Qiao and Zhou *Advances in Difference Equations* (2017) 2017:8
DOI 10.1186/s13662-016-1042-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of positive solutions of singular fractional differential equations with infinite-point boundary conditions

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Abstract

Using height functions of the nonlinear term on some bounded sets and considering integrations of these height functions, we obtain the existence of positive solutions for a class of singular fractional differential equations with infinite-point boundary value conditions.

Keywords: existence; positive solutions; singular fractional differential equation; infinite-point boundary value conditions

1 Introduction

In this paper, we investigate the existence of positive solutions of the following singular fractional differential equations with infinite-point boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha}x(t) + q(t)f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta}x(1) = \sum_{i=1}^{\infty}\alpha_i x(\xi_i), \end{cases} \tag{1.1}$$

where $\alpha > 2$, $n-1 < \alpha \leq n$, $\beta \in [1, \alpha-1]$ is a fixed number, $\alpha_i \geq 0$ $(i = 1, 2, \ldots)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{i-1} < \xi_i < \cdots < 1$, $\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1} > 0$, and $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville derivatives. The nonlinear term f may be singular with respect to both time and space variables.

A function $x \in C[0,1]$ satisfying (1.1) is called a positive solution of (1.1) if $x > 0$ on $(0,1]$. Recently, fractional differential equations have gained considerable attention; for example, see [1–5] and the references therein. Papers [6–9] discussed the existence of positive solutions of fractional boundary value problems.

In [10], the authors investigated the existence of a positive solution for the following fractional boundary value problem in a Banach space E:

$$\begin{cases} D_{0^+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \sum_{i=1}^{m}\alpha_i u(\xi_i). \end{cases} \tag{1.2}$$

In [11], the authors obtained the existence and multiplicity of positive solutions of the following infinite-point fractional differential equations:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i). \end{cases} \tag{1.3}$$

In [12], the author obtained the existence and multiplicity of positive solutions of the following fractional differential equation with infinite-point boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + q(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j). \end{cases} \tag{1.4}$$

Motivated by the works mentioned, in this paper, we consider the boundary value problem (1.1) with more general and more complex boundary conditions. The existence of positive solutions for (1.1) is obtained by means of cone expansion and the compression fixed point theorem.

In order to establish the results, we assume that the following conditions are satisfied:

(A_1) $f \in C((0,1) \times (0,+\infty), [0,+\infty))$.

(A_2) $q \in C((0,1), [0,+\infty))$, and q does not vanish identically on any subinterval of $(0,1)$.

(A_3) For any positive numbers $r_1 < r_2$, there exists a continuous function $p_{r_1,r_2} : (0,1) \to [0,+\infty)$ such that

$$\int_0^1 q(t) p_{r_1,r_2}(t)\,dt < +\infty$$

and

$$f(t,u) \le p_{r_1,r_2}(t), \quad 0 < t < 1, t^{\alpha-1} r_1 \le u \le r_2.$$

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and relevant lemmas. In Section 3, we apply the fixed-point theorem to study the existence of positive solutions for the boundary value problem (1.1). In Section 4, we give some examples to illustrate our main results.

2 Preliminaries and relevant lemmas

For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory. In order to avoid redundancy, for the definitions of the Riemann-Liouville fractional integral and fractional derivative, we refer the readers to [13].

Lemma 2.1 ([14]) *Let $\alpha > 0$, and $u(t)$ be an integrable function. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$ $(i = 1,2,\ldots,n)$, and n is the smallest integer greater than or equal to α.

Lemma 2.2 *Given $y \in C[0,1]$, the unique solution of the problem*

$$\begin{cases} D_{0^+}^{\alpha} x(t) + y(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \\ D_{0^+}^{\beta} x(1) = \sum_{i=1}^{\infty} \alpha_i x(\xi_i), \end{cases} \tag{2.1}$$

is given by

$$x(t) = \int_0^1 G(t,s) y(s)\, ds,$$

where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha) l(0)} \{l(s) t^{\alpha-1}(1-s)^{\alpha-\beta-1} - l(0)(t-s)^{\alpha-1}\}, & 0 \le s \le t \le 1, \\ \frac{1}{\Gamma(\alpha) l(0)} \{l(s) t^{\alpha-1}(1-s)^{\alpha-\beta-1}\}, & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

with $l(s) = \frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha)} \sum_{s \le \xi_i} \alpha_i (\frac{\xi_i - s}{1-s})^{\alpha-1}(1-s)^{\beta}$ $(0 \le s \le 1)$.

