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Uniform ultimate boundedness of solutions of predator-prey system with Michaelis-Menten functional response on time scales

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Abstract

In this paper, a predator-prey system with Michaelis-Menten functional response on time scales is investigated. First of all, we generalize the semi-cycle concept to time scales. Second, we obtain the uniformly ultimate boundedness of solutions of this system. Our results demonstrate that when the death rate of the predator is rather small without prey, whereas the intrinsic growth rate of the prey is relatively large, the species could coexist in the long run. In particular, if $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$, some well-known results have been generalized. In addition, for the continuous case, we provide a new idea to prove its permanence. Finally, a numerical simulation is given to support our main results.

MSC: 34N05; 37N25

Keywords: ω-periodic solutions; uniform ultimate boundedness; semi-cycle; time scales; predator-prey system

1 Introduction

The permanence is based on a global criterion for the coexistence of species, which describes a numerical technique for assembly of ecological communities of Lotka-Volterra form [1]. During the last few years, the permanence of ecological models has been discussed by many authors [2–15]. In [4, 5, 10, 11], the comparison method was used, and some sufficient conditions of permanence of biological systems were established. In [6], by using the semi-cycle and related concepts, they discussed the permanence of a discrete biological system.

To the best of our knowledge, the permanence of biological system on time scales was first discussed by Zhang and Zhang [8]. Using the theory of differential inequality, they obtained the permanence of a cooperation system with feedback controls on time scales. But for the system itself, we could verify that some additional conditions in [8] may not be necessary. For further study, in [15], Li and Wang obtained a permanence result for a multispecies Lotka-Volterra mutualism system by establishing some dynamic inequalities on time scales.

In the following we shall use the notation

$$l = \min\{[0,\infty) \cap \mathbb{T}\}, \qquad I_\omega = [l, l+\omega] \cap \mathbb{T},$$
$$f^M = \sup_{t\in\mathbb{T}} f(t), \qquad f^m = \inf_{t\in\mathbb{T}} f(t), \qquad \bar{f} = \frac{1}{\omega}\int_{I_\omega} f(s)\Delta s,$$

where f is a periodic rd-continuous function with period ω in $\mathbb{T}$. We assume that $s+\omega \in \mathbb{T}$ for any $s \in \mathbb{T}$.

Remark 1.1 For any $t > l$, $t \in \mathbb{T}$, we have $t - \rho(t) \le \omega$.

Proof For any $t > l$, $\rho(t) \in \mathbb{T}$, then $\rho(t)+\omega \in \mathbb{T}$, if $t-\rho(t) > \omega$, then $\rho(t)+\omega < t$, this implies $\omega = 0$, we arrive at a contradiction. Thus $t - \rho(t) \le \omega$ holds. □

As is well known, predator-prey systems play an important role in ecosystems [2–7, 9–14, 16–27]. We find that there are few papers discussing the permanence of these systems on time scales.

In this paper, we are concerned with the following predator-prey system with Michaelis-Menten functional response:

$$\begin{cases} x_1^\Delta(t) = a(t) - b(t)\exp\{x_1(t)\} - \frac{c(t)\exp\{x_2(t)\}}{\beta(t)\exp\{x_1(t)\}+\gamma(t)\exp\{x_2(t)\}}, \\ x_2^\Delta(t) = -d(t) + \frac{f(t)\exp\{x_1(t)\}}{\beta(t)\exp\{x_1(t)\}+\gamma(t)\exp\{x_2(t)\}}, \end{cases} \quad t \in \mathbb{T}^k, \tag{1}$$

with the initial condition

$$x_1(t_0), x_2(t_0) \in \mathbb{R}, \quad t_0 \in \mathbb{T},$$

on time scales $\mathbb{T}$, where $a, b, c, d, f, \beta, \gamma \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ are positive ω-periodic functions. The Δ stands for the delta derivative.

Let $x(t) = \exp\{x_1(t)\}$, $y(t) = \exp\{x_2(t)\}$, $\beta(t) = 1$, if $\mathbb{T} = \mathbb{R}$, then (1) reduces to the continuous predator-prey system

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t) - \frac{c(t)y(t)}{x(t)+\gamma(t)y(t)}], \\ y'(t) = y(t)[-d(t) + \frac{f(t)x(t)}{x(t)+\gamma(t)y(t)}], \end{cases} \quad t \in \mathbb{R}. \tag{2}$$

In [16], Fan *et al.* studied some basic problems of system (2), such as positive invariance, permanence, non-persistence, extinction, dissipativity, and globally asymptotic stability. The methods they used were comparison theorems, coincidence degree theory, and the Lyapunov functional. Much attention has been paid to this predator-prey system or its analogs in [2–4, 17–21]. In [4], Fan and Li considered the permanence of the general delayed ratio-dependent predator-prey model

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t)] - c(t)g(\frac{x(t)}{y(t)})y(t), \\ y'(t) = y(t)[-d(t) + e(t)g(\frac{x(t-\tau)}{y(t-\tau)})], \end{cases} \quad t \in \mathbb{R}, \tag{3}$$

where $g(u)$ is monotonic increasing and there exists a constant k such that $\lim_{u\to+\infty} g(u) = k$, by using comparison theorems. They obtained the following.

