|x\|_{[-\delta, 0]} = \sup_{t \in [-\delta, 0]} \|x(t)\|$. $L^p((0, +\infty), (0, +\infty))$ is the L^p space with the norm $\|x\|_p = (\int_0^{+\infty} |x(t)|^p dt)^{\frac{1}{p}}$ for $1 \leq p < \infty$, and we abbreviate this notation to L^p .

We recall the following basic definitions of the fractional calculus theory. For more details, we refer to [1, 2].

Definition 2.1 ([1, 2]) The fractional integral of order q with the lower limit zero for a function $g \in L^1[0, \infty)$ is defined as

$$I_t^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) ds, \quad t > 0, 0 < q < 1,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([1, 2]) The Caputo derivative of order q for a function $g \in C^1[0, \infty)$ can be written as

$${}^c D_t^q g(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{g'(s)}{(t-s)^q} ds, \quad t > 0, 0 < q < 1.$$

As in [16, 17], we state the concept of almost sectorial operators as follows.

Definition 2.3 Let $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. By $\Theta_\omega^\gamma(X)$ we denote the family of all linear closed operators $A : D(A) \subset X \rightarrow X$ which satisfy

- (1) $\sigma(A) \subset S_\omega = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| \leq \omega\} \cup \{0\}$ and
- (2) for every $\omega < \zeta < \pi$ there exists a constant C_ζ such that

$$\|R(z; A)\|_{L(X)} \leq C_\zeta |z|^\gamma, \quad \text{for all } z \in \mathbb{C} \setminus S_\zeta.$$

A linear operator A will be called an almost sectorial operator on X if $A \in \Theta_\omega^\gamma(X)$.

Remark 2.4 Let $A \in \Theta_\omega^\gamma(X)$, then the definition implies that $0 \in \rho(A)$.

We denote the semigroup associated with A by $W(t)$. For $t \in S_{\frac{\pi}{2}-\omega}^0 = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| < \frac{\pi}{2} - \omega\}$,

$$W(t) = e^{-tz}(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{-tz} R(z; A) dz,$$

forms an analytic semigroup of growth order $1 + \gamma$, here $\omega < \theta < \frac{\pi}{2} - |\arg t|$, the integral contour $\Gamma_\theta := \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$ is oriented counter-clockwise [16, 17].

Proposition 2.5 ([16, 17]) *Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Then*

(i) *There exists a constant $C_0 = C_0(\gamma) > 0$ such that*

$$\|W(t)\|_{L(X)} \leq C_0 t^{-\gamma-1}, \quad \text{for all } t > 0.$$

(ii) *If $\beta > 1 + \gamma$, then $D(A^\beta) \subset \Sigma_T = \{x \in X; \lim_{t \rightarrow 0; t > 0} W(t)x = x\}$.*

(iii) *The functional equation $W(s + t) = W(s)W(t)$ for all $s, t \in S_{\frac{\pi}{2}-\omega}^0$ holds. However, it is not satisfied for $t = 0$ or $s = 0$.*

Consider the function of Wright-type (see [17, 18])

$$\Psi_q(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-qn + 1 - q)} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^n}{(n-1)!} \Gamma(nq) \sin(n\pi q), \quad z \in \mathbb{C}$$

with $0 < q < 1$.

Define operator families $\{S_q(t)\}_{t \in S_{\frac{\pi}{2}-\omega}^0}$ and $\{\mathcal{T}_q(t)\}_{t \in S_{\frac{\pi}{2}-\omega}^0}$ by

$$S_q(t)x = \int_0^\infty \Psi_q(\sigma) W(\sigma t^q) x \, d\sigma, \quad t \in S_{\frac{\pi}{2}-\omega}^0, x \in X,$$

$$\mathcal{T}_q(t)x = \int_0^\infty q\sigma \Psi_q(\sigma) W(\sigma t^q) x \, d\sigma, \quad t \in S_{\frac{\pi}{2}-\omega}^0, x \in X.$$