Proof By Lemma 2.1 we can see that

$$x(t) = -I_{0^+}^{\alpha} y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \cdots + c_n t^{\alpha-n},$$

so, the solution of (2.1) is

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \cdots + c_n t^{\alpha-n}.$$

Since $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$, we have $c_2 = c_3 = \cdots = c_n = 0$.

In addition,

$$\begin{aligned} D_{0^+}^{\beta} x(t) &= -D_{0^+}^{\beta} I_{0^+}^{\alpha} y(t) + c_1 D_{0^+}^{\beta} \left(t^{\alpha-1}\right) \\ &= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)\, ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(n+\alpha-\beta)} \frac{d^n}{dt^n} \left(t^{n+\alpha-\beta-1}\right) \\ &= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)\, ds \\ &\quad + c_1 \frac{\Gamma(\alpha)}{\Gamma(n+\alpha-\beta)} (\alpha-\beta+n-1)(\alpha-\beta+n-2)\cdots(\alpha-\beta) t^{\alpha-\beta-1}, \end{aligned}$$

where $n = [\beta] + 1$.

Combining this with the second boundary value condition of (2.1), we have

$$\begin{aligned} D_{0^+}^{\beta} x(1) &= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \\ &= -\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{\infty} \alpha_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s)\, ds + c_1 \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}. \end{aligned}$$

This yields

$$c_1 = \frac{\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}y(s)\,ds - \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{\infty}\alpha_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)\,ds}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}.$$

Therefore, the solution of (2.1) is

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + \frac{\frac{1}{\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}t^{\alpha-1}y(s)\,ds}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \\ &\quad - \frac{\frac{1}{\Gamma(\alpha)}\int_0^1\sum_{s\le\xi_i}\alpha_i(\xi_i-s)^{\alpha-1}t^{\alpha-1}y(s)\,ds}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}} \\ &= \int_0^1 G(t,s)y(s)\,ds. \end{aligned}$$

The proof is completed. □

It is easy to see that if $l(s)$ is continuous on $[0,1]$, then $G(t,s)$ is continuous on $[0,1]\times[0,1]$.

Lemma 2.3 *The function* $l(s)$ *is positive and nondecreasing on* $[0,1]$.

Proof By the definition of $l(s)$ it follows that

$$l(0) = \frac{1}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1} > 0.$$

By computation we have that

(1) when $0 \le s < \lim_{i\to\infty}\xi_i$,

$$\begin{aligned} l'(s) &= \frac{1}{\Gamma(\alpha)}\left[\sum_{s\le\xi_i}\alpha_i(\alpha-1)\left(\frac{\xi_i-s}{1-s}\right)^{\alpha-2}(1-s)^{\beta-2}(1-\xi_i) + \sum_{s\le\xi_i}\alpha_i\beta\left(\frac{\xi_i-s}{1-s}\right)^{\alpha-1}(1-s)^{\beta-1}\right] \\ &> 0, \quad \text{and} \end{aligned}$$

(2) when $\lim_{i\to\infty}\xi_i \le s \le 1$,

$$l(s) = \frac{1}{\Gamma(\alpha-\beta)}, \qquad l'(s) = 0.$$

So, l is nondecreasing on $[0,1]$, and $l(s) \ge l(0) > 0$, $s\in[0,1]$. The proof is completed. □

Lemma 2.4 *The function* $G(t,s)$ *defined in* (2.2) *admits the following properties*:

(1) $G(t,s) > 0$, $\frac{\partial}{\partial t}G(t,s) > 0$, $0 < t,s < 1$;

(2) $\max_{t\in[0,1]} G(t,s) = G(1,s) = \frac{1}{\Gamma(\alpha)l(0)}[l(s)(1-s)^{\alpha-\beta-1} - l(0)(1-s)^{\alpha-1}]$, $0 \le s \le 1$;

(3) $G(t,s) \ge t^{\alpha-1}G(1,s)$, $0 \le t,s \le 1$.