Theorem 1.1 *Assume*

(C1) $\overline{a} > m\overline{c}$;

(C2) $k\overline{e} > \overline{d}$

hold, here $m = \sup_{z\in[0,+\infty)}\{g(z)/z\}$. *Then system* (3) *is permanent.*

Let $N_1(k) = \exp\{x_1(k)\}$, $N_2(k) = \exp\{x_2(k)\}$, $\beta(k) = 1$, if $\mathbb{T} = \mathbb{Z}$, then (1) is reformulated as

$$\begin{cases} N_1(k+1) = N_1(k)\exp\{a(k) - b(k)N_1(k) - \frac{c(k)N_2(k)}{N_1(k)+\gamma(k)N_2(k)}\}, \\ N_2(k+1) = N_2(k)\exp\{-d(k) + \frac{f(k)N_1(k)}{N_1(k)+\gamma(k)N_2(k)}\}, \end{cases} \quad k \in \mathbb{Z}, \tag{4}$$

this predator-prey system or its other forms has attracted the attention of many authors [5–7]. Fan and Li [5] considered the permanence of a delayed discrete predator-prey model with Holling-type III functional response

$$\begin{cases} N_1(k+1) = N_1(k)\exp\{b_1(k) - a_1(k)N_1(k-[\tau_1]) - \frac{\alpha_1(k)N_1(k)N_2(k)}{N_1^2(k)+m^2(k)N_2^2(k)}\}, \\ N_2(k+1) = N_2(k)\exp\{-b_2(k) + \frac{\alpha_2(k)N_1^2(k-[\tau_2])}{N_1^2(k-[\tau_2])+m^2(k)N_2^2(k-[\tau_2])}\}, \end{cases} \quad k \in \mathbb{Z}, \tag{5}$$

the comparison theorem, and the theory of population equation. They obtained the following.

Theorem 1.2 *Assume*

(B1) $2m\overline{b_1} > \overline{\alpha_1}$;

(B2) $\overline{\alpha_2} > \overline{b_2}$;

(B3) $(b_2(k))^M < 1$

hold. Then system (5) *is permanent.*

In fact, (B3) is not necessary. In [6], by using the semi-cycle and related concepts, Fan and Li considered permanence of the system (4), and obtained the following.

Theorem 1.3 *Assume*

(H1) $\overline{a} > \overline{c/\gamma}$;

(H2) $\overline{f} > \overline{d}$

hold. Then system (4) *is permanent.*

In recent years, the existence of periodic solutions of predator-prey systems on time scales has been obtained by coincidence degree theory in many articles [22–26], since the existence result could be obtained by coincidence degree theory both in the continuous case and the discrete case. Fazly and Hesaaraki [22] obtained the existence of periodic solutions of nonautonomous predator-prey dynamical system with Beddington-DeAngelis functional response by coincidence degree theory. Tong *et al.* [23] investigated the existence of periodic solutions of a predator-prey system with sparse effect and Beddington-DeAngelis or a Holling III functional response. By using a continuation theorem based on coincidence degree theory, they obtained sufficient criteria for the existence of periodic solutions for the system.

Since the permanence of this system has been obtained by the comparison theorem in the continuous case, while it has been obtained by semi-cycle concept in discrete case.

Also, in order to delete the additional condition (B3) and the additional conditions of Theorem 3.1 in [8], we need to extend the semi-cycle concept of the discrete case to that of time scales.

Noticing that system (2) and (4) is derived from (1) by exponential transformations, $x(t) = \exp\{x_1(t)\}$, $y(t) = \exp\{x_2(t)\}$, and $N_1(k) = \exp\{x_1(k)\}$, $N_2(k) = \exp\{x_2(k)\}$, respectively, when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Obviously, when solutions of system (1) are uniformly ultimate bounded, systems (2) and (4) are both permanent, and the contrary also holds true. Thus, our aim is to prove the uniform ultimate boundedness of solutions of system (1) by using the semi-cycle concept on time scales instead of comparison theorems. Our result is an unification and extension of a continuous and discrete analysis.

The rest of the paper is organized as follows. In Section 2, we state some basic properties about time scales, and generalize the semi-cycle concept to time scales. Section 3 is devoted to the uniform ultimate boundedness of solutions of system (1). A discussion is presented in Section 4. The final section of the paper contains a numerical example supporting the result.

2 Preliminary

First we will give some definitions about time scales before presenting our main result (see [28, 29]).

Definition 2.1 A time scale is an arbitrary nonempty closed subset $\mathbb{T}$ of the real number $\mathbb{R}$.

Definition 2.2 For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively.

Throughout this paper we often assume $a \leq b$, $a, b \in \mathbb{T}$, and define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Definition 2.3 Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$, where throughout the paper

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\bigl|\bigl[f\bigl(\sigma(t)\bigr) - f(s)\bigr] - f^{\Delta}(t)\bigl[\sigma(t) - s\bigr]\bigr| \leq \varepsilon\bigl|\sigma(t) - s\bigr| \quad \text{for all } s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger) derivative of f at t.

The following lemmas will be useful to prove our main result. Their proofs are similar to [6], we omit them here.

Lemma 2.1 *The equation*

$$\begin{cases} x^{\Delta}(t) = a(t) - b(t)\exp\{x(t)\}, \\ x(0) = x_0 > 0, \end{cases}$$

has at least one periodic solution $x(t)$ if $a(t)$, $b(t)$ are both ω-periodic rd-continuous functions and $\overline{a} > 0$, $b(t) > 0$; moreover, the inequalities

$$x(t) \le \ln(\overline{a}/\overline{b}) + \overline{a}\omega, \qquad x(t) \ge \ln(\overline{a}/\overline{b}) - \overline{a}\omega$$

hold.

Lemma 2.2 *Assume that d, f, β, γ are all positive ω-periodic rd-continuous functions, then, for any positive constant M, the equation*

$$\begin{cases} x^{\Delta}(t) = -d(t) + \frac{f(t)M}{\beta(t)M+\gamma(t)\exp\{x(t)\}}, \\ x(0) = x_0 > 0, \end{cases}$$

has at least one periodic solution $x(t)$ provided with $\overline{d} < \overline{f/\beta}$; moreover, the inequalities

$$x(t) \le \ln\bigl((\overline{f/\beta}M/\overline{d} - M)/\bigl(\gamma(t)/\beta(t)\bigr)^{m}\bigr) + \overline{d}\omega,$$
$$x(t) \ge \ln\bigl((\overline{f/\beta}M/\overline{d} - M)/\bigl(\gamma(t)/\beta(t)\bigr)^{M}\bigr) - \overline{d}\omega$$

hold.