Theorem 2.6 ([18]) *For each fixed $t \in S_{\frac{\pi}{2}-\omega}^0$, $S_q(t)$ and $\mathcal{T}_q(t)$ are linear and bounded operators on X . Moreover, for all $t > 0$, $-1 < \gamma < 0$, $0 < q < 1$,*

$$\begin{aligned} \|S_q(t)\| &\leq M_1 t^{-q(1+\gamma)}, \\ \|\mathcal{T}_q(t)\| &\leq M_2 t^{-q(1+\gamma)}, \end{aligned} \tag{2.1}$$

where $M_1 = \frac{C_0 \Gamma(-\gamma)}{\Gamma(1-q(1+\gamma))}$ and $M_2 = \frac{qC_0 \Gamma(1-\gamma)}{\Gamma(1-q\gamma)}$. Moreover,

$$\lim_{t \rightarrow \infty} \|S_q(t)\| = 0, \quad \lim_{t \rightarrow \infty} \|\mathcal{T}_q(t)\| = 0. \tag{2.2}$$

Theorem 2.7 ([18], Theorem 3.2) *For $t > 0$, $S_q(t)$ and $\mathcal{T}_q(t)$ are continuous in the uniform operator topology. Moreover, for every $\tilde{r} > 0$, the continuity is uniform on $[\tilde{r}, \infty)$.*

Remark 2.8 ([18], Theorem 3.4) *Let $\beta > 1 + \gamma$. Then, for all $x \in D(A^\beta)$,*

$$\lim_{t \rightarrow 0; t > 0} S_q(t)x = x.$$

When $\phi(0) \in D(A^\beta)$ with $\beta > 1 + \gamma$, we present the definition of mild solution of problem (1.1) as follows.

Definition 2.9 A function $u \in C([-\delta, +\infty), X)$ satisfying the equation

$$u(t) = \begin{cases} \phi(t), & t \in [-\delta, 0], \\ S_q(t)\phi(0) + \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s)f(s, u_s) \, ds, & t > 0, \end{cases} \tag{2.3}$$

is called a mild solution of problem (1.1).

Remark 2.10 In general, mild solutions to problem (1.1) are assumed to have the same kind of singularity at $t = 0$ as the operator $\mathcal{S}_q(t)$. When $\phi(0) \in D(A^\beta)$ with $\beta > 1 + \gamma$, it follows from Remark 2.8 that the mild solution is continuous at $t = 0$.

Let us recall the notion of S-asymptotically T -periodic functions which will come into play later on.

Definition 2.11 ([9]) A function $g \in C_b(\mathbb{R}_+, X)$ is called S-asymptotically T -periodic if there exists $T > 0$ such that $\lim_{t \rightarrow \infty} \|g(t + T) - g(t)\| = 0$. In this case, we say that T is an asymptotic period of g .

Let $SAP_T(X)$ represent the space formed for all the X -valued S-asymptotically T -periodic functions endowed with the uniform convergence norm denoted by $\|\cdot\|_\infty$. Then $SAP_T(X)$ is a Banach space (see Proposition 3.5, [9]). We set $SAP_{T,0}(X) = \{x \in SAP_T(X); x(0) = 0\}$. Clearly, $SAP_{T,0}(X)$ is a closed subspace of $SAP_T(X)$.

Lemma 2.12 Let $u : [-\delta, +\infty) \rightarrow X$ be a function with $u_0 \in C([-\delta, 0], X)$ and $u|_{[0, +\infty)} \in SAP_T(X)$. Then the function $t \rightarrow u_t$ belongs to $SAP_T(C([-\delta, 0], X))$.

Proof Since u_t is continuous on $[-\delta, 0]$ which is compact, there exists $\bar{\theta} \in [-\delta, 0]$ such that

$$\|u_{t+T} - u_t\|_{[-\delta, 0]} = \sup_{-\delta \leq \theta \leq 0} \|u(t + T + \theta) - u(t + \theta)\| = \|u(t + T + \bar{\theta}) - u(t + \bar{\theta})\|.$$

Setting $\tau = t + \bar{\theta}$, we obtain $\lim_{t \rightarrow +\infty} \|u(t + T + \bar{\theta}) - u(t + \bar{\theta})\| = \lim_{\tau \rightarrow +\infty} \|u(\tau + T) - u(\tau)\| = 0$. □

3 Main result

In this section we discuss the existence and uniqueness of S-asymptotically T -periodic solutions for problem (1.1).

The function $f : (0, +\infty) \times C([-\delta, 0], X) \rightarrow X$ satisfies the following conditions:

(H1) There exists a function $\mu(\cdot) \in L^{\frac{1}{p}}(0 < p < -q\gamma)$ such that $\|f(t, \varphi)\| \leq \mu(t)$, for all $\varphi \in C([-\delta, 0], X)$.