Proof The proof is similar to that of Lemma 3 in [12]. □

The following lemma is the main tool in this paper.

Lemma 2.5 ([16]) *Let E be a real Banach space, and $P \subset E$ be a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*:

(1) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, *or*

(2) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let $E = C[0,1]$, so that E is a Banach space with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Set $P = \{x \in E | x(t) \geq t^{\alpha-1}\|x\|, t \in [0,1]\}$. Then P is a cone in E.

Denote $\Omega(r) = \{x \in P : \|x\| < r\}$ and $\partial\Omega(r) = \{x \in P : \|x\| = r\}$ for $r > 0$.

Define the operator $T : P\backslash\{\theta\} \to E$ by

$$(Tx)(t) = \int_0^1 G(t,s)q(s)f\bigl(s,x(s)\bigr)\,ds, \quad t \in [0,1]. \tag{2.3}$$

Lemma 2.6 *Suppose that (A_1)-(A_3) hold and $0 < r_1 < r_2$. Then $T : \overline{\Omega(r_2)}\backslash\Omega(r_1) \to P$ is completely continuous.*

Proof For any $x \in \overline{\Omega(r_2)}\backslash\Omega(r_1)$, it follows from (2.3) and Lemma 2.4 that

$$(Tx)(t) = \int_0^1 G(t,s)q(s)f\bigl(s,x(s)\bigr)\,ds \leq \int_0^1 G(1,s)q(s)f\bigl(s,x(s)\bigr)\,ds, \quad t \in [0,1], \tag{2.4}$$

and

$$\begin{aligned} (Tx)(t) &= \int_0^1 G(t,s)q(s)f\bigl(s,x(s)\bigr)\,ds \\ &\geq t^{\alpha-1}\int_0^1 G(1,s)q(s)f\bigl(s,x(s)\bigr)\,ds, \quad t \in [0,1]. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5) we know that

$$(Tx)(t) \geq t^{\alpha-1}\|Tx\|, \quad t \in [0,1],$$

which means that $T : \overline{\Omega(r_2)}\backslash\Omega(r_1) \to P$.

Noticing the continuity of $G(t,s)$ and (A_1)-(A_3), it is easy to see that T is continuous in $\overline{\Omega(r_2)}\backslash\Omega(r_1)$. Next, we show T is compact. For any $x \in \overline{\Omega(r_2)}\backslash\Omega(r_1)$, we have

$$\bigl|Tx(t)\bigr| = \left|\int_0^1 G(t,s)q(s)f\bigl(s,x(s)\bigr)\,ds\right| \leq \int_0^1 G(1,s)q(s)p_{r_1,r_2}(s)\,ds$$

for $t \in [0,1]$, which implies that $T(\overline{\Omega(r_2)}\backslash\Omega(r_1))$ is uniformly bounded.

Because $G(t,s)$ is continuous on $[0,1] \times [0,1]$, $G(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$. Thus, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\bigl|G(t_1,s) - G(t_2,s)\bigr| < \epsilon$$

if $|t_1 - t_2| < \delta$ and $(t_1, s), (t_2, s) \in [0,1] \times [0,1]$. Then, for any $x \in \overline{\Omega(r_2)} \backslash \Omega(r_1)$ and $t_1, t_2 \in [0,1]$ such that $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} \left|Tx(t_1) - Tx(t_2)\right| &= \left|\int_0^1 \bigl(G(t_1,s) - G(t_2,s)\bigr) q(s) f\bigl(s, x(s)\bigr)\, ds\right| \\ &\leq \int_0^1 \left|G(t_1,s) - G(t_2,s)\right| q(s) p_{r_1,r_2}(s)\, ds \\ &\leq \epsilon \int_0^1 q(s) p_{r_1,r_2}(s)\, ds. \end{aligned}$$

We can see that the functions in $T(\overline{\Omega(r_2)} \backslash \Omega(r_1))$ are equicontinuous. So, $T(\overline{\Omega(r_2)} \backslash \Omega(r_1))$ is relatively compact in $C[0,1]$. Thereby, T is compact in $\overline{\Omega(r_2)} \backslash \Omega(r_1)$, and thus $T : \overline{\Omega(r_2)} \backslash \Omega(r_1) \to P$ is completely continuous. □