Similar to the definition of semi-cycle in discrete case (see [30]), we give the definition of a semi-cycle on time scales.

Definition 2.4 A positive semi-cycle of a rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ consists of a 'string' of terms $\{f(t), t \in [s,t], s,t \in \mathbb{T}\}$, all greater than or equal to 0. A negative semi-cycle of a rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ consists of a 'string' of terms $\{f(t), t \in [p,q], p,q \in \mathbb{T}\}$, all less than or equal to 0.

3 The uniform ultimate boundedness

Before giving our main result, we list the definition of uniform ultimate boundedness.

Definition 3.1 Solutions of (1) are said to be uniformly ultimate bounded if there exist two constants λ_1 and λ_2 such that, for any initial condition $(x_1(0), x_2(0))^{\mathbb{T}} \in \mathbb{R}^2$,

$$\lambda_1 \le \lim_{t\to\infty} \inf x_i(t) \le \lim_{t\to\infty} \sup x_i(t) \le \lambda_2, \quad i = 1, 2.$$

Here we say 'uniformly', because λ_1 and λ_2 are independent on $(x_1(0), x_2(0))^{\mathbb{T}}$.

We now state the main result as follows.

Theorem 3.1 *Assume*

(A1) $\overline{a} > \overline{c/\gamma}$;

(A2) $\overline{d} < \overline{f/\beta}$

hold. Then the solutions of system (1) *are uniformly ultimate bounded.*

Proof We will prove $\lambda_1 \leq \liminf_{t\to\infty} x_i(t) \leq \limsup_{t\to\infty} x_i(t) \leq \lambda_2$, λ_1 and λ_2 are constants, $i = 1, 2$. Thus, we divide the proof into four parts.

Part 1. $\limsup_{t\to\infty} x_1(t) \leq K_1$, that is to say, $x_1(t)$ is uniformly ultimate bounded above.

From (1), when t is sufficiently large, we know

$$x_1^{\Delta}(t) \leq a(t) - b(t)\exp\{x_1(t)\}, \tag{6}$$

and construct the following auxiliary equation:

$$y_1^{\Delta}(t) = a(t) - b(t)\exp\{y_1(t)\}. \tag{7}$$

Note that $a(t)$, $b(t)$ are positive ω-periodic functions, we can see $\overline{a} > 0$. By Lemma 2.1, (7) has at least one ω-periodic solution, denote it as $y_1^*(t)$, we have

$$y_1^*(t) \leq \ln(\overline{a}/\overline{b}) + \overline{a}\omega, \qquad y_1^*(t) \geq \ln(\overline{a}/\overline{b}) - \overline{a}\omega. \tag{8}$$

From (6) and (7), we have

$$\begin{aligned}\bigl(x_1(t) - y_1^*(t)\bigr)^{\Delta} &\leq -b(t)\exp\{x_1(t)\} - \bigl(-b(t)\exp\{y_1^*(t)\}\bigr)\\ &= -b(t)\bigl(\exp\{x_1(t)\} - \exp\{y_1^*(t)\}\bigr)\\ &= -b(t)\exp\{y_1^*(t)\}\bigl(\exp\{x_1(t) - y_1^*(t)\} - 1\bigr),\end{aligned}$$

let $u_1(t) = x_1(t) - y_1^*(t)$, then

$$\bigl(u_1(t)\bigr)^{\Delta} \leq -b(t)\exp\{y_1^*(t)\}\bigl(\exp\{u_1(t)\} - 1\bigr). \tag{9}$$

Now, the proof has two cases according to the oscillating property of $u_1(t)$. First we assume that $u_1(t)$ does not oscillate about zero, then $u_1(t)$ will be either eventually positive or eventually negative. If the latter holds, then

$$x_1(t) < y_1^*(t) \leq \ln(\overline{a}/\overline{b}) + \overline{a}\omega. \tag{10}$$

If the former holds, then $(u_1(t))^{\Delta} < 0$, which means that $u_1(t)$ is eventually decreasing, also in terms of its positivity, we know that $\lim_{t\to\infty} u_1(t)$ exists. Then (9) yields $\lim_{t\to\infty} u_1(t) = 0$, which leads to

$$\lim_{t\to\infty}\sup x_1(t) \leq \ln(\overline{a}/\overline{b}) + \overline{a}\omega. \tag{11}$$

Now we assume $u_1(t)$ oscillates about zero, by (7), we know that $u_1(t) \geq 0$ implies $(u_1(t))^{\Delta} \leq 0$. Thus, by the semi-cycle concept, we let $u_1(t) \geq 0$, for $t \in [s_\alpha, t_\alpha]$, $s_\alpha, t_\alpha \in \mathbb{T}$, $\alpha \in \mathcal{L}$, where $\mathcal{L}$ is an index set, and the interval $[s_\alpha, t_\alpha]$ satisfies:

(a1) For any $\alpha, \beta \in \mathcal{L}$, if $\alpha \neq \beta$, $[s_\alpha, t_\alpha] \cap [s_\beta, t_\beta] = \varnothing$.
(b1) If s_α is left-scattered, then $u_1(\rho(s_\alpha)) < 0$.
(c1) If s_α is left-dense, then there exists a hollow left neighborhood $\mathring{U}_-(s_\alpha)$ of s_α such that $u_1(t) < 0$, for $t \in \mathring{U}_-(s_\alpha)$.
(d1) If t_α is right-scattered, then $u_1(\sigma(t_\alpha)) < 0$.
(e1) If t_α is right-dense, then there exists a hollow right neighborhood $\mathring{U}_+(t_\alpha)$ of t_α such that $u_1(t) < 0$, for $t \in \mathring{U}_+(t_\alpha)$.