(H2) There exists a function $\eta : (0, +\infty) \rightarrow (0, +\infty)$ such that

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leq \eta(t)\|\psi_1 - \psi_2\|, \quad \text{for all } t > 0, \psi_1, \psi_2 \in C([-\delta, 0], X)$$

and

$$\Lambda := \sup_{t \geq 0} \int_0^t \frac{\eta(s)}{(t-s)^{1+q\gamma}} ds < \frac{1}{M_2}. \tag{3.1}$$

(H3) $K := \sup_{t \geq 0} \int_0^t \frac{\|f(s, 0)\|}{(t-s)^{1+q\gamma}} ds < \infty$.

(H4) There exists a function $\xi : (0, +\infty) \rightarrow (0, +\infty)$ such that

$$\lim_{t \rightarrow \infty} \frac{\|f(t + T, \varphi) - f(t, \varphi)\|}{\xi(t)} = 0, \quad \text{for all } \varphi \in C([-\delta, 0], X),$$

$$\text{and } \sup_{t \geq 0} \int_0^t \frac{\xi(s)}{(t-s)^{1+q\gamma}} ds < \infty.$$

For $v > p, 0 \leq t_0 < t$, it follows from the Hölder inequality that

$$\int_{t_0}^t (t-s)^{v-1} \mu(s) ds \leq \left(\int_{t_0}^t (t-s)^{\frac{v-1}{1-p}} ds \right)^{1-p} \|\mu\|_{L^{\frac{1}{p}}} = \delta_v (t-t_0)^{v-p}, \tag{3.2}$$

where $\delta_v = (\frac{1-p}{v-p})^{1-p} \|\mu\|_{L^{\frac{1}{p}}}$.

Theorem 3.1 *Assume that (H1)-(H4) hold. Then, for every $\phi(0) \in D(A^\beta)$ with $\beta > 1 + \gamma$, the problem (1.1) has a unique S -asymptotically T -periodic mild solution.*

Proof For $\phi \in C([-\delta, 0], X)$, we define the function $y(t) = \phi(t)$ for $t \in [-\delta, 0]$, $y(t) = \mathcal{S}_q(t)\phi(0)$ for $t > 0$, then $y \in C([-\delta, \infty), X)$. Set $u(t) = x(t) + y(t), t \in [-\delta, +\infty)$. It is obvious that u satisfies (2.3) if and only if x satisfies $x_0 = 0$ and for $t > 0$,

$$x(t) = \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f(s, x_s + y_s) ds.$$

We write

$$\tilde{C}_b(X) = \{x \in C_b([-\delta, +\infty), X); x|_{t>0} \in C_b(\mathbb{R}_+, X), x|_{[-\delta, 0]} = 0\},$$

endowed with the norm $\|x\|_\infty := \sup_{t>0} \|x(t)\| + \|x_0\|_{[-\delta, 0]} = \sup_{t>0} \|x(t)\|$.

For $x \in \tilde{C}_b(X)$, setting $C_1 := \|x\|_\infty + \sup_{t>0} \|\mathcal{S}_q(t)\phi(0)\| + \|\phi\|_{[-\delta, 0]}$, we have

$$\begin{aligned} \|x_t + y_t\|_{[-\delta, 0]} &\leq \sup_{-\delta \leq \theta \leq 0} \|x(t+\theta)\| + \sup_{-\delta \leq \theta \leq 0} \|y(t+\theta)\| \\ &\leq \sup_{0 < \tau \leq t} \|x(\tau)\| + \sup_{0 < \tau \leq t} \|\mathcal{S}_q(\tau)\phi(0)\| + \|\phi\|_{[-\delta, 0]} \leq C_1, \end{aligned} \tag{3.3}$$

then

$$\|f(t, x_t + y_t)\| \leq \eta(t) \|x_t + y_t\|_{[-\delta, 0]} + \|f(t, 0)\| \leq C_1 \eta(t) + \|f(t, 0)\|. \tag{3.4}$$

We consider the operator \mathcal{F} on $\tilde{C}_b(X)$ as follows:

$$(\mathcal{F}x)(t) = \begin{cases} 0, & t \in [-\delta, 0], \\ \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f(s, x_s + y_s) ds, & t > 0. \end{cases}$$