3 Main result

We introduce the following height functions to control the growth of the nonlinear term $f(t,x)$. Let

$$\begin{aligned} \varphi(t,r) &= \max\bigl\{f(t,x) : t^{\alpha-1} r \leq x \leq r\bigr\} \quad (0 < t <, r > 0), \\ \psi(t,r) &= \min\bigl\{f(t,x) : t^{\alpha-1} r \leq x \leq r\bigr\} \quad (0 < t < 1, r > 0). \end{aligned}$$

Theorem 3.1 *Suppose that* (A_1)-(A_3) *hold and there exist two positive numbers* $a < b$ *such that one of the following conditions is satisfied*:

(B_1) $a \leq \int_0^1 G(1,s) q(s) \psi(s,a)\, ds < +\infty$ *and* $\int_0^1 G(1,s) q(s) \varphi(s,b)\, ds \leq b$;

(B_2) $\int_0^1 G(1,s) q(s) \varphi(s,a)\, ds \leq a$ *and* $b \leq \int_0^1 G(1,s) q(s) \psi(s,b)\, ds < +\infty$.

Then the boundary value problem (1.1) *has at least one strictly increasing positive solution* $x^* \in P$ *such that* $a \leq \|x^*\| \leq b$.

Proof Without loss of generality, we only prove (B_1).

If $x \in \partial\Omega(a)$, then $\|x\| = a$ and $t^{\alpha-1} a \leq x(t) \leq a$, $0 \leq t \leq 1$. By the definition of $\psi(t,a)$ we know that

$$f\bigl(t, x(t)\bigr) \geq \psi(t,a), \quad 0 < t < 1. \tag{3.1}$$

By (3.1) and Lemma 2.4 we have that

$$\begin{aligned} \|Tx\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) q(s) f\bigl(s, x(s)\bigr)\, ds \\ &\geq \max_{t\in[0,1]} t^{\alpha-1} \int_0^1 G(1,s) q(s) f\bigl(s, x(s)\bigr)\, ds \\ &= \int_0^1 G(1,s) q(s) f\bigl(s, x(s)\bigr)\, ds \\ &\geq \int_0^1 G(1,s) q(s) \psi(s,a)\, ds \geq a = \|x\|. \end{aligned} \tag{3.2}$$

If $x \in \partial\Omega(b)$, then $\|x\| = b$ and $t^{\alpha-1}b \leq x(t) \leq b$, $0 \leq t \leq 1$. By the definition of $\varphi(t,b)$ we get that

$$f\bigl(t,x(t)\bigr) \leq \varphi(t,b), \quad 0 < t < 1. \tag{3.3}$$

By (3.3) and Lemma 2.4 we have that

$$\begin{aligned} \|Tx\| &= \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)f\bigl(s,x(s)\bigr)\,ds \leq \int_0^1 G(1,s)q(s)f\bigl(s,x(s)\bigr)\,ds \\ &\leq \int_0^1 G(1,s)q(s)\varphi(s,b)\,ds \leq b = \|x\|. \end{aligned} \tag{3.4}$$

By Lemma 2.5, T has a fixed point $x^* \in \overline{\Omega(b)}\backslash\Omega(a)$. From Section 2 we know that x^* is a solution of (1.1) and $a \leq \|x^*\| \leq b$. Because $x^*(t) \geq t^{\alpha-1}\|x^*\| \geq at^{\alpha-1} > 0, 0 < t \leq 1$, we get that x^* is a positive solution for (1.1).

From Lemma 2.4 we have that

$$\bigl(x^*\bigr)'(t) = \bigl(Tx^*\bigr)'(t) = \int_0^1 \frac{\partial}{\partial t}G(t,s)q(s)f\bigl(s,x^*(s)\bigr)\,ds > 0,$$

which shows that x^* is a strictly increasing positive solution. The proof is completed. □

4 Examples

Example 4.1 Consider the boundary value problem

$$\begin{cases} D_{0^+}^{\frac{7}{2}}x(t) + \frac{1}{4\sqrt{1-t}}(x^5 + \frac{1}{2\sqrt[3]{x}}) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = x^{(2)}(0) = 0, \\ D_{0^+}^{\frac{3}{2}}x(1) = \sum_{i=1}^{\infty} \frac{2}{i^2}x(1 - \frac{1}{i+1}). \end{cases} \tag{4.1}$$