Notice that $\limsup_{t\to\infty} u_1(t) = \limsup_{\alpha\to\infty} u_1(s_\alpha)$. If s_α is left-scattered, by integrating (9) over the set $[\rho(s_\alpha), s_\alpha]$, we have

$$u_1(s_\alpha) - u_1\bigl(\rho(s_\alpha)\bigr) \le \int_{\rho(s_\alpha)}^{s_\alpha} \bigl[-b(t)\exp\bigl\{y_1^*(t)\bigr\}\bigl(\exp\bigl\{u_1(t)\bigr\} - 1\bigr)\bigr]\Delta t,$$

by (b1), it follows that

$$u_1(s_\alpha) \le \int_{\rho(s_\alpha)}^{s_\alpha} b(t)\exp\bigl\{y_1^*(t)\bigr\}\Delta t < \infty, \tag{12}$$

if and only if

$$s_\alpha - \rho(s_\alpha) \le \omega, \tag{13}$$

it is easy to see (13) holds from Remark 1.1.

If s_α is left-dense, we choose $t_1 \in \mathring{U}_-(s_\alpha)$, such that $s_\alpha - t_1 \le \omega$. By integrating (9) over the set $[t_1, s_\alpha]$, we have

$$u_1(s_\alpha) - u_1(t_1) = \int_{t_1}^{s_\alpha} \bigl[-b(t)\exp\bigl\{y_1^*(t)\bigr\}\bigl(\exp\bigl\{u_1(t)\bigr\} - 1\bigr)\bigr]\Delta t,$$

notice that $u_1(t_1) < 0$, it follows that

$$u_1(s_\alpha) \le \int_{t_1}^{s_\alpha} b(t)\exp\bigl\{y_1^*(t)\bigr\}\Delta t < \infty. \tag{14}$$

Then from (12) and (14), $u_1(t) = x_1(t) - x_1^*(t)$ is uniformly ultimate bounded above, thus

$$\lim_{t\to\infty}\sup x_1(t) \le \ln(\overline{a}/\overline{b}) + \overline{a}\omega + \overline{a}\omega\exp(\overline{a}\omega) := K_1. \tag{15}$$

Therefore from (10), (11), and (15), $x_1(t)$ is uniformly ultimate bounded above.

Part 2. $k_1 \le \liminf_{t\to\infty} x_1(t)$, that is to say, $x_1(t)$ is uniformly ultimate bounded below.

Also from (1), when t is sufficiently large, we know

$$x_1^\Delta(t) \ge a(t) - c(t)/\gamma(t) - b(t)\exp\bigl(x_1(t)\bigr), \tag{16}$$

and construct the following auxiliary equation:

$$z_1^\Delta(t) = a(t) - c(t)/\gamma(t) - b(t)\exp\bigl(z_1(t)\bigr). \tag{17}$$

From (A1) and Lemma 2.1, (17) has at least one ω-periodic solution, denote it as $z_1^*(t)$, we have

$$z_1^*(t) \le \ln\bigl((\overline{a} - \overline{c/\gamma})/\overline{b}\bigr) + \overline{a}\omega, \qquad z_1^*(t) \ge \ln\bigl((\overline{a} - \overline{c/\gamma})/\overline{b}\bigr) - \overline{a}\omega. \tag{18}$$

From (16) and (17), we have

$$\begin{aligned}\bigl(x_1(t) - z_1^*(t)\bigr)^\Delta &\ge -b(t)\exp\bigl\{x_1(t)\bigr\} - \bigl(-b(t)\exp\bigl\{z_1^*(t)\bigr\}\bigr)\\ &= -b(t)\bigl(\exp\bigl\{x_1(t)\bigr\} - \exp\bigl\{z_1^*(t)\bigr\}\bigr)\\ &= -b(t)\exp\bigl\{z_1^*(t)\bigr\}\bigl(\exp\bigl\{x_1(t) - z_1^*(t)\bigr\} - 1\bigr),\end{aligned}$$

let $u_2(t) = x_1(t) - z_1^*(t)$, then

$$\bigl(u_2(t)\bigr)^\Delta \ge -b(t)\exp\bigl\{z_1^*(t)\bigr\}\bigl(\exp\bigl\{u_2(t)\bigr\} - 1\bigr), \tag{19}$$

from (15) and (18), we know when t is sufficiently large,

$$u_2(t) \le \ln\bigl(\overline{a}/(\overline{a} - \overline{c/\gamma})\bigr) + 2\overline{a}\omega + \overline{a}\omega\exp(\overline{a}\omega) := M_1. \tag{20}$$

Now, the proof has two cases according to the oscillating property of $u_2(t)$. First we assume that $u_2(t)$ does not oscillate about zero, then $u_2(t)$ will be either eventually positive or eventually negative. If the former holds, then

$$x_1(t) > z_1^*(t) \ge \ln\bigl((\overline{a} - \overline{c/\gamma})/\overline{b}\bigr) - \overline{a}\omega. \tag{21}$$

If the latter holds, then $(u_2(t))^\Delta > 0$, which means that $u_2(t)$ is eventually increasing, also in terms of its negativity, we know that $\lim_{t\to\infty} u_2(t)$ exists. Then (26) yields $\lim_{t\to\infty} u_2(t) = 0$, which leads to

$$\liminf_{t\to\infty} x_1(t) \ge \ln\bigl((\overline{a} - \overline{c/\gamma})/\overline{b}\bigr) - \overline{a}\omega. \tag{22}$$

Now we assume $u_2(t)$ oscillates about zero, by (19), we know that $u_2(t) \le 0$ implies $(u_2(t))^\Delta \ge 0$. Thus, by the semi-cycle concept, we let $u_2(t) \le 0$, for $t \in [p_\alpha, q_\alpha]$, $p_\alpha, q_\alpha \in \mathbb{T}$, $\alpha \in \mathcal{L}$, where $\mathcal{L}$ is an index set, and the interval $[p_\alpha, q_\alpha]$ satisfies:

(a2) For any $\alpha, \beta \in \mathcal{L}$, if $\alpha \ne \beta$, $[p_\alpha, q_\alpha] \cap [p_\alpha, q_\alpha] = \varnothing$.