We will show initially that $\mathcal{F}x \in \tilde{C}_b(X)$. Let $h > 0$, we have

$$\begin{aligned} &\left\| \int_0^{t+h} (t+h-s)^{q-1} \mathcal{T}_q(t+h-s) f(s, x_s + y_s) ds \right. \\ &\quad \left. - \int_0^t (t-s)^{q-1} \mathcal{T}_q(t-s) f(s, x_s + y_s) ds \right\| \\ &\leq \left\| \int_0^t [(t+h-s)^{q-1} - (t-s)^{q-1}] \mathcal{T}_q(t+h-s) f(s, x_s + y_s) ds \right\| \end{aligned}$$

$$\begin{aligned}
 & + \left\| \int_0^t (t-s)^{q-1} [\mathcal{T}_q(t+h-s) - \mathcal{T}_q(t-s)] f(s, x_s + y_s) ds \right\| \\
 & + \left\| \int_t^{t+h} (t+h-s)^{q-1} \mathcal{T}_q(t+h-s) f(s, x_s + y_s) ds \right\|.
 \end{aligned}$$

For $\varepsilon > 0$ small enough, we conclude

$$\begin{aligned}
 & \left\| \int_0^t [(t+h-s)^{q-1} - (t-s)^{q-1}] \mathcal{T}_q(t+h-s) f(s, x_s + y_s) ds \right\| \\
 & \leq M_2 \int_0^{t-\varepsilon} |(t+h-s)^{q-1} - (t-s)^{q-1}| (t+h-s)^{-q(1+\gamma)} \mu(s) ds \\
 & \quad + M_2 \int_{t-\varepsilon}^t \left[(t+h-s)^{-1-q\gamma} + \frac{(t-s)^{q-1}}{(t+h-s)^{q(1+\gamma)}} \right] \mu(s) ds,
 \end{aligned}$$

taking $h \rightarrow 0$, $\varepsilon \rightarrow 0$, and using (3.2), the right-hand side of the above inequality tends to zero.

Moreover, by (3.2), we have

$$\begin{aligned}
 \left\| \int_t^{t+h} (t+h-s)^{q-1} \mathcal{T}_q(t+h-s) f(s, x_s + y_s) ds \right\| & \leq M_2 \int_t^{t+h} (t+h-s)^{-q\gamma-1} \mu(s) ds \\
 & \rightarrow 0, \quad \text{as } h \rightarrow 0.
 \end{aligned}$$

For $\varepsilon > 0$ small enough, noting that (2.1) and (3.2), we obtain

$$\begin{aligned}
 & \left\| \int_0^t (t-s)^{q-1} [\mathcal{T}_q(t+h-s) - \mathcal{T}_q(t-s)] f(s, x_s + y_s) ds \right\| \\
 & \leq \int_0^{t-\varepsilon} (t-s)^{q-1} \|\mathcal{T}_q(t+h-s) - \mathcal{T}_q(t-s)\|_{L(X)} \mu(s) ds \\
 & \quad + \int_{t-\varepsilon}^t (t-s)^{q-1} \|\mathcal{T}_q(t+h-s) - \mathcal{T}_q(t-s)\| \mu(s) ds \\
 & \leq \sup_{s \in [0, t-\varepsilon]} \|\mathcal{T}_q(t+h-s) - \mathcal{T}_q(t-s)\|_{L(X)} \cdot \int_0^{t-\varepsilon} (t-s)^{q-1} \mu(s) ds \\
 & \quad + M_2 \int_{t-\varepsilon}^t \left(\frac{(t-s)^{q-1}}{(t+h-s)^{q(1+\gamma)}} + \frac{(t-s)^{q-1}}{(t-s)^{q(1+\gamma)}} \right) \mu(s) ds.
 \end{aligned}$$

This, together with Theorem 2.7, shows that the right-hand side tends to zero as $h \rightarrow 0$ and $\varepsilon \rightarrow 0$.

Moreover, from (H2), (H3), (3.1), and (3.4), we have

$$\left\| \int_0^t \frac{\mathcal{T}_q(t-s) f(s, x_s + y_s)}{(t-s)^{1-q}} ds \right\| \leq M_2 C_1 \int_0^t \frac{\eta(s)}{(t-s)^{1+q\gamma}} ds + M_2 K < C_1 + M_2 K.$$

Now, the operator $\mathcal{F} : \tilde{C}_b(X) \rightarrow \tilde{C}_b(X)$ is well defined. It is clear that the fixed points of \mathcal{F} are mild solutions to problem (1.1).