Let $\alpha = \frac{7}{2}, \beta = \frac{3}{2}$, $f(t,x) = x^5 + \frac{1}{2\sqrt[3]{x}}$, $q(t) = \frac{1}{4\sqrt{1-t}}, \alpha_i = \frac{2}{i^2}, \xi_i = 1 - \frac{1}{i+1}$. Obviously, $f \in C((0,1) \times (0,+\infty), [0,+\infty))$ and $q \in C((0,1), [0,+\infty))$. It is not difficult to calculate that $\sum_{i=1}^{\infty} \alpha_i\xi_i^{\alpha-1} \approx 1.643$, $\Gamma(\alpha) = \Gamma(\frac{7}{2}) \approx 3.3234$, $\Gamma(\alpha - \beta) = \Gamma(2) = 1$, $l(0) \approx 0.8169$. For any positive numbers $r_1 < r_2$, we can see that (A_1)-(A_3) hold for $p_{r_1,r_2}(t) = r_2^5 + \frac{1}{2}t^{-\frac{5}{6}}r_1^{-\frac{1}{3}}$.

The height functions $\varphi(t,r)$ and $\psi(t,r)$ satisfy the following inequalities:

$$\begin{aligned} \varphi(t,r) &= \max\left\{x^5 + \frac{1}{2\sqrt[3]{x}} : t^{\frac{5}{2}}r \leq x \leq r\right\} \leq r^5 + \frac{1}{2}t^{-\frac{5}{6}}r^{-\frac{1}{3}}, \\ \psi(t,r) &= \min\left\{x^5 + \frac{1}{2\sqrt[3]{x}} : t^{\frac{5}{2}}r \leq x \leq r\right\} \geq t^{\frac{25}{2}}r^5 + \frac{1}{2}r^{-\frac{1}{3}}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_0^1 G(1,s)q(s)\varphi(s,1)\,ds &\leq \frac{1}{\Gamma(\frac{7}{2})l(0)} \int_0^1 \bigl[l(s)(1-s)^{\alpha-\beta-1}\bigr]\frac{1}{4\sqrt{1-s}}\left(1 + \frac{1}{2}s^{-\frac{5}{6}}\right)ds \\ &\leq \frac{1}{\Gamma(\frac{7}{2})l(0)} \int_0^1 \Gamma(2)(1-s)\frac{1}{4\sqrt{1-s}}\left(1 + \frac{1}{2}s^{-\frac{5}{6}}\right)ds < 1 \end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 G(1,s)q(s)\psi\left(s,\frac{1}{100}\right)ds \\
&\geq \frac{1}{\Gamma(\frac{7}{2})l(0)}\int_0^1\left[l(s)(1-s)^{\alpha-\beta-1}-l(0)(1-s)^{\alpha-1}\right] \\
&\quad\times\frac{1}{4\sqrt{1-s}}\left[s^{\frac{25}{2}}\left(\frac{1}{100}\right)^5+\frac{1}{2}\left(\frac{1}{100}\right)^{-\frac{1}{3}}\right]ds \\
&\geq \frac{1}{4\Gamma(\frac{7}{2})l(0)}\int_0^1\left[l(0)(1-s)-l(0)(1-s)^{\frac{5}{2}}\right](1-s)^{-\frac{1}{2}}\left[s^{\frac{25}{2}}\left(\frac{1}{100}\right)^5+\frac{1}{2}\left(\frac{1}{100}\right)^{-\frac{1}{3}}\right]ds \\
&> \frac{1}{100}.
\end{aligned}$$

According to Theorem 3.1, we get that (4.1) has at least one strictly increasing positive solution x^* and $\frac{1}{100} \leq \|x^*\| \leq 1$.

5 Conclusions

Infinite-point boundary value conditions are the classical boundary value conditions. The existence of positive solutions for singular fractional differential problem with infinite-point boundary conditions is established by height functions of the nonlinear term on some bounded sets.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

Acknowledgements
This article was supported by the National Natural Science Foundation of China (Grant No. 11371027) and Anhui Provincial Natural Science Foundation (1608085MA12). The authors would like to thank the referees for their valuable suggestions and comments.

Received: 4 September 2016 Accepted: 18 November 2016 Published online: 09 January 2017

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