(b2) If p_α is left-scattered, then $u_2(\rho(p_\alpha)) > 0$.

(c2) If p_α is left-dense, then there exists a hollow left neighborhood $\mathring{U}_-(p_\alpha)$ of p_α such that $u_2(t) > 0$, for $t \in \mathring{U}_-(p_\alpha)$.

(d2) If q_α is right-scattered, then $u_2(\sigma(q_\alpha)) > 0$.

(e2) If q_α is right-dense, then there exists a hollow right neighborhood $\mathring{U}_+(q_\alpha)$ of q_α such that $u_2(t) > 0$, for $t \in \mathring{U}_+(q_\alpha)$.

Notice that $\liminf_{t\to\infty} u_2(t) = \liminf_{\alpha\to\infty} u_2(p_\alpha)$. If p_α is left-scattered, by integrating (19) over the set $[\rho(p_\alpha), p_\alpha]$, we have

$$u_2(p_\alpha) - u_2\bigl(\rho(p_\alpha)\bigr) \ge \int_{\rho(p_\alpha)}^{p_\alpha} \bigl[-b(t)\exp\bigl\{z_1^*(t)\bigr\}\bigl(\exp\bigl\{u_2(t)\bigr\} - 1\bigr)\bigr]\Delta t,$$

by (b2), from (20), it follows that

$$u_2(p_\alpha) \geq \int_{\rho(p_\alpha)}^{p_\alpha} \left[-b(t)\exp\{M_1\}\exp\left\{z_1^*(t)\right\}\right]\Delta t,$$

from Remark 1.1, we can see

$$p_\alpha - \rho(p_\alpha) \leq \omega,$$

then $u_2(t_n)$ is lower bounded.

If p_α is left-dense, we choose $t_2 \in \mathring{U}_-(p_\alpha)$, such that $p_\alpha - t_2 \leq \omega$. Notice that $u_2(t_2) > 0$, by integrating (19) over the set $[t_2, p_\alpha]$, we have

$$u_2(p_\alpha) - u_2(t_2) \geq \int_{t_2}^{p_\alpha} \left[-b(t)\exp\left\{z_1^*(t)\right\}\left(\exp\left\{u_2(t)\right\} - 1\right)\right]\Delta t,$$

from (20), we have

$$u_2(p_\alpha) \geq \int_{t_2}^{p_\alpha} \left[-b(t)\exp\{M_1\}\exp\left\{z_1^*(t)\right\}\right]\Delta t,$$

in this case, $u_2(p_\alpha)$ is also lower bounded.

Then $u_2(t) = x_1(t) - z_1^*(t)$ is uniformly ultimate bounded below, hence

$$\lim_{t\to\infty} \inf x_1(t) \geq \ln\left((\overline{a} - \overline{c/\gamma})/\overline{b}\right) - \overline{a}\omega - (\overline{a} - \overline{c/\gamma})\omega\exp(\overline{a}\omega + M_1) := k_1. \tag{23}$$

Thus from (21), (22), and (23), $x_1(t)$ is uniformly ultimate bounded below. Then Part 2 holds.

Therefore, from Part 1 and Part 2, $x_1(t)$ is uniformly ultimate bounded, we can assume $m \leq \exp\{x_1(t)\} \leq M$ for any $t \in [T_0, \infty) \cap \mathbb{T}$, where T_0 is sufficiently large.

Part 3. $k_1 \leq \liminf_{t\to\infty} x_2(t)$, that is to say, $x_2(t)$ is uniformly ultimate bounded below.

From (1), when t is sufficiently large, we know

$$x_2^\Delta(t) \geq -d(t) + \frac{f(t)m}{\beta(t)m + \gamma(t)\exp\{x_2(t)\}}, \tag{24}$$

and we construct the following auxiliary equation:

$$z_2^\Delta(t) = -d(t) + \frac{f(t)m}{\beta(t)m + \gamma(t)\exp\{z_2(t)\}}, \tag{25}$$

from Lemma 2.2 and (A2), we find that (25) has at least one ω-periodic solution, denote it as $z_2^*(t)$, we have

$$z_2^*(t) \leq \ln\left((\overline{f/\beta}m/\overline{d} - m)/\left(\gamma(t)/\beta(t)\right)^m\right) + \overline{d}\omega,$$

$$z_2^*(t) \geq \ln\left((\overline{f/\beta}m/\overline{d} - m)/\left(\gamma(t)/\beta(t)\right)^M\right) - \overline{d}\omega.$$

From (25) and (24), we have

$$\begin{aligned}
\big(x_2(t)-z_2^*(t)\big)^{\Delta} &\geq \frac{f(t)m}{\beta(t)m+\gamma(t)\exp\{x_2(t)\}}-\frac{f(t)m}{\beta(t)m+\gamma(t)\exp\{z_2^*(t)\}}\\
&=\frac{f(t)m\gamma(t)(\exp\{z_2^*(t)\}-\exp\{x_2(t)\})}{(\beta(t)m+\gamma(t)\exp\{x_2(t)\})(\beta(t)m+\gamma(t)\exp\{z_2^*(t)\})}\\
&=\frac{f(t)m\gamma(t)\exp\{x_2(t)\}(\exp\{z_2^*(t)-x_2(t)\}-1)}{(\beta(t)m+\gamma(t)\exp\{x_2(t)\})(\beta(t)m+\gamma(t)\exp\{z_2^*(t)\})}.
\end{aligned}$$