Next, we will show that \mathcal{F} is $SAP_{T,0}(X)$ -valued, where we identify the element $v \in SAP_{T,0}(X)$ with its extension to $[-\delta, +\infty)$ given by $v_0 = 0$. We will prove for any $x \in SAP_{T,0}(X)$, $\mathcal{F}x \in SAP_{T,0}(X)$. Obviously, (2.2) implies that $y|_{[0,\infty)} \in SAP_T(X)$, then from

Lemma 2.12, the function $t \rightarrow y_t$ belongs to $SAP_T(C([-δ, 0], X))$. Now $x_t + y_t \in SAP_T(C([-δ, 0], X))$, this means that, for each $\varepsilon > 0$, there is a positive constant $L_1 > 0$ such that

$$\| (x_{t+T} + y_{t+T}) - (x_t + y_t) \|_{[-\delta, 0]} \leq \varepsilon, \quad \text{for every } t \geq L_1. \tag{3.5}$$

Moreover, (H4) implies that there is a positive constant $L_2 > 0$ such that

$$\| f(t + T, x_{t+T} + y_{t+T}) - f(t, x_{t+T} + y_{t+T}) \| < \xi(t)\varepsilon, \quad \text{for every } t \geq L_2.$$

Then, for $t > L + T$, $L := \max\{L_1, L_2\}$, we have

$$\begin{aligned} & \| (\mathcal{F}x)(t + T) - (\mathcal{F}x)(t) \| \\ &= \left\| \int_0^T \frac{\mathcal{T}_q(t + T - s)}{(t + T - s)^{1-q}} f(s, x_s + y_s) ds + \int_T^{t+T} \frac{\mathcal{T}_q(t + T - s)}{(t + T - s)^{1-q}} f(s, x_s + y_s) ds \right. \\ &\quad \left. - \int_0^t (t - s)^{q-1} \mathcal{T}_q(t - s) f(s, x_s + y_s) ds \right\| \\ &\leq M_2 \left[\int_0^T (t + T - s)^{-1-q\gamma} (C_1 \eta(s) + \|f(s, 0)\|) ds \right] \\ &\quad + \int_0^t (t - s)^{q-1} \| \mathcal{T}_q(t - s) \| \| f(s + T, x_{s+T} + y_{s+T}) - f(s, x_{s+T} + y_{s+T}) \| ds \\ &\quad + M_2 \int_0^t (t - s)^{-1-q\gamma} \| f(s, x_{s+T} + y_{s+T}) - f(s, x_s + y_s) \| ds \\ &= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

Noting that $t + T - s \geq \frac{t+T}{T}(T - s)$, we have

$$\int_0^T \frac{C_1 \eta(s) + \|f(s, 0)\|}{(t + T - s)^{1+q\gamma}} ds \leq \left(\frac{T}{t + T} \right)^{1+q\gamma} \int_0^T \frac{C_1 \eta(s) + \|f(s, 0)\|}{(T - s)^{1+q\gamma}} ds,$$

which implies $I_1(t) \rightarrow 0$ as $t \rightarrow \infty$. From (3.4),

$$\| f(s + T, x_{s+T} + y_{s+T}) - f(s, x_{s+T} + y_{s+T}) \| \leq C_1 [\eta(s + T) + \eta(s)] + \|f(s + T, 0)\| + \|f(s, 0)\|.$$

Denoting $\sup_{t \geq 0} \int_0^t \frac{\xi(s)}{(t-s)^{1+q\gamma}} ds := \tilde{M}$ and noting that (H2), (H3), (2.2), and $t - s \geq \frac{t}{L}(L - s)$, we have

$$\begin{aligned} I_2(t) &\leq M_2 \int_0^L \frac{C_1 \eta(s + T) + \|f(s + T, 0)\| + C_1 \eta(s) + \|f(s, 0)\|}{(t - s)^{1+q\gamma}} ds \\ &\quad + M_2 \varepsilon \int_L^t (t - s)^{-1-q\gamma} \xi(s) ds \\ &\leq M_2 \left[\int_T^{L+T} \frac{C_1 \eta(s) + \|f(s, 0)\|}{(t + T - s)^{1+q\gamma}} ds + \int_0^L \frac{C_1 \eta(s) + \|f(s, 0)\|}{(t - s)^{1+q\gamma}} ds \right] + M_2 \tilde{M} \varepsilon \\ &\leq M_2 \left(\frac{L + T}{t + T} \right)^{1+q\gamma} \left[\int_0^{L+T} \frac{(C_1 \eta(s) + \|f(s, 0)\|) ds}{(L + T - s)^{1+q\gamma}} \right] \end{aligned}$$

$$\begin{aligned}
 &+ M_2 \left(\frac{L}{t}\right)^{1+q\gamma} \left[\int_0^L \frac{(C_1\eta(s) + \|f(s,0)\|) ds}{(L-s)^{1+q\gamma}} \right] + M_2 \tilde{M}\varepsilon \\
 &\leq 2M_2 \left(\frac{L+T}{t+T}\right)^{1+q\gamma} [C_1\Lambda + K] + M_2 \tilde{M}\varepsilon.
 \end{aligned}$$

From (H2), (3.3), and (3.5), it follows that

$$\begin{aligned}
 I_3(t) &\leq 2M_2 C_1 \int_0^L \frac{\eta(s)}{(t-s)^{1+q\gamma}} ds + M_2 \varepsilon \int_L^t \frac{\eta(s)}{(t-s)^{1+q\gamma}} ds \\
 &\leq M_2 \Lambda \left[2C_1 \left(\frac{L}{t}\right)^{1+q\gamma} + \varepsilon \right].
 \end{aligned}$$

Now we can see $\|(\mathcal{F}x)(t+T) - (\mathcal{F}x)(t)\| \rightarrow 0$ as $t \rightarrow \infty$. As a result, $\mathcal{F}(\text{SAP}_{T,0}(X)) \subseteq \text{SAP}_{T,0}(X)$.

For $x, \tilde{x} \in \text{SAP}_{T,0}(X)$, we have

$$\|(\mathcal{F}x)(t) - (\mathcal{F}\tilde{x})(t)\| \leq M_2 \int_0^t (t-s)^{-1-q\gamma} \eta(s) \|x_s - \tilde{x}_s\|_{[-\delta,0]} ds \leq M_2 \Lambda \|x - \tilde{x}\|_\infty,$$

then $\|\mathcal{F}x - \mathcal{F}\tilde{x}\|_\infty \leq M_2 \Lambda \|x - \tilde{x}\|_\infty$. Then \mathcal{F} is a contraction mapping, the proof now can be finished by using the contraction mapping principle. \square

Example 3.2 Let Ω be a bounded domain in \mathbb{R}^N ($N \geq 1$) with boundary $\partial\Omega$ of class C^4 . Let $X = C^{\frac{4}{5}}(\overline{\Omega})$,

$$\tilde{A} = \Delta, \quad D(\tilde{A}) = \{u \in C^{\frac{14}{5}}(\overline{\Omega}); u|_{\partial\Omega} = 0\}.$$

It follows from Example 2.3 [17] that there exist $\varsigma, \omega > 0$, such that

$$\tilde{A} + \varsigma \in \Theta_{\frac{\pi}{2}-\omega}^{-\frac{3}{5}}(C^{\frac{4}{5}}(\overline{\Omega})),$$

then $\tilde{A} + \varsigma$ is an almost sectorial operator and generates a semigroup $\{W(t)\}_{t \geq 0}$ with $\|W(t)\| \leq C_0 t^{-\gamma-1}$ ($\gamma = -\frac{3}{5}, C_0 > 0$).

We consider the following fractional differential problem:

$$\begin{cases}
 {}^c D_t^{\frac{5}{6}} v(t, x) = \Delta v(t, x) + 2v(t, x) + \frac{\alpha \cos v(t+\tau, x)}{(t+t^4)^{\frac{1}{4}}} + \frac{\sin 2\pi t}{\sqrt{t+1}}, & t > 0, x \in \Omega, \\
 v|_{\partial\Omega} = 0, \\
 v(t, x) = \phi(t, x), & t \in [-1, 0],
 \end{cases} \tag{3.6}$$

where $\alpha \in \mathbb{R}$ and $\phi : [-1, 0] \times \Omega \rightarrow X$ is a continuous function.