Similarly, let $v_2(t)=x_2(t)-z_2^*(t)$, then

$$\big(v_2(t)\big)^{\Delta}\geq\frac{f(t)m\gamma(t)\exp\{x_2(t)\}(\exp\{-v_2(t)\}-1)}{(\beta(t)m+\gamma(t)\exp\{x_2(t)\})(\beta(t)m+\gamma(t)\exp\{z_2^*(t)\})}. \tag{26}$$

Now, the proof has two cases according to the oscillating property of $v_2(t)$. First we assume that $v_2(t)$ does not oscillate about zero, similar to $u_2(t)$ in Part 2, we obtain

$$\lim_{t\to\infty}\inf x_2(t)\geq\ln\big(\overline{(f/\beta}m/\overline{d}-m)/\big(\gamma(t)/\beta(t)\big)^{M}\big)-\overline{d}\omega. \tag{27}$$

Now we assume $v_2(t)$ oscillates about zero, by (26), we know that $v_2(t)\leq 0$ implies $(v_2(t))^{\Delta}\geq 0$. Thus, by the semi-cycle concept, we let $v_2(t)\leq 0$, for $t\in[p_\alpha,q_\alpha]$, $p_\alpha,q_\alpha\in\mathbb{T}$, $\alpha\in\mathcal{L}$, where $\mathcal{L}$ is an index set, the interval $[p_\alpha,q_\alpha]$ satisfies (a2)-(e2) by replacing $u_2(t)$ in Part 2 with $v_2(t)$.

Notice that $\limsup_{t\to\infty}v_2(t)=\limsup_{\alpha\to\infty}v_2(p_\alpha)$. By a similar analysis to Part 2, if p_α is left-scattered, integrating inequality (26) from $\rho(p_\alpha)$ to p_α, we have

$$v_2(p_\alpha)-v_2\big(\rho(p_\alpha)\big)\geq\int_{\rho(p_\alpha)}^{p_\alpha}\frac{f(t)m\gamma(t)\exp\{x_2(t)\}(\exp\{-v_2(t)\}-1)}{(\beta(t)m+\gamma(t)\exp\{x_2(t)\})(\beta(t)m+\gamma(t)\exp\{z_2^*(t)\})}\Delta t,$$

by (b2), it follows that

$$\begin{aligned}
v_2(p_\alpha)&\geq\int_{\rho(p_\alpha)}^{p_\alpha}\bigg[-\frac{f(t)m\gamma(t)\exp\{x_2(t)\}}{(\beta(t)m+\gamma(t)\exp\{x_2(t)\})(\beta(t)m+\gamma(t)\exp\{z_2^*(t)\})}\bigg]\Delta t\\
&\geq\int_{\rho(p_\alpha)}^{p_\alpha}\bigg[-\frac{f(t)m}{\gamma(t)\exp\{z_2^*(t)\}}\bigg]\Delta t,
\end{aligned}$$

by Remark 1.1, $v_2(p_\alpha)$ is lower bounded.

If p_α is left-dense, we choose $t_3\in\mathring{U}_-(p_\alpha)$, such that $p_\alpha-t_3\leq\omega$. By integrating (26) over the set $[t_3,p_\alpha]$, we have

$$v_2(p_\alpha)-v_2(t_3)=\int_{t_3}^{p_\alpha}\frac{f(t)m\gamma(t)\exp\{x_2(t)\}(\exp\{-v_2(t)\}-1)}{(\beta(t)m+\gamma(t)\exp\{x_2(t)\})(\beta(t)m+\gamma(t)\exp\{z_2^*(t)\})}\Delta t,$$

notice that $v_2(t_3)>0$, hence

$$\begin{aligned}
v_2(p_\alpha)&\geq\int_{t_3}^{p_\alpha}\bigg[-\frac{f(t)m\gamma(t)\exp\{x_2(t)\}}{(\beta(t)m+\gamma(t)\exp\{x_2(t)\})(\beta(t)m+\gamma(t)\exp\{z_2^*(t)\})}\bigg]\Delta t\\
&\geq\int_{t}^{p_\alpha}\bigg[-\frac{f(t)m}{\gamma(t)\exp\{z_2^*(t)\}}\bigg]\Delta t,
\end{aligned}$$

in this case, $v_2(p_\alpha)$ is also lower bounded.

Then $v_2(t) = x_2(t) - z_2^*(t)$ is uniformly ultimate bounded below, hence

$$\begin{aligned}\lim_{t\to\infty}\inf x_2(t) &\geq \ln\bigl((\overline{f/\beta}m/\overline{d} - m)/\bigl(\gamma(t)/\beta(t)\bigr)^M\bigr) - \overline{d}\omega \\ &\quad - m\overline{f/\gamma}\omega\bigl(\gamma(t)/\beta(t)\bigr)^M\exp(\overline{d}\omega)/(\overline{f/\beta}m/\overline{d} - m) := k_2. \end{aligned} \tag{28}$$

Thus from (27) and (28), $x_2(t)$ is uniformly ultimate bounded below.

Part 4. $\limsup_{t\to\infty} x_2(t) \leq K_2$, that is to say, $x_2(t)$ is uniformly ultimate bounded above.

Also from (1), when t is sufficiently large, we know

$$x_2^\Delta(t) \leq -d(t) + \frac{f(t)M}{\beta(t)M + \gamma(t)\exp\{x_2(t)\}}, \tag{29}$$

and we construct the following auxiliary equation:

$$y_2^\Delta(t) = -d(t) + \frac{f(t)M}{\beta(t)M + \gamma(t)\exp\{y_2(t)\}}, \tag{30}$$

from Lemma 2.2 and (A2), we find that (30) has at least an ω-periodic solution, denote it as $y_2^*(t)$, we have

$$\begin{aligned} &y_2^*(t) \leq \ln\bigl((\overline{f/\beta}M/\overline{d} - M)/\bigl(\gamma(t)/\beta(t)\bigr)^m\bigr) + \overline{d}\omega, \\ &y_2^*(t) \geq \ln\bigl((\overline{f/\beta}M/\overline{d} - M)/\bigl(\gamma(t)/\beta(t)\bigr)^M\bigr) - \overline{d}\omega. \end{aligned} \tag{31}$$

From (29) and (30), we have

$$\begin{aligned}\bigl(x_2(t) - y_2^*(t)\bigr)^\Delta &\leq \frac{f(t)M}{\beta(t)M + \gamma(t)\exp\{x_2(t)\}} - \frac{f(t)M}{\beta(t)M + \gamma(t)\exp\{y_2^*(t)\}} \\ &= \frac{f(t)M\gamma(t)(\exp\{y_2^*(t)\} - \exp\{x_2(t)\})}{(\beta(t)M + \gamma(t)\exp\{x_2(t)\})(\beta(t)M + \gamma(t)\exp\{y_2^*(t)\})} \\ &= \frac{f(t)M\gamma(t)\exp\{x_2(t)\}(\exp\{y_2^*(t) - x_2(t)\} - 1)}{(\beta(t)M + \gamma(t)\exp\{x_2(t)\})(\beta(t)M + \gamma(t)\exp\{y_2^*(t)\})}. \end{aligned}$$

Similarly, let $v_1(t) = x_2(t) - y_2^*(t)$, then

$$\bigl(v_1(t)\bigr)^\Delta \leq \frac{f(t)M\gamma(t)\exp\{x_2(t)\}(\exp\{-v_1(t)\} - 1)}{(\beta(t)M + \gamma(t)\exp\{x_2(t)\})(\beta(t)M + \gamma(t)\exp\{y_2^*(t)\})}, \tag{32}$$

from (28) and (31), we know, when t is sufficiently large,

$$\begin{aligned} v_1(t) &\geq \ln\biggl(\frac{\overline{f/\beta}m/\overline{d} - m}{(\gamma(t)/\beta(t))^{M-m}(\overline{f/\beta}M/\overline{d} - M)}\biggr) - 2\overline{d}\omega \\ &\quad - m\overline{f/\gamma}\omega\bigl(\gamma(t)/\beta(t)\bigr)^M\exp(\overline{d}\omega)/(\overline{f/\beta}m/\overline{d} - m) := m_1. \end{aligned} \tag{33}$$

Now, the proof has two cases according to the oscillating property of $v_1(t)$. First we assume that $v_1(t)$ does not oscillate about zero, similar to $u_1(t)$ in Part 1, we obtain

$$\lim_{t\to\infty}\sup x_2(t) \leq \ln\bigl((\overline{f/\beta}M/\overline{d} - M)/\bigl(\gamma(t)/\beta(t)\bigr)^m\bigr) + \overline{d}\omega. \tag{34}$$

Now we assume $v_1(t)$ oscillates about zero, by (32), we know that $v_1(t) \geq 0$ implies $(v_1(t))^{\Delta} \leq 0$. Thus, by the semi-cycle concept, we let $v_1(t) \geq 0$, for $t \in [s_\alpha, t_\alpha]$, $s_\alpha, t_\alpha \in \mathbb{T}$, $\alpha \in \mathcal{L}$, where $\mathcal{L}$ is an index set, the interval $[s_\alpha, t_\alpha]$ satisfies (a1)-(e1) by replacing $u_1(t)$ in Part 1 with $v_1(t)$.

Notice that $\limsup_{t\to\infty} v_1(t) = \limsup_{\alpha\to\infty} v_1(s_\alpha)$. By a similar analysis to Part 1, if s_α is left-scattered, integrating inequality (32) from $\rho(s_\alpha)$ to s_α, we have

$$v_1(s_\alpha) - v_1\bigl(\rho(s_\alpha)\bigr) \leq \int_{\rho(s_\alpha)}^{s_\alpha} \frac{f(t)M\gamma(t)\exp\{x_2(t)\}(\exp\{-v_1(t)\}-1)}{(\beta(t)M+\gamma(t)\exp\{x_2(t)\})(\beta(t)M+\gamma(t)\exp\{y_2^*(t)\})}\Delta t,$$

by (b1), from (33), it follows that

$$\begin{aligned} v_1(s_\alpha) &\leq \int_{\rho(s_\alpha)}^{s_\alpha} \frac{f(t)M\gamma(t)\exp\{x_2(t)\}\exp\{-m_1\}}{(\beta(t)M+\gamma(t)\exp\{x_2(t)\})(\beta(t)M+\gamma(t)\exp\{y_2^*(t)\})}\Delta t \\ &\leq \int_{\rho(s_\alpha)}^{s_\alpha} \frac{f(t)M\exp\{-m_1\}}{\gamma(t)\exp\{y_2^*(t)\}}\Delta t < \infty. \end{aligned} \tag{35}$$

If s_α is left-dense, we choose $t_4 \in \mathring{U}_-(s_\alpha)$, such that $s_\alpha - t_4 \leq \omega$. By integrating (32) over the set $[t_4, s_\alpha]$, we have

$$v_1(s_\alpha) - v_1(t_4) = \int_{t_4}^{s_\alpha} \frac{f(t)M\gamma(t)\exp\{x_2(t)\}(\exp\{-v_1(t)\}-1)}{(\beta(t)M+\gamma(t)\exp\{x_2(t)\})(\beta(t)M+\gamma(t)\exp\{y_2^*(t)\})}\Delta t,$$

notice that $v_1(t_4) < 0$, hence

$$\begin{aligned} v_1(s_\alpha) &\leq \int_{t_4}^{s_\alpha} \frac{f(t)M\gamma(t)\exp\{x_2(t)\}\exp\{-m_1\}}{(\beta(t)M+\gamma(t)\exp\{x_2(t)\})(\beta(t)M+\gamma(t)\exp\{y_2^*(t)\})}\Delta t \\ &\leq \int_{t_4}^{s_\alpha} \frac{f(t)M\exp\{-m_1\}}{\gamma(t)\exp\{y_2^*(t)\}}\Delta t < \infty. \end{aligned} \tag{36}$$