Problem (3.6) can be written in the abstract form as follows:

$$\begin{aligned}
 {}^c D_t^q v(t) &= Av(t) + F(t, v_t), \quad t > 0, \\
 v(t) &= \phi(t), \quad t \in [-1, 0],
 \end{aligned}$$

where $q = \frac{5}{6}, Av = (\tilde{A} + 2)v, F(t, v_t) = \frac{\alpha \cos v_t}{(t+t^4)^{\frac{1}{4}}} + \frac{\sin 2\pi t}{\sqrt{t+1}}$.

For $\varphi, \tilde{\varphi} \in C([-1, 0], X)$, we can easily see

$$\begin{aligned} \|F(t, \varphi)\| &\leq \frac{|\alpha|}{(t+t^4)^{\frac{1}{4}}} + \frac{1}{\sqrt{t+1}} := \mu(t), \\ \|F(t, \varphi) - F(t, \tilde{\varphi})\| &\leq \frac{|\alpha|}{(t+t^4)^{\frac{1}{4}}} \|\varphi - \tilde{\varphi}\| := \eta(t) \|\varphi - \tilde{\varphi}\|. \end{aligned}$$

Noting that

$$\begin{aligned} \int_0^1 \left(\frac{1}{t+t^4}\right)^{\frac{3}{4}} dt &\leq \int_0^1 \left(\frac{1}{t}\right)^{\frac{3}{4}} dt = 4, \\ \int_1^{+\infty} \left(\frac{1}{t+t^4}\right)^{\frac{3}{4}} dt &\leq \int_1^{+\infty} \left(\frac{1}{t^4}\right)^{\frac{3}{4}} dt = \int_1^{+\infty} \frac{1}{t^3} dt = \frac{1}{2}, \\ \int_0^{+\infty} \left(\frac{1}{1+t}\right)^{\frac{3}{2}} dt &= 2, \end{aligned}$$

we can obtain $\mu(t) \in L^{\frac{1}{p}}(p = \frac{1}{3})$.

Noting that $-1 - q\gamma = -\frac{1}{2}$, $(t+t^4)^{-\frac{1}{4}} \leq t^{-\frac{1}{4}}$, $\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = \pi$, we have

$$\begin{aligned} \Lambda &:= \sup_{t \geq 0} \int_0^t (t-s)^{-\frac{1}{2}} \eta(s) ds = |\alpha| \sup_{t \geq 0} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{4}} \cdot (1+s^3)^{-\frac{1}{4}} ds \\ &= |\alpha| \sup_{t \geq 0} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \cdot s^{\frac{1}{4}} (1+s^3)^{-\frac{1}{4}} ds \\ &\leq |\alpha| \pi. \end{aligned} \tag{3.7}$$

Moreover, $F(t, 0) = \frac{\alpha}{(t+t^4)^{\frac{1}{4}}} + \frac{\sin 2\pi t}{\sqrt{t+1}}$, noting that (3.7) and

$$\int_0^t (t-s)^{-\frac{1}{2}} \frac{|\sin 2\pi s|}{\sqrt{s+1}} ds \leq \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = \pi,$$

we get

$$\sup_{t \geq 0} \int_0^t (t-s)^{-\frac{1}{2}} \|F(s, 0)\| ds < +\infty.$$

We take $\xi(t) = t^{-\frac{1}{2}}$ then

$$\begin{aligned} \frac{\|F(t+1, \varphi) - F(t, \varphi)\|}{t^{-\frac{1}{2}}} &\leq |\alpha| t^{\frac{1}{2}} \left[\frac{1}{(t+t^4)^{\frac{1}{4}}} + \frac{1}{[(t+1) + (t+1)^4]^{\frac{1}{4}}} \right] \\ &\quad + t^{\frac{1}{2}} \left[\frac{1}{\sqrt{t+1}} - \frac{1}{\sqrt{t+2}} \right] \\ &\rightarrow 0, \quad \text{as } t \rightarrow +\infty, \end{aligned}$$

and $\sup_{t \geq 0} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = \pi < \infty$.

Noting that $M_2 = \frac{qC_0\Gamma(1-\gamma)}{\Gamma(1-q\gamma)} = \frac{5C_0\Gamma(\frac{8}{5})}{6\Gamma(\frac{3}{2})}$ and (3.7), in association with Theorem 3.1, if $|\alpha| < \frac{6\Gamma(\frac{3}{2})}{5\pi C_0\Gamma(\frac{8}{5})}$, then the problem (3.6) has a unique S-asymptotically 1-periodic mild solution for $\phi(0, x) \in D((A + 2)^{\frac{3}{5}})$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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