Then from (35) and (36), $v_1(t) = x_2(t) - y_2^*(t)$ is uniformly ultimate bounded above, thus

$$\begin{aligned} \lim_{t\to\infty}\sup x_2(t) &\leq \ln\bigl((\overline{f/\beta}M/\overline{d} - M)/\bigl(\gamma(t)/\beta(t)\bigr)^m\bigr) + \overline{d}\omega \\ &\quad + M\overline{f/\gamma}\omega\bigl(\gamma(t)/\beta(t)\bigr)^M \exp(\overline{d}\omega - m_1)/(\overline{f/\beta}M/\overline{d} - M) := K_2. \end{aligned} \tag{37}$$

Thus from (34) and (37), $x_2(t)$ is uniformly ultimate bounded above.

Therefore, from Part 3 and Part 4, $x_2(t)$ is uniformly ultimate bounded.

Finally, we choose $\lambda_1 = \min\{k_1, k_2\}$, $\lambda_2 = \max\{K_1, K_2\}$. This completes the proof of Theorem 3.1. □

4 Discussion

In Theorem 3.1, if we let $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, then the result is exactly changed into Theorem 1.1 (by the comparison theorem) and Theorem 1.3 (by the semi-cycle theory). That is, we provide a unified method to study the permanence for the continuous system and discrete system. In addition, we give a new method to investigate the permanence for the continuous system.

From the proof of Theorem 3.1, we can easily see that our methods can also be used to study the following generalized predator-prey system with functional response:

$$\begin{cases} x_1^{\Delta}(t) = a(t) - b(t)\exp\{x_1(t)\} - c(t)h(\exp\{x_1(t) - x_2(t)\}), \\ x_2^{\Delta}(t) = -d(t) + e(t)g(\exp\{x_1(t) - x_2(t)\}), \end{cases} \tag{38}$$

here $h(u) = g(u)/u$, where $g(u)$ is monotonic increasing and there exists a constant k such that $\lim_{u\to+\infty} g(u) = k$. The coefficient functions are all bounded.

As is well known, the permanence of the periodic biological system is closely associated with the existence for the periodic solutions of the system, in general, when the periodic system is permanent, then there must exist at least one positive periodic solution. By a similar analysis to that in Fan and Wang [27], we can obtain the following remark.

Remark 4.1 Assume (A1), (A2) hold, then system (1) has at least one ω-periodic solution.

This shows that the conditions for uniform ultimate boundedness of solutions are the same as that for the existence of periodic solutions of the system.

5 Numerical example

In this section, we give a numerical example to support our main result. Assume that $a(t) = 1.2 + 0.1\sin(\pi t)$, $b(t) = 0.006 + 0.005\sin(\pi t)$, $c(t) = 0.4 + 0.1\cos(\pi t)$, $d(t) = 0.5 + 0.3\sin(\pi t)$, $f(t) = 0.5 + 0.05\cos(\pi t)$, $\gamma(t) = 0.5 + 0.1\cos(\pi t)$, $\beta(t) = 0.45 + 0.05\cos(\pi t)$, and $\mathbb{T} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$. In this case, we can see

$$\begin{cases} x_1(2k) = x_1(2k-1) + a(2k-1) - b(2k-1)\exp\{x_1(2k-1)\} \\ \qquad - \frac{c(2k-1)\exp\{x_2(2k-1)\}}{\beta(2k-1)\exp\{x_1(2k-1)\}+\gamma(2k-1)\exp\{x_2(2k-1)\}}, \\ x_2(2k) = x_2(2k-1) - d(2k-1) + \frac{f(2k-1)\exp\{x_1(2k-1)\}}{\beta(2k-1)\exp\{x_1(2k-1)\}+\gamma(2k-1)\exp\{x_2(2k-1)\}}, \end{cases}$$

here $k = 1, 2, 3, \ldots$. In applying a numerical analysis using Matlab, we assume that $x_1(0) = 4.7$, $x_2(0) = 4.8$, and then obtain Figure 1. It is easy to see $x_1(t)$, $x_2(t)$ are uniformly ultimate bounded. We also obtain the relationship between $x_1(t)$ and $x_2(t)$ (see (B) of Figure 1). Our numerical simulation supports our theoretical findings (see the figures). We conclude that it is valid for any initial condition $(x_1(0), x_2(0))^{\mathbb{T}} \in \mathbb{R}^2$.

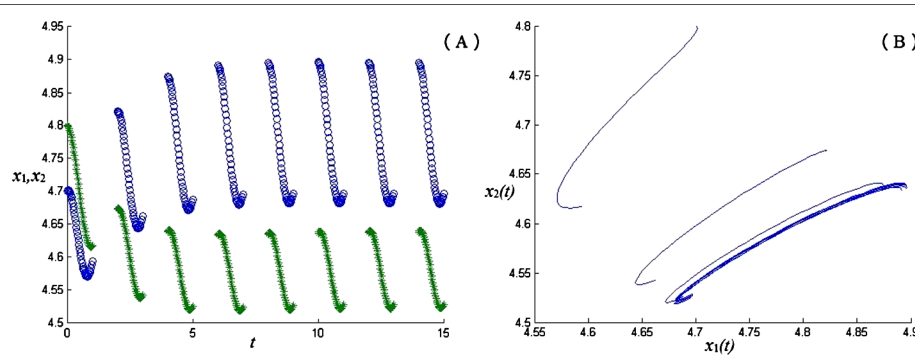

Figure 1 For the initial conditions $x_1(0) = 4.7$, $x_2(0) = 4.8$. 'o' represents $x_1(t)$, '∗' represents $x_2(t)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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